INTERVAL VALUED TYPE 2 FUZZY SETS, MULTI-VALUED MAPS, AND ROUGH SETS

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Abstract: - There is a unification of three formal approaches known as interval-valued Type 2 fuzzy sets, multi-valued maps and rough sets. In this unification, we investigate Türkşen's interpretation of Zadeh's CWW that generate interval-valued Type 2 fuzzy sets, Dempster's upper and lower set definitions based on multi-valued maps and Pawlak's upper and lower set approximations known as rough sets. First, it can be shown that there is a natural transformation between Dempster's and Pawlak's construction schemas. Secondly, Dempster's and Pawlak's upper set formulation schema is modified. Thirdly, multi-valued maps are restricted to a special well known subset of information granules. This modification and restriction forms the "Dempster-Pawlak-Zadeh" unification (Türkşen, 2001). With this unification, we generate FDCF and FCCF expressions for concept combinations in Zadeh's CWW with the application of linguistic operators such as "AND", "OR", etc. Finally, this opens the way, for new interpretation of fuzzy measures such as Belief and Plausibility, etc., over Interval-valued Type 2 fuzzy sets.

Key words: - Interval-valued Type 2 Fuzzy sets, multi-valued maps, rough sets.

1 Introduction

In most current investigations, fuzzy set representations and their logical combinations are based on Type 1 schema for both the knowledge representation and approximate reasoning. First Type 1 representation is a “reductionist” approach for it discards the spread of membership values by averaging or curve fitting techniques and hence, camouflages the “uncertainty” embedded in the spread of membership values. Secondly, Type 1 approximate reasoning relies just on the fuzzified version of the "shortest" forms of the classical Boolean Normal Forms, with the "assumption" that the linguistic “AND” corresponds to a t-norm and linguistic “OR” corresponds to a t-conorm in a one to one mapping. Thus Type 1 representation and reasoning is "reductions" and "myopic" and does not allow CWW a rich platform.

A more natural representation and reasoning are provided by interval-valued Type 2 fuzzy theory. These issues were discussed in our previous papers. In particular, it was shown that interval- valued Type 2 representation and reasoning is generated in the combination of Type 1 fuzzy sets with linguistic "AND", "OR", etc., operators. As well, it was shown that combination of linguistic variables when combined with linguistic connectives generate FDCF and FCCF expression, i.e., Fuzzy Disjunctive Canonical Forms and Fuzzy Conjunctive Canonical Forms, respectively. In this paper, we show that infact these FDCF and FCCF expressions can be directly obtained from a new constructive schema, which we call, "T-formalism", for short, based on "Dempster-Pawlak-Zadeh" Unification.

2 Upper and Lower Set Formulas

In a recent article [Türkşen, 2001], it is shown that there is a natural transformation between Dempster's multi-valued maps [Dempster, 1997] and Pawlak's rough sets [Pawlak, 1982, 1991]. In this article, it is shown that a modified and restricted Dempster-Pawlak constructs, named "T-formalism", generate upper fuzzy sets, and lower fuzzy sets which correspond to FCCF's and FDCF's obtained from fuzzy truth tables [Türkşen, 2001]. Here, we summarize very briefly this recent development which sheds further light on canonical forms and multi-valued maps.

2.1 Information Granules in Sets

Our aim is to demonstrate how we could generate FDCF and FCCF expression directly with the use of information granules and with the proposed approach without resorting to the Truth Table derivation that was discussed in our previous papers. (Türkşen, 1999, 2001)

Let $\mathbf{G_2}$ be the family of information granules that are the 8 possible combinations of any two predicates A and B under conjunction, disjunction and complementation operations:

$\mathbf{G_2}$ = {G_1, G_2,..., G_8},

where G_1=A B,..., G_8=c(A) c(B), such that

$\mathbf{G_2}$ = {A B, c(A) B, A c(B), c(A) c(B),
A B, c(A) B, A c(B), c(A) c(B)}.

It should be noted that the first four of these information granules, G_1,...G_4, i.e., A B, c(A) B, A c(B), and c(A) c(B), form a disjoint partition of the universe, whereas the last four of these information granules, (G_5,...,G_8), i.e., A B, c(A) B, A c(B), and c(A) c(B) have overlaps. In fact, we observe that (G_5,...,G_8) are the complements of (G_4,...,G_1) in that order.

