The Constraint Optimisation in Subband Image Coding using Fuzzy Iterative Algorithm

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Abstract: - In the paper the constraint optimisation of approximately convex function, which appears in optimal bit-quantization allocation in subband image coding is presented. A novel modelling for graphical illustration of optimisation is introduced. A simple single-variable fuzzy iterative search algorithm is used.

Key-Words: - constraint optimisation, convex functions, fuzzy iterative algorithm, subband image coding

1 Introduction

A frequent problem in scientific and engineering computation is to determine the optimum (the maximum or minimum) and the corresponding arguments of a real-valued function $f(x_1, x_2, ..., x_n)$ of n real variables over a set S in an n-dimensional space:

$$\max_{\mathbf{x}\in S}(f(\mathbf{x})); \quad \mathbf{x} = (x_{1,} x_2, ..., x_n) \qquad (1)$$

If the set S is the entire n-dimensional space, the optimisation problem is said to be unconstrained. Otherwise, the problem is constrained by whatever conditions define the set. Usually, the set is defined by a set of non-linear functions satisfying equality or inequality conditions (equality or inequality constrained optimization):

$$\mathbf{g}(\mathbf{x}) \geq \mathbf{b} \text{ or } g_i(\mathbf{x}) \geq b_i, \quad i = 1, ..., m; \quad m < n \quad . \qquad (2)$$

It is easier to find a local maximum of function than it is to find the global maximum over the entire domain. In practice, for $n > 2$, the only way to find a global maximum is to have information from the problem itself about the location of such a maximum and then search for the local maximum. Follows a short overview over unconstrained and constrained multi-variable optimisation methods in order, to better understand the proposed solution in presented image coding application [1],[2].

1.1 Unconstraint optimization

1.1.1 Simple multivariable searches

Some multivariable search methods use a sequence of single-variable searches to achieve an optimum. The problem is how to choose a sensible direction in which to search for the optimum of function.

An old method is *method of steepest descent*, proposed by A. Chauchy in 1845. Assume, that function $f(\mathbf{x})$ has continuous partial derivatives of several orders. The gradient $\mathbf{gr}(\mathbf{x})$ of f at $\mathbf{x}$ is a vector, whose components are $\partial f / \partial x_i$. Then $\mathbf{gr}(\mathbf{x})$ defines the direction of steepest descent of f at $\mathbf{x}$. Under weak hypotheses, this method will converge to the local minimum of f in whose basin the first $\mathbf{x}$ is located. The analysis show that this method converges very slow in some cases, even for small value of n (for example, if the function is constant on ellipses or large eccentricity).

For a general strictly quadratic function f on n variables we can compute the minimum $\mathbf{x}_s$ at an arbitrary point $\mathbf{x}$ by:

$$\mathbf{x}_s = \mathbf{x} - \mathbf{H} \cdot \mathbf{gr}(\mathbf{x}); \qquad \mathbf{H} = \mathbf{B}^{-1} \qquad (3)$$

where the *Hessian* matrix $\mathbf{B}$ consists of second partial derivatives of f at $\mathbf{x}_s$: $\partial^2 f / (\partial x_i \cdot \partial x_j)$.

1.1.2 Advanced multivariable searches

If a smooth function is not quadratic, we can use an modified iteration of the form:

$$\mathbf{x}_{k+1} = \mathbf{x} - \alpha_k \cdot \mathbf{H}_k \cdot \mathbf{gr}(\mathbf{x}_k), \qquad (4)$$

where $\mathbf{H}_k$ is the k-th approximation of the $\mathbf{H}$. Methods of this type are called *variable metric* methods or *quazi-Newton* methods.

1.2 Constraint optimization

The **equality constraints** problem ($\mathbf{g}(\mathbf{x}) = \mathbf{b}$) might be solved by suitable insertion of constraint equation in the function f to get unconstraint problem. For the general equality constraint problem one can form the Lagrange function

$$L(\mathbf{x}, \lambda_1, ..., \lambda_m) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \left[b_i - g_i(\mathbf{x}) \right] \quad (5)$$

to obtain unconstraint problem. By setting partial derivatives of L to zero, we obtain the necessary conditions for the extremum of $f(\mathbf{x})$, subject to $\mathbf{g}(\mathbf{x}) = \mathbf{b}$:

