

A Model Analysis of Temporally Asymmetric Hebbian Learning

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Abstract: - Among a lot of models for learning in neural networks, Hebbian and anti-Hebbian learnings might be the most familiar ones. Although there are many variants, the most typical paradigms are such that when pre- and post-synaptic activations (firing) occur at the same time, synaptic efficacy is increased (Hebbian) or decreased (anti-Hebbian). According recent neurophysiological observations, however, synaptic modification in neurons depends on the precise temporal relation between pre-synaptic and post-synaptic activities. Namely, pre-synaptic spikes that precede postsynaptic firing lead to synaptic potentiation, while those that follow postsynaptic firing elicit synaptic depression. This kind of asymmetric feature of neural plasticity seems to play a very important role in temporal behavior of animals. The purpose of this study is to address certain potential implications of the asymmetric Hebbian (anti-Hebbian) learning through a mathematical model and computer simulations. Although the neuron model and the learning rule used in this study are extremely simplified compared with those of real neurons, they can explain some salient features of temporal behavior in animals.

Key words: - Hebbian learning, Anti-Hebbian learning, unsupervised learning

1. Introduction

Among a lot of models for learning in neural networks, Hebbian and anti-Hebbian learning might be the most familiar ones in the community of artificial neural networks. Although there are many variants, the most typical paradigms are such that when pre- and post-synaptic activations occur at the same time, synaptic efficacy is increased (Hebbian) or decreased (anti-Hebbian).[2] According recent neuro-physiological observations, however, synaptic modification in neurons depends on the precise temporal relation between pre-synaptic and post-synaptic firing.[4] Namely, pre-synaptic spikes that precede postsynaptic firing lead to synaptic

potentiation, while those that follow postsynaptic firing elicit synaptic depression. This kind of asymmetric feature of neural plasticity seems to play a very important role in temporal learning involved in the behavior of animals.

The purpose of this study is to address some potential implications of the asymmetric Hebbian learning through a mathematical model. Although the neuron model and the learning rule considered in this paper might be extremely simplified compared with those of real neurons, they can explain some salient features of temporal behavior in animals. For example, memory of temporal patterns, self-organization of coincident detectors, and generation of very slow temporal patterns

(oscillations) can be explained in the framework of the present model.

2. Mathematical Formulation

2.1 A simple circuit dynamics and a learning dynamics

We start with a ‘static’ system with N inputs and a single cell, whose input-output relation is given by

$$y(t) = \sum_{i=1}^N w_i(t) s_i(t) \quad (1)$$

or in a vector form

$$y(t) = \mathbf{w}(t) \mathbf{s}(t), \quad (2)$$

where $\mathbf{s}(t) = [s_1(t), \dots, s_N(t)]^T$ and $\mathbf{w}(t) =$

$[w_1(t), \dots, w_N(t)]$. The learning equation is

$$\Delta w_i(t) \triangleq w_i(t+1) - w_i(t) = \mathbf{a} y(t) s_i(t) \quad (3)$$

or

$$\Delta \mathbf{w}(t) = \mathbf{a} y(t) \mathbf{s}^T(t), \quad (4)$$

where \mathbf{a} is a learning coefficient, being a small positive or negative constant. When \mathbf{a} is positive / negative, eqn (4) represents a Hebbian learning / anti-Hebbian learning. According to algorithm (4) by itself, the magnitude of $\mathbf{w}(t)$ diverges to infinity or shrinks to zero (corresponding to the signs), so some normalization operation is usually introduced, for example,

$$\mathbf{w}(t+1) = \frac{\mathbf{w}(t) + \Delta \mathbf{w}(t)}{\|\mathbf{w}(t) + \Delta \mathbf{w}(t)\|}. \quad (5)$$

This operation maintains the magnitude (or norm) of $\mathbf{w}(t)$ to be zero through the entire process. It is well known that the weight vector converges to the most dominant or the least dominant eigenvector of the ‘instantaneous’ correlation matrix of $\mathbf{s}(t)$, i.e.

$\mathbf{R} = E[\mathbf{s}(t) \mathbf{s}^T(t)]$. Namely, the network becomes a

principal or minor component analyzer. Of course, eqn (4) with (5) is just an instance among a lot of models representing Hebb-type learning.

We want to extend the above ‘static’ model to a more sophisticated one that involves two kinds of

temporal dynamics:

- (i) To incorporate a temporal dynamics into the circuit equation (1);
- (ii) To incorporate a temporal characteristics into the learning equation (4).