Let us identify the target sets, T's, to be usual definition of "AND", "OR", etc., in two-valued set and logic expressions. For each target set T, we identify the upper and lower subsets to be information granules that have the following properties in general:

l (T) = {G G **G**, G T}, and
u(T) = {G G **G**, T G}. (1)

These initial, general, expressions of identification have an ambiguity because " " in both directions includes "the equality", =. This ambiguity is resolved by additional restrictions depending on whether we are forming linguistic "AND", "OR" compositions or other combinations. This will be discussed later in the sequal.

It is to be observed that each element of l (T) comes from the subsets of the conjunctive information granules, i.e., A B, c(A) B, A c(B), and c(A) c(B) whereas each element of u(T) comes from the disjunctive information granules, i.e., A B, c(A) B, A c(B), and c(A) c(B).

With these lower and upper information granules, we determine the lower and upper set formulas of the target set T in the proposed T-formalism as:

L(T) = l (T) and **U**(T) = u(T) (2)

It is to be noted that the disjunction of l (T)'s are taken to form the lower set **L**(T) since they are all contained in the target set and they are disjoint among themselves. Thus **L**(T) forms "the greatest lower bound". But the conjunction of u(T)'s are taken to form the upper set, **U**(T), since they all contain the target set and they are not disjoint. Thus **U**(T) forms "the least upper bound".

It is clear that **L**(T) T **U**(T) by the construction schema. Furthermore these definitions and relations apply whether the concepts are crisp or fuzzy. However, in the crisp case, we get **L**(T) T **U**(T)

However, there is the ambiguity to be resolved as indicated above. In equation (1), we observe that G T for l (T) and T G for u(T) for G **G**. Thus there are possible G's that may belong to l (T) and u(T) both. That is, we need to identify which G's are taken for the equality and which are taken for containment with respect to linguistic operators, "AND", "OR" and other special cases. This will be sorted out and clarified in Sections 2.2, 2.3 and 2.4 below because they depend whether a meta-linguistic concept is formed by "AND" or "OR" or other linguistic connectives.

2.2 Five Meta-Linguistic Expression that have "AND" Composition

The usual, commonly used, target set T is "A B" for "A AND B". With T-formalism, we identify the set of information granule that is contained in T to be equal to T itself as follows:

l (T) = {A B} and thus the lower set formula is:

L(T) = {A B}=l (T).

The set of information granules that strictly contain T are

u(T) = {A B, c(A) B, A c(B)}

and thus the upper set formula is:

U(T)= *u*(T)=(A B)   (c(A) B)   (A c(B)).

It is to be observed that l (T) = {A B} with the choice coming from the usual classical expression. It so happens that it is one of the information granules itself. Thus **L**(T) =l (T), i.e., **L**(T)={G   G **G**, G=T}. We generalize this for all the five out of sixteen combination of concepts, i.e., the particular meta-linguistic expressions, known as; "empty set", "not A and not B", "A and B", "A and not B", and "not A and B", that admit the linguistic "AND" connective in their meta-linguistic composition, respectively. Therefore, the rule is

l (T) = {G   G **G**, G = T} for the targets sets , c(A) c(B), A B, A c(B), c(A) B,

where the usual target set T itself forms the greatest lower bound.

This in turn entailes, the rule for **U**(T) to be **U**(T) =   {G   G **G**, T G} for these five out of sixteen combination of concepts that admit the linguistic "AND" connective in their meta-linguistic concept combination. This clarification resolves the anomaly generated by equations (1) for the cases of the linguistic "AND" connective.

It is to be noted that

L(T) = FDCF (A AND B), and

U(T) = FCCF (A AND B)

and therefore,

we have FDCF (A AND B)   FCCF (A AND B) by the construction.

This fact holds to be true for both the crisp and fuzzy sets and for all the t-norms and t-conorms due to construction of **L**(T)=T **U**(T) for the "AND" composition in the combination of linguistic concepts constructed with linguistic "AND" operators. It is to be noted that if we apply Law of Contradiction, LC, after the axioms of commutativity and distributivity, we get FDCF (A AND B) = FCCF (A AND B) in two-valued set and logic theory, which is known as DNF(.)=CNF(.).

