Complementary Extremum Principles for Isoperimetric Optimisation Problems

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Abstract: - An isoperimetric problem of the calculus of variations is reviewed. An integral functional, dependent on a function $u(x)$, its derivative $u'(x)$ and on the independent variable x, is minimised. The minimisation is subject to the isoperimetric constraint that the length of the integration interval remains constant. The well-known Euler equation of the fundamental problem of the calculus of variations is recovered with an additional relationship connecting the values of the function u and its derivative u' at the ends of the interval. A new complementary extremum principle is derived, that offers an algorithm for determining lower bounds on the minimum value of the original functional. A special treatment of the degenerate case where there is linear dependence on u is presented. Examples of each case are given and the scope for future work is discussed.

Key-Words: - Isoperimetric, Calculus of Variations, Complementary Principle, Variable End-points.

1 Introduction

Interest in isoperimetric optimisation problems dates back to Queen Dido of Carthage. Her enemies taunted her by offering her as much land as she could enclose with the hide of an ox. She confounded them by cutting the hide into strips, making a rope from them, and laying it in a large semi-circular with ends on the shore. The canny queen knew that the circle is the solution of the problem of maximising the area contained by a curve of given length. Legend has it that she was allowed to found Carthage in the large area enclosed.

Isoperimetric optimisation problems remain of interest to day as they arise in diverse applications. One may immediately cite the problem of minimising the drag on a body of given mass or volume as an example of importance. Indeed, this paper lays the groundwork for addressing one example of this problem – the required extension to treat higher dimension problems and general constraints will form the subject of a following paper.

We are concerned with the mathematics of isoperimetric problems where it is desired to optimise a functional of a field variable that satisfies a differential equation within a domain having a variable boundary. The aim is to perform the optimisation with respect to the variable boundary, or in other words, to choose the best domain to maximise or minimise the quantity of interest as determined by the functional. We shall focus on problems where the governing differential equation is itself the Euler equation of an extremum principle, this Euler equation being derived by the methods of the calculus of variations. Although this narrows the applicability of the analysis somewhat, there is a considerable body of problems in mathematical physics for which such extremum principles exist. Much work on solving the differential equations within a prescribed fixed domain has been done. However, there is far less literature on problems where there is an unknown boundary. The solution of such problems can be challenging even for the simplest governing equations. Usually there is a requirement to undertake numerical computation in order to predict the unknown boundary and calculate the desired optimal value of the functional. Under such circumstances it is very desirable to have the capability of obtaining reasonably accurate upper and lower bounds on the desired value by a simple analytical method. Such bounds can serve as a check on a detailed numerical solution scheme. Moreover, the bounds could assist in the provision of useful initial values where an iterative numerical scheme is required to obtain the exact solution.

Optimal boundary problems have been studied by a number of authors. Tuck [1], Watson [2], Pironneau [3], Bourot [4], Mironov [5] and more recently Richardson [6]_considered the problem of minimising the drag on an axisymmetric body of given volume moving in the direction of its axis in slow viscous flow. The analogous problem for flow at higher Reynolds numbers was considered first by Pironneau [7] and then by Glowinski and Pironneau [8]. Polya [9] and subsequently Banichuk [10] and Curtis and Walpole [11] considered the optimisation of elastic bars and shafts in torsion. Banichuk and Karihaloo [12] and later Parberry and Karihaloo [13] studied the similar problem of the minimum weight design of bars subject to constraints on their torsional and flexural rigidities.

Gurvitch [14] considered the problem of determining the domain that minimises the electrostatic capacity functional in the class of plane doubly connected domains having the same area and outer boundary. Payne [15] reviewed the literature on the problem of minimising the average virtual mass in potential flow theory. Curtis [16] investigated the problem of minimising the rate of heat loss from a body surrounded by a layer of homogeneous insulation of given volume. The minimisation was with respect to variation of the shape of the outer surface of the insulation layer.

Many of the above authors used the calculus of variations to obtain necessary conditions for optimality. The calculus of variations is applied in this paper too. The reader desiring more background on the calculus of variations is referred to Bolza [17] and Hildebrandt and Tromba [18]. We show how one may exploit the standard theory to obtain approximations to the optimal value of the functional by the development of complementary extremum principles.

The focus of the paper is the natural starting point of the fundamental one-dimensional problem of the calculus of variations. We suppose that the well-known Euler equation applies in a given domain and gives a minimum of the functional concerned for that domain. Then the end points are allowed to vary subject to the constraint that the length of the domain remain fixed. The minimum value of all the minima is sought by variation of the domain limits while satisfying this isoperimetric constraint.