In the new model described below, the dynamics of the weight vector $\mathbf{w}(t)$ will depend on the auto-correlation matrix $\mathbf{R}(\mathbf{t}) = E[\mathbf{s}(t) \mathbf{s}^T(t+\mathbf{t})]$

($\mathbf{t} = \dots, -1, 0, 1, \dots$).

2.2 A generalized model

As for (i), we consider a more general model.

$$y(t) = \sum_{i=1}^N w_i(t) \sum_{\mathbf{t}=0}^L h_i(\mathbf{t}) s_i(t-\mathbf{t}). \quad (6)$$

Here, $h_i(\mathbf{t})$ ($\mathbf{t} = 0, 1, \dots, L$) is a fixed weighting function. Note that $\{w_i(t) h_i(0), \dots, w_i(t) h_i(L)\}$ is an impulse response at time t for input $s_i(t)$. Rewriting this in a matrix form, we have

$$y(t) = \mathbf{w}(t) \sum_{\mathbf{t}=0}^L \mathbf{H}(\mathbf{t}) \mathbf{s}(t-\mathbf{t}), \quad (7)$$

where $\mathbf{H}(\mathbf{t}) = \text{diag}\{h_1(\mathbf{t}), \dots, h_N(\mathbf{t})\}$.

The basic learning equation (3) can be expressed as

$$w_i(t+1) = w_i(0) + \mathbf{a} \sum_{t'=0}^t y(t') s_i(t'). \quad (8)$$

Here, \mathbf{a} is a small positive constant. The above equation implies that weight $w_i(t)$ at time t is an accumulation of the product of input $s_i(t')$ and the output $y(t')$ in the past. We here generalize this equation to

$$w_i(t+1) = w_i(0) + \mathbf{a} \sum_{t'=0}^t \sum_{t''=0}^t y(t') s_i(t'') g_i(t''-t'). \quad (9)$$

Namely, this equation implies that $w_i(t)$ depends on the temporal relation between the input and the output. The kernel $g_i(\mathbf{t})$ is considered zero for

large \mathbf{t} , i.e., $g_i(\mathbf{t}) = 0$ for $|\mathbf{t}| > K$. An example of

$g_i(\mathbf{t})$ is shown in Fig.1. This implies, in terminology of neurophysiology, that pre-synaptic activation that precedes the postsynaptic activations lead to synaptic potentiation (Hebbian learning), while those that follow postsynaptic firing elicit

synaptic depression (anti-Hebbian learning). We call this kind of learning ‘asymmetric Hebbian learning’.

From eqn (9) we have

$$\begin{aligned}\Delta w_i(t) &= \mathbf{a} \left\{ \sum_{t'=0}^t y(t') s_i(t) g_i(t-t') + \sum_{t'=0}^{t-1} y(t') s_i(t') g_i(t'-t) \right\} \\ &= \mathbf{a} \left\{ \sum_{t=0}^t y(t-t) s_i(t) g_i(t) + \sum_{t=-t}^{-1} y(t) s_i(t+t) g_i(t) \right\}\end{aligned}$$

Expressing this equation in matrix form, we have

$$\Delta \mathbf{w}(t) =$$

$$\mathbf{a} \left\{ \sum_{t=0}^t y(t-t) \mathbf{s}^T(t) \mathbf{G}(t) + \sum_{t=-t}^{-1} y(t) \mathbf{s}^T(t+t) \mathbf{G}(t) \right\}$$

where $\mathbf{g}(t) = \text{diag}\{g_1(t), \dots, g_N(t)\}$. After $t \geq K$,

we can write this as

$$\Delta \mathbf{w}(t) = \mathbf{a} \left\{ \sum_{t=0}^K y(t-t) \mathbf{s}^T(t) \mathbf{G}(t) + \sum_{t=-K}^{-1} y(t) \mathbf{s}^T(t+t) \mathbf{G}(t) \right\} \quad (10)$$

This is the algorithm that we want to investigate in the paper, though some variations will appear in the next section, depending on each application.