Whereas in fuzzy set and logic theory FDCF and FCCF provide lower and upper set formulas, respectively, for the "AND" combination of any two fuzzy concepts, A and B, i.e., fuzzy predicates. This containment is true, since the containment of the lower set in the upper set holds due to the construction. Furthermore, it holds for all t-norms and conorms because of the monotonicity axioms of the norms.

2.3 Five Meta-Linguistic Expression that have "OR" Composition

The usual, commonly used, target set T is "A B" for "A OR B". Again with T-formalism, we identify the set of information granules that are contained in T as:

l (T) = {A B, c(A) B, A c(B)},

and thus the lower set formula is:

L(T) =   l (T) = (A B) (c(A) B) (A c(B)).

The set of information granule that contain T is:

u(T) = {A B},

and thus the upper set formula is:

U(T) = {A B}=u(T).

In an analogous manner, we observe that u(T) = {A B} which is the target set itself and thus **U**(T) = {G G **G**, T=G}. Again we generalize this for all the five cases out of sixteen combination of concepts that admit the linguistic "OR" connective in their combination. Therefore the rule is u(T) = {G G **G**, T=G} for the particular target sets I, A B, c(A) c(B), c(A) B, A c(B), i.e., the meta-linguistic expressions known as: "universe", "A or B", "not A or not B", "A implies B", and "A or not B", where the usual target set itself forms "the least upper bound".

This in turn entailes, the rule for **L**(T) to be **L**(T) = {G G **G**, G T} for these five out of sixteen cases that admit "OR" connective in their meta-linguistic expressions. Again, this clarification resolves the anomaly introduced by equation (1) for the cases of "OR" connective.

Again it is to be noted that

L(T) = FDDCF(A OR B), and

U(T) = FCCF(A OR B)

and therefore, we have FDCF(A OR B) FCCF(A OR B).

This fact again holds to be true for both the crisp and fuzzy sets and for all the t-norms and t-conorms due to the construction of **L**(T) T = **U**(T).

Again it is to be noted that if we apply the commutativity and distributivity, first and then this time the Law of Excluded Middle, LEM, we get FDCF(A OR B) = FCCF(A OR B) in two-valued set and logic theory, which is known as DNF(.)=CNF(.).

Whereas in fuzzy set and logic theory FDCF and FCCF provide lower and upper set formulas, respectively, for the "OR" combination of two fuzzy concepts, A and B, i.e., fuzzy predicates for all the t-norms and conorms due to construction scheme and the monotonicity axiom.

2.4 Other Six Meta-Linguistic Expressions

The remaining six meta-linguistic expressions, i.e., "A iff B", "A xor B", "A", "not A", "B", and "not B" are treated in a slightly different manner. Let us investigate, for example, the meta-linguistic expression known as the bi-conditional, "A IF AND ONLY IF B". Its usual target set is symbolically "A B". It is clear that

l (T) = {A B, c(A) c(B)} with the property that

l (T) = {G G **G**, G T}.

Thus we have

L(T) = l (T) = (A B) (c(A) c(B)).

As well, it is clear that u(T) = {c(A) B, A c(B)}, with the property that u(T) = {G G **G**, T G}.

Thus, we have

U(T) = u(T) = (c(A) B) (A c(B)).

This inturn entails the rule for the other five meta-linguistic expressions, i.e., "A xor B", "A", "not A", "B", and "not B" to be

l (T) = {G G **G**, G T} and

u(T) = {G G **G**, T G}.

2.5 Generalization

The schema developed for the determination of upper and lower set identification for any two sets A and B, crisp or fuzzy, can be generalized to n sets. Suppose there are n concepts that are represented by n predicates $A_1,...,A_n$, crisp or fuzzy. We can write $\mathbf{G}_n$ with the formation of 2^{n+1} primitives, i.e., information granules, derived from conjunction, disjunction and complementation of these n concepts as:

G_n = {A_1 A_2 ... A_n,...,c(A_1) c(A_2) ... c(A_n)}

Then we can apply the proposed formalism as shown in its application to two sets A and B, to determine the upper and lower set formulas of any meta-linguistic expression, say Y, made up of these n concepts. Thus the lower set formula will be FDCF(Y) and its upper set formula will be FCCF(Y).

3 Extension

These developments in turn open the way to a re-assessment of fuzzy measures. In particular, Belief and Plausibility and Probability measures over Interval-valued Type 2 fuzzy sets can be determined by way of extending previous results with α-cuts rule of combination over FDCF and FCCF.

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