$$\frac{\partial f}{\partial x_k} + \sum_{i=1}^{m} \lambda_i \cdot \frac{\partial g_i}{\partial x_k} = 0; \quad k = 1, 2, ..., n, \quad (6)$$

where $\lambda_1, \lambda_2, ..., \lambda_m$ are the *Lagrange multipliers.* These n equations must be solved together with the m constraints $g_i = b_i$. There are n+m equations in the n+m unknowns $x_1, x_2, ..., x_n$ and $\lambda_1, \lambda_2, ..., \lambda_m$. The method is known as Lagrange-multiplier method. For the two-variable problem $f(x_1, x_2)$ subject to single constraint $g(x_1, x_2) = b$, the mathematical conditions became much simpler:

$$\frac{\partial f}{\partial x_k} + \lambda \cdot \frac{\partial g}{\partial x_k} = 0; \;\; b - g(x_1, x_2) = 0; \; k = 1, 2. \quad (7)$$

Illustrated geometrically, the solution is the crossing point between curves $g(x_1, x_2) = b$ and the $f(x_1, x_2) = \text{constant}$ in the plain x_1, x_2, at which the curves have the common tangent with slope λ.

For the case of **inequality constraints** ($\mathbf{g}(\mathbf{x}) \geq \mathbf{b}$) Kuhn and Tucker proved the following theorem. To maximize the function $f(\mathbf{x})$ subject to $g(\mathbf{x}) \geq \mathbf{b}$, the equivalent conditions to (6) are:

$$\frac{\partial f}{\partial x_k} + \sum_{i=0}^{m} \lambda_i \cdot \frac{\partial g_i}{\partial x_k} = 0; \;\; k = 1, ..., n \quad (8)$$

$$\lambda_i \cdot (b_i - g_i) = 0, \; \lambda_i \leq 0, \; b_i - g_i \geq 0 \}; \; i = 1, ..., m \; (9)$$

According to $\lambda_i \cdot (b_i - g_i) = 0$ there are two alternatives: a) $g_i = 0$, in which case the constraint is “active” and the corresponding $\lambda_i < 0$, or b) $\lambda_i = 0$ and $g_i > 0$, so that the optimum is away from this constraint and the Lagrange multiplier is not necessary.

2 Optimal Bit (Quantization) Allocation in Subband Image Coding

The standard subband transform image coding scheme is implemented [3], containing separable discrete wavelet transform as subband transform, uniform scalar quantizer of subbands and entropy coder. The Johnston's 16b filters with near perfect reconstruction property are used.

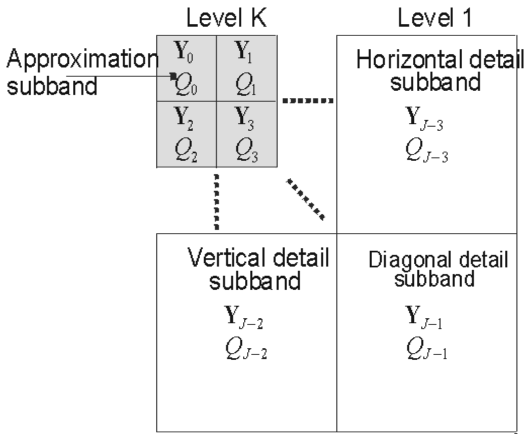


Fig. 1: The disposition and quantizing of the transformed subbands

In the quantization process to the each subband $\boldsymbol{Y}_i$, $i = 1, \cdots, J-1$, the quantization step Q_i is allocated, see Fig. 1. The quantization steps constitutes the allocation vector $\mathbf{Q}$:

$$\mathbf{Q} = \left[Q_0, Q_1, ..., Q_{J-1} \right]. \quad (10)$$

For entropy encoding adaptive recency rank coder is employed [6]. The rate-distortion relation of the entropy coded uniform scalar quantizer at high bit rates (low distortion) can be modelled as [4]:

$$D_i(R_i) = h_i \cdot \sigma_i^2 \cdot 2^{-2 \cdot R_i}, \quad (11)$$

where D is the distortion, R bit-rate, σ^2 is the variance and i is the subband index. The constant h_i depends on statistics of subband $\boldsymbol{Y}_i$. (1.2 for Laplacian distribution). The distortion is related to the quantization step as [4]:

$$D_i = \frac{Q_i^2}{\beta_i (Q_i / \sigma_i)}, \quad (12)$$

where relationship $\beta_i (Q_i / \sigma_i)$ depends also on the statistic of the subbands $\boldsymbol{Y}_i$. At low ratio Q_i / σ_i for β_i practically constant value 12 can be assumed:

$$D_i = \frac{Q_i^2}{12} \quad (13)$$

For higher ratio of Q_i / σ_i this assumption is not valid, especially in higher frequency subbands, because most natural images have energy concentrated at low frequencies. When the quantization steps approaches the signal variance, the distortion saturates and is approximately equal signal variance. The $\beta_i \approx Q_i / \sigma_i^2$. In the modelling it is appropriate to consider also the well known relations for orthogonal subband coding:

$$R = \sum_{i=0}^{J-1} \alpha_i \cdot R_i; \; D = \sum_{i=0}^{J-1} \alpha_i \cdot D_i; \; \sum_{i=0}^{J-1} \alpha_i = 1 \quad (14)$$

where are R the total bit rate, R_i subband bit rates, D total distortion, D_i distortions of subbands, and a_i normalized areas of subbands $\boldsymbol{Y}_i$.

The problem of optimal bit allocation is to minimize the total distortion and not to exceed the constraint (available) total bit-rate R_{con}:

$$\begin{aligned} &\min_{\mathbf{R}} D(\mathbf{R}) = \min_{\mathbf{R}} \sum_{i=0}^{J-1} D_i(R_i), \\ &R(\mathbf{R}) = \sum_{i=0}^{J-1} \alpha_i \cdot R_i \le R_{con}; \; \mathbf{R} = \left[R_1, R_2, \ldots R_{J-1}\right] \end{aligned} \quad (15)$$

Weighted total distortion can also be minimized:

$$D_w(\mathbf{R}) = \sum_{i=0}^{J-1} w_i \cdot \alpha_i \cdot D_i(R_i) \quad (16)$$

where w_i are perceptual weights. Optimal bit allocation is in general a complex non-linear problem, which must be solved numerically. Assuming that multivariable function $D(\mathbf{R})$ is strictly convex and anywhere differentiable, the condition for the optimum can be expressed as:

$$\frac{\partial D}{\partial R_i} = \frac{\partial D}{\partial R_j}; \; \forall i, j. \quad (17)$$

Furthermore, if orthogonal subband transform is used, i.e. the subband distortion D and bit rate R are expresses by (15), the condition for optimum became more simple:

$$\frac{\partial D_i(R_i)}{\partial R_i} = \frac{\partial D_i(R_j)}{\partial R_j} = -\lambda; \quad \forall i, j. \quad (18)$$

The method is known as constant slope method. It is equivalent to the following Lagrange multiplier method. Assuming low distortion model (11) and equations (14), then the problem (16) can be solved analytically introducing Lagrange function:

$$L = D(\mathbf{R}) + \lambda \cdot R(\mathbf{R}); \; R(\mathbf{R}) \le R_{con} \quad (19)$$

In practice, the exact relations $D_i(R_i)$ are not known and the condition of convexity is not fulfilled. One can still use similar ideas on operative rate distortion functions. The classical are methods, introduced by Shoham (1998) and Westernik (1988). They not assume the knowledge of the statistics of subbands, but are computation demanding and not appropriate for fast real time applications.

We propose a simple single-variable iterative searching algorithm, which is not optimal in general, especially at higher distortions, but is computational efficient and has small number of iterations. However, to obtain the analytically calculated optimal bit rates R_i, the quantiaztion steps for each subband must be adjusted. In fact, the distortion and bit rate are controlled via quantization:

$$D_i = D_i(Q_i); \quad R_i = R_i(Q_i) \quad (20)$$

Therefore, the problem of bit allocation is then reformulated as problem of quantization allocation:

$$\begin{aligned} &\min_{\mathbf{Q}} D(\mathbf{Q}) = \sum_{i=0}^{J-1} \alpha_i \cdot D_i(Q_i) \\ &R(\mathbf{Q}) = \sum_{i=0}^{J-1} \alpha_i \cdot R_i(Q_i) \le R_{con} \end{aligned} \quad (21)$$

The distortion can be expressed as:

$$D(\mathbf{Q}) = \sum_{i=0}^{J-1} \alpha_i \cdot \frac{Q_i^2}{\beta_i(Q_i / \sigma_i)}. \quad (22)$$