2.3 An analysis of the asymmetric learning

Substituting eqn (7) into eqn (10), we have

$$\begin{aligned}\Delta \mathbf{w}(t) &= \mathbf{a} \left\{ \sum_{t=0}^K \sum_{t'=0}^L \mathbf{w}(t-t) \mathbf{H}(t') \mathbf{s}(t-t') \mathbf{s}^T(t') \mathbf{G}(t) \right. \\ &\quad \left. + \sum_{t=-K}^{-1} \sum_{t'=0}^L \mathbf{w}(t) \mathbf{H}(t') \mathbf{s}(t-t') \mathbf{s}^T(t+t') \mathbf{G}(t) \right\} \\ &\quad (11)\end{aligned}$$

Here, we assume that $\mathbf{s}(t)$ is a stationary random process and moreover that \mathbf{a} is very small and hence $\mathbf{w}(t)$ varies very slowly, implying that $\mathbf{w}(t+t) \approx \mathbf{w}(t)$ for $|t| \leq M$. Then, the behavior of $\mathbf{w}(t)$ can be approximated as

$$\Delta \mathbf{w}(t) = \mathbf{a} \mathbf{w}(t) \sum_{t=-K}^K \sum_{t'=0}^L \mathbf{H}(t') \mathbf{R}(t+t') \mathbf{G}(t), \quad (12)$$

where $\mathbf{R}(t) = E[\mathbf{s}(t) \mathbf{s}^T(t+t)]$. Matrix $\mathbf{R}(t)$ is the auto-correlation matrix of $\mathbf{s}(t)$, satisfying $\mathbf{R}^T(t) = \mathbf{R}(-t)$. Defining further

$$\bar{\mathbf{R}} = \sum_{t=-K}^K \sum_{t'=0}^L \mathbf{H}(t') \mathbf{R}(t+t') \mathbf{G}(t), \quad (13)$$

we have

$$\Delta \mathbf{w}(t) = \mathbf{a} \mathbf{w}(t) \bar{\mathbf{R}}. \quad (14)$$

In order to avoid the divergence of the weight $\mathbf{w}(t)$, we may introduce the same normalizing operation as eqn (5) again:

$$\mathbf{w}(t+1) = \frac{\mathbf{w}(t) + \Delta \mathbf{w}(t)}{\|\mathbf{w}(t) + \Delta \mathbf{w}(t)\|} = \frac{\mathbf{w}(t) + \mathbf{a} \mathbf{w}(t) \bar{\mathbf{R}}}{\|\mathbf{w}(t) + \mathbf{a} \mathbf{w}(t) \bar{\mathbf{R}}\|}. \quad (15)$$

So far, we have considered a discrete-time model. The continuous-time model corresponding to eqn (10) becomes

$$\begin{aligned}\frac{d\mathbf{w}(t)}{dt} &= \int_0^\infty y(t-t) \mathbf{s}^T(t) \mathbf{G}(t) dt \\ &\quad + \int_{-\infty}^0 y(t) \mathbf{s}^T(t+t) \mathbf{G}(t) dt.\end{aligned} \quad (16)$$

When trying to perform a mathematical analysis, the continuous model is more convenient. It can be shown that the continuous form of eqn (14) with eqn (15) becomes

$$\frac{d\mathbf{w}(t)}{dt} = \mathbf{w}(t) \bar{\mathbf{R}} - \frac{1}{2} \mathbf{w}(t) (\bar{\mathbf{R}} + \bar{\mathbf{R}}^T) \mathbf{w}^T(t) \mathbf{w}(t). \quad (17)$$

Note that in the continuous-time model, matrix $\bar{\mathbf{R}}$ has been defined as

$$\bar{\mathbf{R}} = \int_{-K}^K \int_0^L \mathbf{H}(t') \mathbf{R}(t+t') \mathbf{G}(t) dt' dt. \quad (18)$$

We find that if $\|\mathbf{w}(0)\|=1$, relation $\|\mathbf{w}(t)\|=1$ holds for every time t because

$$\frac{d\|\mathbf{w}(t)\|^2}{dt} = \mathbf{w}(t) (\bar{\mathbf{R}} + \bar{\mathbf{R}}^T) \mathbf{w}^T(t) (1 - \|\mathbf{w}(t)\|^2) = 0.$$

Namely, $\mathbf{w}(t)$ continues to stay on the surface of the unit sphere if the initial condition is given as $\|\mathbf{w}(0)\|=1$.

From now, we deal with only the case $\mathbf{G}(t) = \mathbf{I}$ ($t=0$) and $=\mathbf{O}$ ($t \neq 0$); in every example in the next section, diagonal matrix $\mathbf{G}(t)$ will be set so. Then, we have

$$\bar{\mathbf{R}} = \int_{-K}^K \mathbf{R}(t) \mathbf{G}(t) dt. \quad (19)$$

Here we consider two special cases.