We consider this one-dimensional problem with the aim of establishing the basic principles prior to considering more complex problems in several variables. In the present treatment we derive necessary conditions for the existence of the lower bounds by investigating first-order variations only. Consideration of sufficient conditions is deferred to later work. From an engineering viewpoint these are perhaps not of great importance, since an optimal solution is often demonstrably better than the alternatives. However, confirmation of local optimality is always desirable in seeking confidence that one has found a global optimum.

Watson (Ioc.cit.) was among the first to consider extremum principles with domain boundary variation. He encountered difficulties in the treatment of what he termed the degenerate case where the functional had linear dependence on the unknown solution of the boundary-value problem. These difficulties prevented the derivation of lower bounds on the minimum drag on a body in slow viscous flow.

This paper commences with a discussion of the well-known fundamental problem of the calculus of variations extended to consider variable end points satisfying the isoperimetric constraint. A complementary variational principle is derived in the general case. An example is discussed. Then the analysis is extended to consider the degenerate case where there is at most linear dependence on the unknown solution of the underlying boundary value problem. An example is presented.

We discuss the generalisation of the method to consider problems of higher dimensions, arbitrary rather than isoperimetric constraints, and the derivation of sufficient conditions for optimality.

2 Fundamental Isoperimetric Problem

Let us define a functional

$$I[u;a,b] = \int_a^b F(x,u,u')dx \ , \quad (1)$$

where $u = u(x)$ is a twice differentiable function defined in the interval $[a, b]$, the superscript prime denoting differentiation with respect to x. Suppose these endpoints are allowed to vary subject to the constraint that the length of the interval remains constant. Thus

$$b - a = k \ , \quad (2)$$

where k is constant.

Consider the following minimisation problem: minimise $I\ [u;\ a,\ b]$ subject to the constraint (2) and the boundary conditions

$$u = U(a) \ \text{at} \ x = a \ ,$$
$$u = U(b) \ \text{at} \ x = b \quad (3)$$

Application of standard techniques of the calculus of variations yields necessary conditions satisfied by a solution

$$u = u_o, \ a = a_o, \ b = b_o, \quad (4)$$

producing a minimum value d_o. The function u_o satisfies the well-known Euler equation

$$F_{u_o} - \frac{d}{dx}\left(F_{u'_o}\right) = O, \quad (5)$$

holding in $[a_o, b_o]$, and is subject to the boundary conditions

$$u_o(a_o) = U(a_o) \ , \quad (6)$$

and

$$u_o(b_o) = U(b_o) \ . \quad (7)$$

In equation (5) the subscripts u_o, u'_o denote partial derivatives of F with respect to u and u' respectively, evaluated at the solution $u = u_o$.

The necessary condition

$$[F_o + (U' - u'_o)\,F_{u'_o}]_{a_o}^{b_o} = O \ , \quad (8)$$

must also hold, where F_o denotes the function $F(x, u_o, u_o')$ and the square brackets denote the difference between their content evaluated at b_o and a_o. It must be noted that a solution (4) of equations and conditions (5) to (8) is not always available. For certain forms of the functions F and U constraints on allowed values of a or b or the form of the function u may be present and the minimum value of I may occur where one or more of these constraints apply.

We do not address this situation here, but are concerned with the case where the solution (4) can be found by solving conditions (5) to (8). This is not a severe limitation, because in many physical problems such extremum principles do exist.

Consider the following example. Let

$$F(x, u, u') = u'^2 + u^2 - 2u + x^2 + 1 \ ,$$

$$U(a) = U(b) = 1,$$

and

$$k = 1 \ .$$

The Euler equation (5) is

$$u_o'' = u_o - 1 \ ,$$

subject to the boundary conditions

$$u_o(a_o) = 1 \ , \ u_o(b_o) = 1 \ .$$

Equation (8) yields

$$\left[-u_o'^2 + u_o^2 - 2u_o + x^2 + 1\right]_{a_o}^{b_o} = O \quad .(9)$$

It is easy to show that the solution $u_o = 1$ satisfies the Euler equation and boundary conditions. The condition (9) becomes

$$b_o^2 - a_o^2 = O.$$

This may be solved with the isoperimetric constraint (2) to yield the solution

$$a_o = -\tfrac{1}{2}, \quad b_o = \tfrac{1}{2}.$$

The minimum value of I, d_o, is then readily evaluated as $d_o = \frac{1}{12}$.

It is apparent that the condition (8) supplies the second relationship between the interval limits if the solution of the Euler equation can be explicitly expressed in terms of them.