At small distortion (23) become:

$$D(\mathbf{Q}) = \sum_{i=0}^{J-1} \frac{\alpha_i \cdot Q_i^2}{12} \quad (23)$$

Using efficient entropy encoding, the constraint can be rewritten in the form:

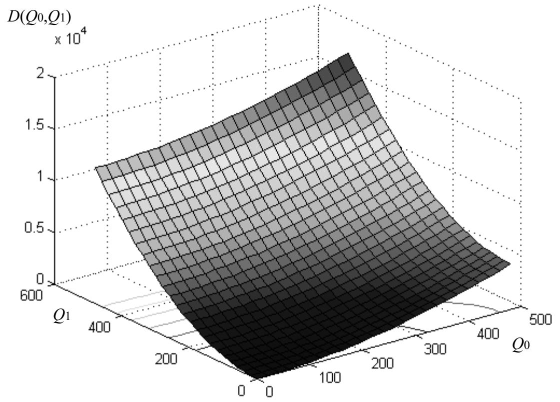
$$g(\mathbf{Q}) = \sqrt{\prod_{i=0}^{J-1} Q_i^{\alpha_i}} \ge Q_{con}, \quad (24)$$

where Q_{con} is the constraint value. The optimisation process for simple two variable case, obtained using only one resolution level of wavelet decomposition ($J = 4$) and equal quantization steps in detail subbands $\mathbf{Q} = \left[Q_0, \, Q_1 = Q_2 = Q_3\right]$ is graphically explained in Fig. 2. The relative areas are α_i=*1/4* for all i=1,…,3. Assuming small ratios Q_i / σ_i and orthogonal transform, the distortion is:

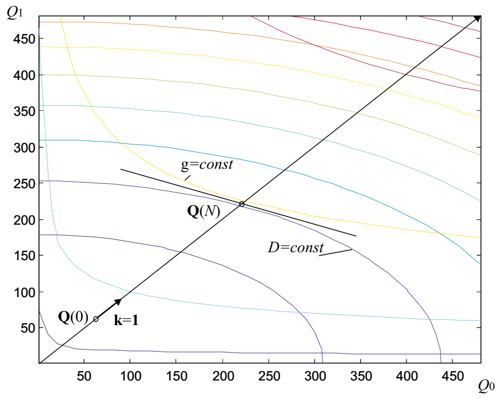
$$D(Q_1,Q_2)=\sum_{i=0}^{3}\alpha_i\cdot\frac{Q^2{}_i}{12}=\left(\frac{1}{4}\cdot\frac{Q_0{}^2}{12}+\frac{3}{4}\cdot\frac{Q_1{}^2}{12}\right) \quad (25)$$

Employing efficient entropy encoding the constraint is:

$$g(\mathbf{Q})=g(Q_1,Q_2)=\sqrt{Q_0{}^{1/4}\cdot Q_1{}^{3/4}}\geq Q_{con}\,. \quad (26)$$



a)



b)

Fig. 2: Ideal coding model using $\beta=12$.

a) Distortion as function of quantization steps

b) Graphical illustration of the optimization method

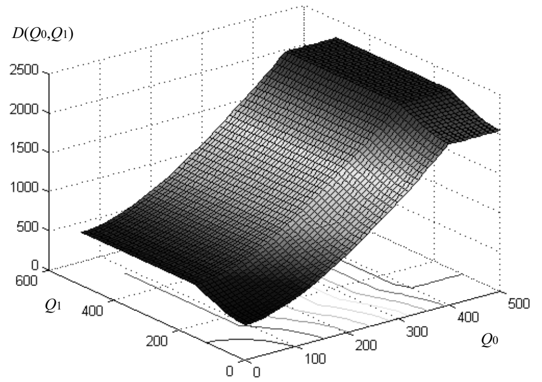
Figure 3 shows characteristics obtained by using more realistic coding model with variable $\beta_i(Q_i/\sigma_i)$, assuming roughly Laplacian distribution in all subbands. We used the following simplified mathematical model:

$$\beta_i(Q_i/\sigma_i)=\begin{cases}2.6\cdot(Q_i/\sigma_i)+12; & 0<Q_i/\sigma_i<5;\\ (Q_i/\sigma_i)^2; & 5<Q_i/\sigma_i\end{cases}.(27)$$