When $g_i(t)$ are even functions, i.e.

$\mathbf{G}(-t) = \mathbf{G}(t)$, then $\bar{\mathbf{R}}$ becomes a symmetric matrix and $\mathbf{w}(t)$ converges to a stable equilibrium \mathbf{w} satisfying

$$\mathbf{w}\bar{\mathbf{R}} = \frac{1}{2}\mathbf{w}(\bar{\mathbf{R}} + \bar{\mathbf{R}}^T)\mathbf{w}^T \cdot \mathbf{w} \quad (20)$$

This implies that \mathbf{w} is an eigenvector of $\bar{\mathbf{R}}$ and its eigenvalue is $\frac{1}{2}\mathbf{w}(\bar{\mathbf{R}} + \bar{\mathbf{R}}^T)\mathbf{w}^T$. Since $\bar{\mathbf{R}}$ is a symmetric matrix with real entries, $\bar{\mathbf{R}}$ has real eigenvalues.

When $g_i(t)$ are odd functions, i.e.

$\mathbf{G}(-t) = -\mathbf{G}(t)$, then $\bar{\mathbf{R}}$ becomes a skew-symmetric matrix. In this case eqn (17) has no equilibrium on the surface of the unit sphere, implying that weight vector $\mathbf{w}(t)$ continues moving on the surface.

It should be noted that the formulation described above is just a particular representation of the generalized (asymmetric) Hebbian learning. A lot of modifications are conceivable in consideration of biological plausibility. Some of the followings may appear in the next section.

- (i) The output of the neuron is always positive; $y(t) \geq 0$;
- (ii) Weight $w_i(t)$ is always positive / negative, if it is excitatory / inhibitory;
- (iii) Normalization (15) of $\mathbf{w}(t)$ may not exist.

3. Some Examples

3.1 Slow oscillation

Fig. 2 shows a slow oscillation generated by a neuron that obeys an asymmetric learning. A (linear) neuron receives two Gaussian white signals, one of which is generated by shifting the other. Kernel $g_i(t)$ was given as in Fig.1,

Although the time constant of the neuron's activity is some milliseconds, the behavioral-level time constant in animals is much greater than that. If the synaptic plasticity is involved in the neural

dynamics, the network can produce a very slow neural activity.

3.2 Bimodal distribution of weights

Some researcher pointed out that the distribution of synaptic weights of a neuron is bimodal.[9] This property can be realized in our model. In this simulation the neuron receives twenty input signals (impulse trains with different time lags). Fig.3 shows the distribution.

3.3 Neural integrator

A neural integrator can be built based on the proposed model.[1] The function of maintaining neural activation, depending on the input, plays very important roles in various phases. Fig. 4 shows a simple realization of neural (leaky) integrator. The neuron receives three inputs: a constant input (constant firing rate), and an excitatory signal (impulse) and an inhibitory signal (impulse). The synaptic weight involved in the constant input is variable.

3.4 Habituation and dishabituation

As an interesting example, we can show a model that mimics the memory of temporal pattern observed in the crayfish.[8] Although the mechanism of the memory apparently seems to be sophisticated, it can be explained by a very simple learning neural net. It only consists of a simple memory that transforms a temporal pattern to a spatial pattern of the synaptic weights. The behavior of the model is quite similar to habituation and dishabituation observed in the visual system of the crayfish.

4. Conclusion

We have shown a model for a generalized Hebbian / anti-Hebbian learning. Although the is a very simple model, it can explain some salient features observed in real neurons of animals.

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References

[1] R. Kempter, W. Gerstner, and J. L. van Hemmen, Intrinsic stabilization of Output rates by spike-based Hebbian Learning, *Neural. Comput.*, 2001.

[2] K. Matsuoka: A general theory of a class of linear neural nets for principal minor component analysis. *Artificial Life and Robotics*, 3, 246-254, 1999.

[3] P. Munro: LTD facilitates learning in a noisy Environment, *Advances in Neural Information Processing Systems 12*, MIT Press, 1999.

[4] O. Paulsen and T. J. Sejnowski, Natural patterns of activity and long-term synaptic plasticity, *Current Opinion in Neurobiology* 2000, Vol. 10, pp.172-179, 2000/

[5] P. D. Roberts, Dynamics of temporal learning rules, *Physical Review E*, Vol. 62, No. 3, 2000.