3 Complementary extremum principle

In seeking a complementary extremum principle we extend a method for the case of a fixed interval presented by Courant and Hilbert [19].

Let us define a new functional

$$I^*[u, p, \lambda \; ; a, b] = \int_a^b \begin{pmatrix} F(x,u,p) \\ + \lambda(u' - p) \end{pmatrix} dx , \quad (10)$$

with the function F being the same as in equation (1). It is clear that setting $p = u'$ in equation (10) recovers the original functional I. Suppose we do not apply this constraint and minimise I^* with respect to its arguments u, p, a, b subject to the isoperimetric constraint (2) and boundary conditions (3), obtaining the minimum value d_λ. It must follow that

$$d_\lambda \le d_o , \quad (11)$$

because the minimisation has been undertaken over the wider set of functions.

We seek the value of d_λ by application of the standard technique of the calculus of variations to obtain the solution

$$u = u_o^* , \; p = p_o^* , \; a = a_o^* , \; b = b_o^* . \quad (12)$$

The Euler equations are

$$F_{u_o *} - \frac{d\lambda}{dx} = O , \quad (13)$$

and

$$F_{p_o *} - \lambda = 0 , \quad (14)$$

to hold in $[a_o^* , b_o^*]$, subject to the boundary conditions

$$u_o^*(a_o^*) = U(a_o^*) , \quad 15)$$

and

$$u_o^*(b_o^*) = U(b_o^*) . \quad (16)$$

Here the subscripts denote partial derivatives of F with respect to u and p evaluated at the solution. The Euler equation and boundary conditions are to be solved with the necessary condition

$$\left[F_o^* + \lambda(U' - p_o^*)\right]_{a_o^*}^{b_o^*} = O , \quad (17)$$

where F_o^* denotes $F(x, u_o^*, p_o^*)$.

A proof that the minimum d_λ exists and is given by the solution (12) if the solution (4) exists is not presently available to us. No counter-examples have been found. It is possible that the imposition of suitable conditions on the forms of the function F and the boundary condition function U would allow an existence proof. It is our intention to address this aspect of the problem in subsequent work.

Let us return to the example considered in the preceding section. We have

$$F(x,u,p) = p^2 + u^2 - 2u + x^2 + 1 , \quad (18)$$

so that equation (13) becomes

$$2(u_o^* - 1) = \frac{d\lambda}{dx} , \quad (19)$$

while equation (14) reduces to

$$2p_o^* = \lambda . \quad (20)$$

Let λ be an unknown constant. Then equation (19) gives $u_o^* = 1$, which satisfies the boundary conditions (15) and (16). From equations (18) and (20)

$$F_o^* = \tfrac{1}{4}\lambda^2 + x^2 \quad , \qquad (21)$$

and substitution of this relation into the condition (17) gives

$$b_o^{*2} - a_o^{*2} = O \quad . \qquad (22)$$

Combination of this with the isoperimetric constraint (2) yields the solutions

$$a_o^* = -\tfrac{1}{2}, \quad b_o^* = \tfrac{1}{2} \quad . \qquad (23)$$

The minimum value of I* is readily calculated as

$$d_\lambda = \tfrac{1}{12} - \tfrac{1}{4}\lambda^2 \le d_o = \tfrac{1}{12} \quad , \qquad (24)$$

the exact solution being recovered when $\lambda = O$. Thus for every value of λ a lower bound on d_o is generated.

The simple nature of the above example has allowed us to derive a family of lower bounds, the maximum of which happens to coincide with the exact solution of the original problem. Under most circumstances it will not be possible to generate solutions satisfying the boundary conditions (15) and (16) so readily. The choice of suitable trial functions λ will be guided by the individual problem under consideration. Note that there is no requirement to solve differential equations in calculating the bound: only the need to solve the functional relationships (13) and (14) for u_o^* and p_o^*.

The above is an example of the general case where u_o^* and p_o^* can be evaluated from the pair of equations (13) and (14). In the next section we consider the degenerate case where this is not possible.