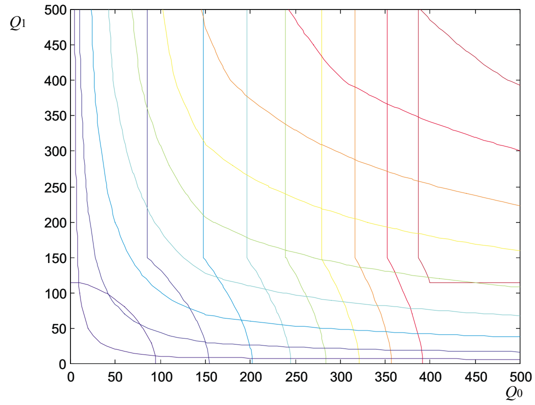
The curve $\beta_i(Q_i/\sigma_i)$ for subband 0 with σ_0=80 is shown in Fig. 3c. At small distortions (small Q_i/σ_i), the model behaves as ideal model in Figure

2. The total distortion is expressed by equation (22). The following constraint equation is obtained:

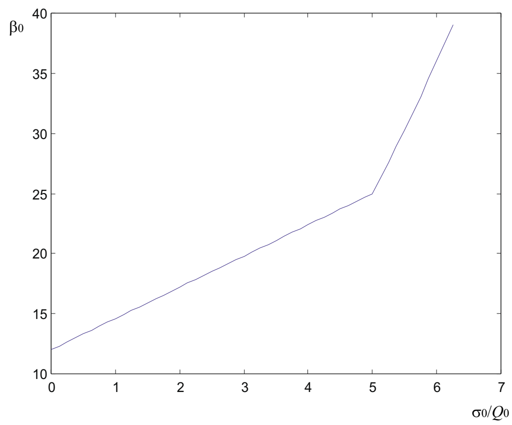
$$\prod_{i=0}^{J-1}\left(\frac{Q_i}{\beta_i(Q_i/\sigma_i)}\right)^{\alpha_i}\geq Q_{con}\,. \quad (28)$$



a)



b)



c)

Fig. 3: More realistic coding model with variable $\beta_i(Q_i/\sigma_i)$. The subband variances are: $\sigma_0=80$, $\sigma_1=30$, $\sigma_2=30$, $\sigma_3=30$.

a) Distortion as function of quantization steps.

b) Graphical illustration of the optimization method

c) β as function of Q/σ for subband 0.

3 The fuzzy iterative solution

For solving the practical bit allocation optimization problem, where function is no more strictly convex (quadratic), we still follow the idea from ideal case, represented in Fig. 2. The optimisation problem can be solved using simple one-variable search, if the right direction is find. For this purpose, we introduced the constant direction vector **k:**

$$\mathbf{k} = \left[k_0, k_1, ...k_{J-1}\right]; \;\; k_i = \frac{Q_i}{Q_0}, \; i = 0,1,...,J-1 \quad (29)$$

The vector **Q** is then $\mathbf{Q} = k \cdot Q_0$. Choosing the vector **k**, the distortion depends simple only on single scalar variable Q_0:

$$D = D(\mathbf{Q}) = D(\mathbf{k} \cdot Q_0). \qquad (30)$$

From Fig. 2 we concluded, that the optimum point is reached using equal quantization steps for all subbands, i.e. **k=1**. This is well known result for small distortions, which can be obtained analytically using Lagrange-multiplier method. We choose this direction as near-optimal direction also for images, by which distortion is no more quadratic function at higher values of variables Q_i. Instead of searching minimal distortion, we simple prescribe the desired distortion. Using iterative algorithm the quantization step Q_0 (and via **k** indirectly **Q)** is adjusted until desired distortion is obtained with prescribed tolerance. The iterative algorithm pushes the point $\mathbf{Q}(n)$ along direction defined by **k** (in ideal case along diagonal), n is iterating index. Vector **k** therefore determines the direction of successive approaching to the end (probably optimal) point $\mathbf{Q}(N)$, N is the total number of iterations. In the case of minimizing weighted distortion, the optimal direction is shifted to the direction determined by $k_i = \sqrt{w_i} / \sqrt{w_j}$, where w_i are selected distortion (perceptual) weights. For coarser quantization, above determined directions are no longer optimal. Generally, the distortion or compression ratio should be considered. The searching for optimal point in the direction **k** can be viewed as iterative or control algorithm for achieving the desired (minimum) distortion. It can be considered also as the algorithm for finding the crossing point between function $D(Q_0)$ and line D_{des}. Different iterative algorithms can be used, usually bisection algorithm is employed. It is stable, but has small convergence rate. We introduce the fuzzy iterative algorithm. It can be by rules user friendly adapted to the gradients in searching direction for particular images. The instantaneous error is:

$$e(n) = D(n) - D_{des} \qquad (31)$$

The algorithm starts with small initial value $Q_0(0)$. In each iteration the new quantization value is obtained as an old value plus increment $dQ(n)$:

$$Q(n) = Q(n-1) + dQ(n). \qquad (32)$$

The increment $dQ(n)$ is generated by the fuzzy rule system (fuzzy controller). Fuzzy rules linguistic expressed the approaching of the iteration point to the desired value. In the simplest form [5], the fuzzy output depends on instantaneous error:

1. If (e is *NB*) then (dQ_0 is *PB*)
2. If (e is *NS*) then (dQ_0 is *PS*)
3. If (e is *ZE*) then (dQ_0 is *ZE*) (33)
4. If (e is *PS*) then (dQ_0 is *NS*)
5. If (e is PB) then (dQ_0 is *NB*),

where *NB* means negative big*,* *NS* negative small, *ZE* zero, *PS* positive small and *PB* positive big. For efficient implementation of the centre of gravity defuzzyfication method, the crisp values for output membership functions are used. Three triangular membership functions for input variable *e* are used. Operational $D(Q_0)$-curves differ for different images. To adapt the fuzzy system to the slope of particular curve and further reduce the number of iterations, the width of middle input membership function is adapted to the curve's slope. The slope can be used as additional input variable of the fuzzy logic system. The instantaneous slope dD / dQ_0 is estimated in each iteration using approximation $\Delta D(n) / \Delta Q_0(n)$ If the slope is smaller at same error, the change dQ_0 must be also smaller. This is incorporated in fuzzy rule-system.

4 Experimental results

Fig. 5 and 6 show results obtained with typical 8-bit grey-scale images "Random" (white noise) and "Lena", using the two dimensional grid $Q_0, Q_1 \in$ {1, 25, 50, 100, 250, 500}. In Fig. 5a and 5b the relationship $D(Q_0, Q_1)$ and contour plain for image "Random" are presented. The results are comparable with those, obtained by modelling in Fig. 2 and 3. The *R-D*-characteristics for images "Random" and "Lena" are shown in Figures 5c in 6. The smallest distortions (circles in diagrams) at given bit rates were obtained with equal quantization steps even at higher distortions and images by which the subband variances decrease by increasing the frequency. The fuzzy iterative algorithm decreases the average number of iterations for different images for about 30% in comparison with bisection algorithm.