[6] M. C. W. van Rossum, G. Q. Bi, and G. G. Turrigiano, Stable Hebbian learning from spike timing-dependent plasticity, *The Journal of Neuroscience*, 20 (23), pp. 8812-8821, 2000.

[7] J. Rubin, D. D. Lee, and H. Sompolinski, Equilibrium properties of temporally asymmetric Hebbian plasticity, *Physical Review Letters*, Vol. 86, No. 2, pp. 364-367, 2001.

[8] T. Shimozawa et al.: Response entrainment and memory of temporal pattern by movement fibers in the crayfish visual system. *Journal of Comparative Psychology. A.*, 114, 267-287, 1977.

[9] X. Xie and H. S. Seung, Spike-based learning and stabilization of persistent neural activity, *Advances in Neural Information Processing Systems 12*, pp. 199-205, 1999.

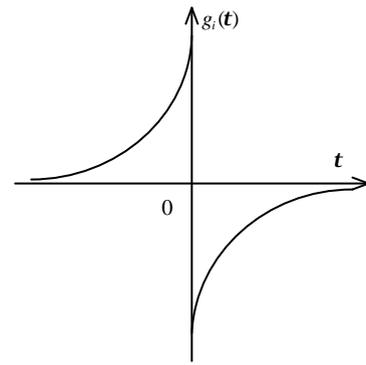


Fig. 1 A kernel function in the asymmetric Hebbian learning.

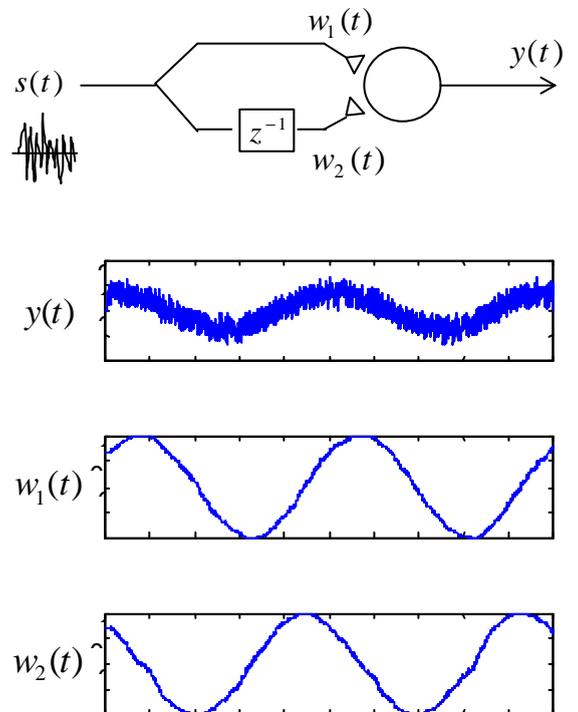


Fig.2 Slow oscillation generalized by an asymmetric Hebbian learning.