4 Degenerate case

In the degenerate case the function *F* takes the particular form

$$F(x,u,u') = G(x,u') + h(x)u \quad , \qquad (25)$$

where *G* and *h* are differentiable functions. The Euler equation (13) in this case reduces to the degenerate form

$$h(x) - \frac{d\lambda}{dx} = O \quad . \qquad (26)$$

Thus, if minimisation of *I** is to be accomplished, the functional form of λ is constrained to within a constant by equation (26). After integration,

$$\lambda = \int^{\lambda} h(x)du + C \quad , \qquad (27)$$

where *C* is a constant. Note the contrast with the non-degenerate case where any form of the function λ is acceptable provided it is compatible with the boundary conditions (15) and (16). The solution p_o^* is expressed in terms of λ and hence *C* by means of equation (14). Then condition (17) can be written in terms of *C*, a_o^*, and b_o^*. With the isoperimetric constraint, a_o^* and b_o^* may be expressed in terms of *C*. The minimum value d_λ is given by

$$d_\lambda = \int_{a_o^*}^{b_o^*} \begin{pmatrix} G(x, p_o^*) + h(x)u_o^* \\ + \lambda(u_o^{*\prime} - p_o^*) \end{pmatrix} dx \quad . \qquad .(28)$$

After an integration by parts, this may be reduced to

$$d_\lambda = \int_{a_o^*}^{b_o^*} \left(G(x, p_o^*) - \lambda p_o^* \right) dx + \lambda(b_o^*)U(b_o^*) - \lambda(a_o^*)U(a_o^*) \quad , \qquad (29)$$

following use of equation (26) and the boundary conditions (15) and (16). The right hand side of equation (29) is a function of the constant *C*; thus

$$d_\lambda = g(C) \quad . \qquad (30)$$

The maximum value of $g(C)$ with respect to C coincides with d_o. Thus, in the degenerate case the complementary extremum principle is replaced by the problem of maximising a function of a single variable. This offers a simpler approach to the evaluation of the solution of the original isoperimetric minimization problem. This is illustrated by the following example.

Let

$$G(x,u') = \tfrac{1}{2}u'^2 \; , \; h(x) = 1 \; , \; U(x) = x^2 .$$

Equation (26) becomes

$$\frac{d\lambda}{dx} = 1 \; ,$$

with solution given by equation (27)

$$\lambda = x + C \; . \qquad (31)$$

Equation (14) yields

$$p_o^* - \lambda = O \; . \qquad (32)$$

Condition (17) is

$$\left[\tfrac{1}{2} p_o^{*2} + h(x)\; U + \lambda\left(U' - p_o^*\right)\right]_{a_o^*}^{b_o^*} = O \; ,$$

which reduces to

$$5\left(b_o^* + a_o^*\right) + 2C = O \; .$$

Use of the isoperimetric constraint yields

$$a_o^* = -\tfrac{C}{5} - \tfrac{1}{2}, \; b_o^* = -\tfrac{C}{5} + \tfrac{1}{2} \; . \qquad (33)$$

Substitution of relations (31), (32) and (33) into equation (29) yields

$$d_\lambda = g(C) = \tfrac{5}{24} - \tfrac{3C^2}{5} \le d_o = \tfrac{5}{24} \; , \qquad (34)$$

the exact solution being recovered when $C = O$ in this case. It is apparent that whatever the value of C chosen, d_λ is a lower bound on d_o.

In this particular example it has been straightforward to express a_o^* and b_o^* in terms of C explicitly as a result of the simple form of the function $G(x, u')$. It is likely in the general case that numerical solution of the equations would be required. However, note also that equation (14) is not a differential equation in p_o^* : it expresses p_o^* as a function of λ, whether implicitly or explicitly. The function λ selected must satisfy the differential equation (26) and numerical integration may be necessary to evaluate it.

5 Discussion

We have demonstrated how lower bounds on the minimum value of a functional defined over a one-dimensional domain of constrained length but with variable end points may be obtained. This is an application of the theory for the fundamental problem of the calculus of variations with variable end-points. The present paper has sought to describe the basic theory in a simple manner. In a subsequent paper it will be shown that the method may be extended to consider problems of higher dimensions and greater physical interest where e.g. a surface area or volume is to be held constant. More generally it is expected that the isoperimetric constraint may be replaced or supplemented by an arbitrary constraint in which there is dependence on the other arguments of the functional.

Although the technique is at present confined to problems with an existing variational extremal principle, the range of such problems is wide in engineering. Methods of bounding quantities can provide useful estimates and offer a valuable quality check on numerical calculations. We aim to illustrate this by extension of the basic theory presented here to address an example from slow viscous flow in a subsequent paper.

A mathematical problem of interest is that of determining sufficient conditions for the equations (4) to (8) inclusive to yield a minimum d_o of the functional (1). Similarly it is of interest to determine sufficient conditions for equations (12) to (17) or (12), (26), (14) to (17) to yield a minimum value of d_λ in the non-degenerate and degenerate cases respectively. In the case of fixed end-points a and b it is simple to show that joint convexity of the function *F* in *u* and u' is sufficient. One or more conditions on both *F* and *U* will be required to derive the sufficient conditions for the case of variable end points.

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