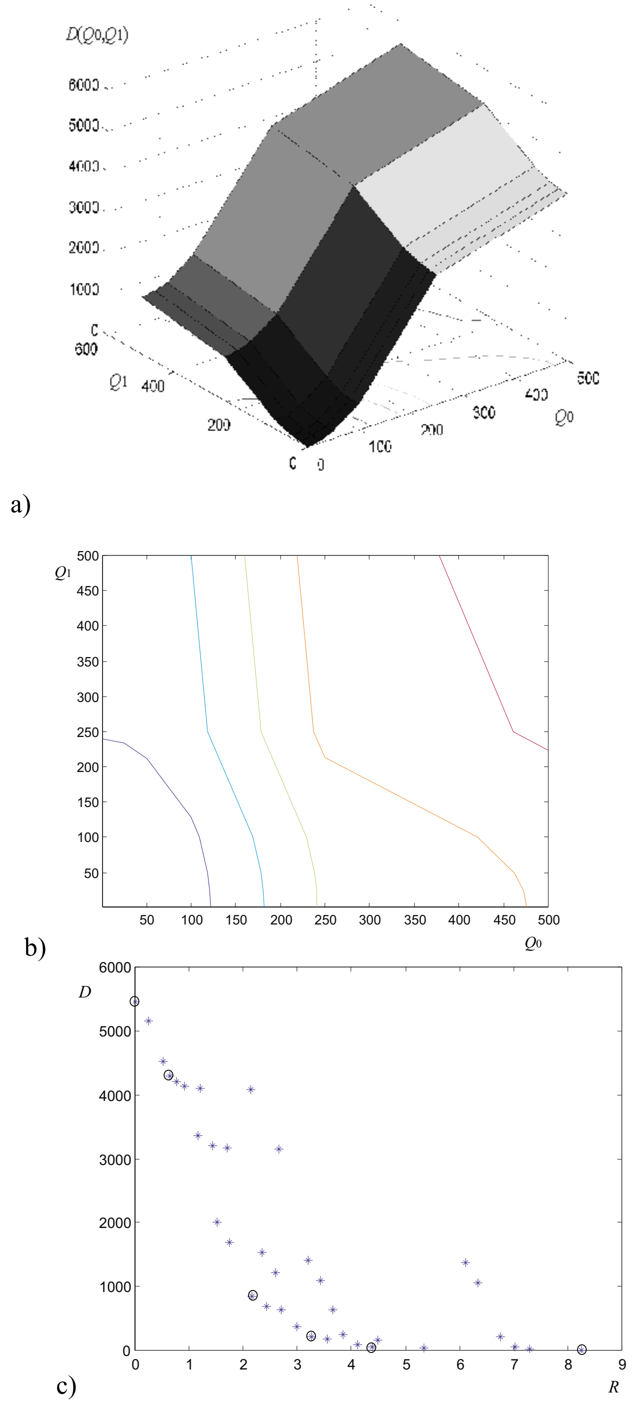


Fig.5: Experiments with image “Random” $\sigma^2 = 5461.9, \sigma_0 = 74.04, \sigma_1 = 73.81, \sigma_2 = 74.01$ $\sigma_3 = 73.78$.
a) Distortion as function of quantization steps
b) Contour plane c) Rate-distortion characteristic

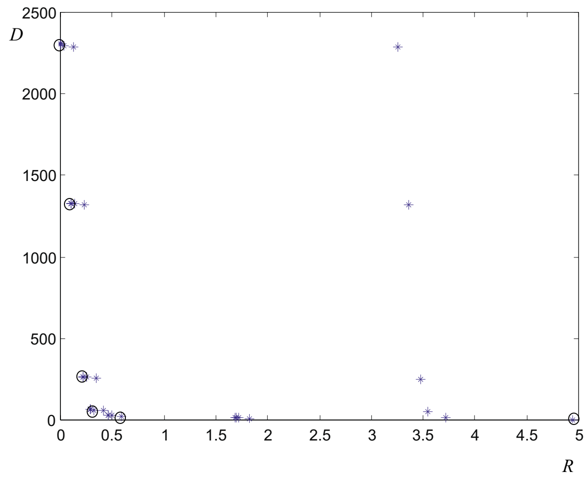


Fig. 6: Experiments with image “Lena”
$\sigma^2 = 2298.1, \sigma_0 = 95.43, \sigma_1 = 6.59, \sigma_2 = 4.08$ $\sigma_3 = 3.03$
Rate-distortion characteristic.

5 Conclusion

The results confirm the applicability of used subband coding modelling and our decision for searching the near-optimum in the directions with equal quantization steps. The incorporation of fuzzy iterative algorithm enable user friendly adaptation to the slopes in searching direction for particular image using fuzzy rules and so to speed up the convergence.

References:

[1] G. F. Forsythe, M. A. Malcolm, C. B. Moler, *Computer Methods for Mathematical Computations*, Prentice Hall, 1977.

[2] Glyn James, *Advanced Modern Engineering Mathematics*, Prentice Hall, 1999.

[3] P. Filip, “An Efficient Method for Image Compression in the Wavelet Transform Domain,” *Proc. of the 38nd SPIE’s Int. Symp. on Optics, Imaging and Instrumentation*, San Diego, LA, pp. 11-16, July 1993.

[4] H. M. Hang and J. J. Chen, “Source model for Transform Video Coder and Its Application - Part I: Fund. Theory”, *IEEE Trans. Circuits and Syst. Video Technol.*, Vol.7, No.2, pp. 87-298.

[5] Peter Planinšič, B. Gergič, B. Banjanin, D. Gleich, Ž. Čučej, “Efficient algorithms for control and optimisation of distortion or rate at wavelet-subband image coding”, *Proc. Picture coding symposium’99*, Portland, Oregon, April, 1999, pp. 407-411.