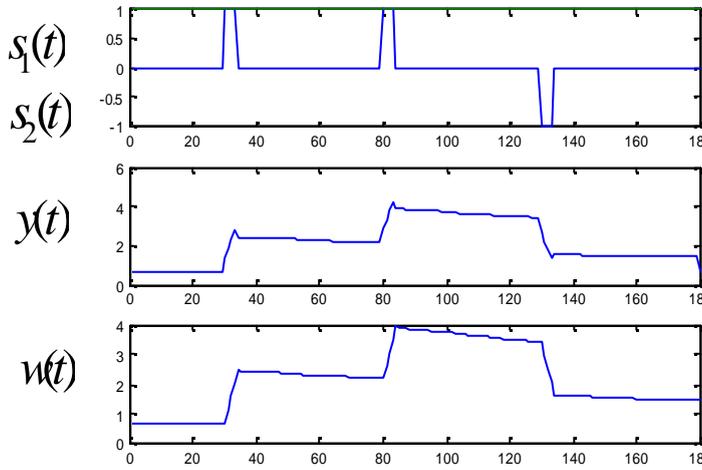
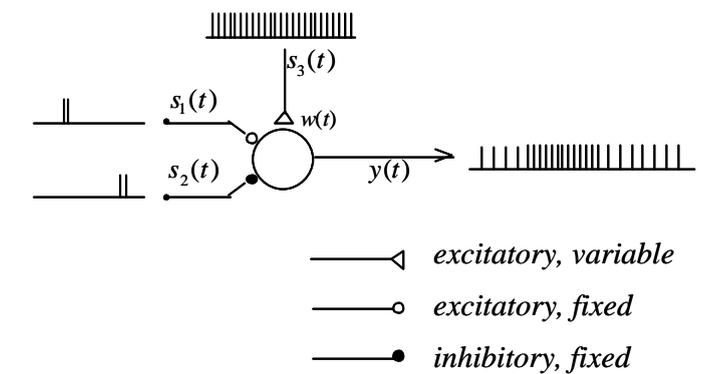
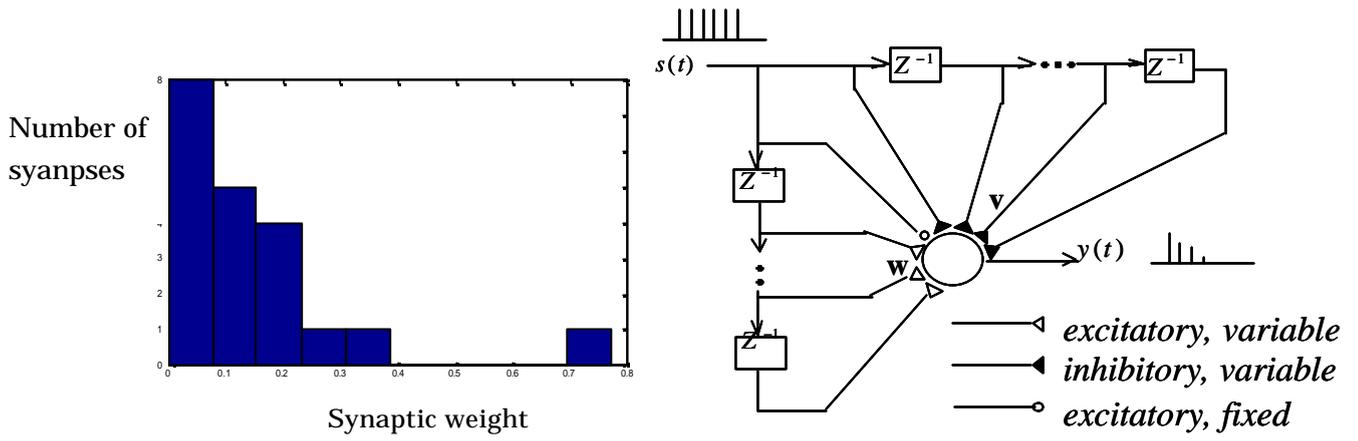


Fig.4 Neural integrator

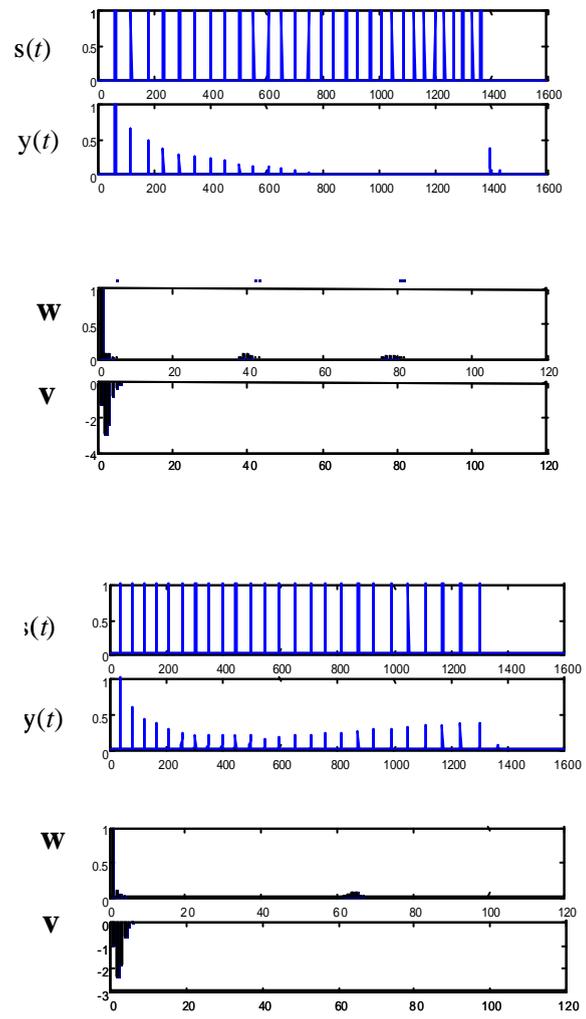


Fig. 5 Habituation and dishabituation