Control of a Public Transport Network by the Max-Plus Algebra: case of a system constrained by maximal connection times

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Abstract: - The theory of linear systems in the max-plus algebra is developed for the analysis of discrete event systems. The timed event graphs are one of the tools used for modelling these systems, and their behaviour can be described by (max, plus)-linear equations. This paper proposes a control structure for a public transport system. We mainly based the research of this control on the Residuation theory. This control aims at conceiving a timetable of buses such that: the connection time at interchange points respects an upper bound corresponding to a tolerance for passengers, and it limits the number of buses required to ensure the connections.

Key-Words: - Public transport system, Petri Nets, Max-Plus algebra, Discrete Event Systems, Control, Connection time.

1 Introduction

The design, the modelling and the analysis of public transport network become increasingly important. They have been studied in order to meet needs of users (see for example [1], [2] and [3]), for instance:

- the displacement between two places under the best conditions (minimum waiting time at the interchange points,…) and better safety;
- the increasing requirements of passengers for information and comfort.

The main objective of these studies is the improvement of the service's quality. As for the production area, and like computer networks, manufacturing systems, communication networks, it appears that public transport networks may be seen as Discrete Event Dynamic Systems (DEDS) [5], [7]. For such networks it is important to exploit the properties and characteristics of appropriate tools of modelling and analysis. For the modelling of a public transport network we will use the graphical tool: Timed Event Graph (TEG) which constitutes a subclass of Petri Nets (PN) [6]. The behaviour of these TEG may be described by a system of linear equations in the (max, plus) algebra [5]. These linear equations allow us to analyse the synchronisation and concurrency phenomena observed. They also enable us to study some properties of the proposed network.

In this paper we are interested in the design of a timetable for a bus network. We try to control this system such as:

a) the connection times remain always lower than a maximal limit fixed beforehand;
b) the necessary number of buses on the network to make all connections is reduced

This problem consists in finding the system control by using the Residuation theory in the (max, plus) algebra.

2 The transport network studied

We are interested in modelling a public transport network, and in particular the management of its connection points in order to improve the service's quality. Let us consider a bus network composed by several lines and several stops, these latter are connection stops or simple stops. In this network, one principal line connects two important interchange points and secondary lines are connected to the principal one. In this paper we limit our study to a simple network composed of two lines with a single common connection stop. Data network like the frequencies of lines, the moving times between two stops, the number of buses on each line…, are known. We are interested in passengers who get on a bus somewhere along the principal line (noted thereafter L_p), make the connection in the interchange stop to reach finally a stop of the secondary line (noted L_s). Our main objective is to minimise the connection times during the travel of those passengers. To reach this goal it is necessary to analyse the network behaviour. Then we model it by a timed event graph from which we deduce a linear (max, +)

model. The analysis of this mathematical model helps us to evaluate and minimise the connection times.

3 A modelling for a bus network

3.1 Modelling by a timed event graph

The timed event graph that models our transport network is given by figure 1:

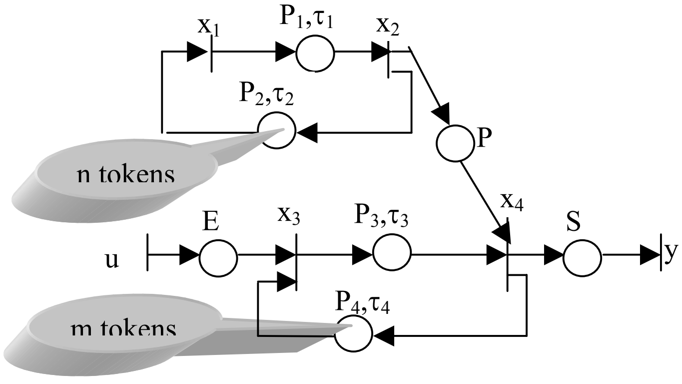

Fig.1. A Timed Event Graph model

In this TEG the circuit (x_1, x_2, x_1) (resp. (x_3, x_4, x_3)) models the line L_p (resp. L_s). Each place is associated with a moving time between two stops (except P that models the passengers waiting for a bus at the connection stop). Transitions model the bus stops:

- x_1 represents the departure stop of line L_p;
- the transitions x_2, x_4 and y are associated to the connection stop;
- the input transition u and x_3 are associated to the departure stop of line L_s.

The temporisation associated to each place represents the moving time between two stops including the time needed for passengers to board and/or get off the bus (we assume that the temporisation associated with E, P and S are null by default). The n tokens ($n\in\angle^*$) in the place P_2 (resp. the m tokens ($m\in\angle^*$) in P_4) represent the buses which circulate on the line L_p (respectively L_s). We assume that the buses of each line are initially at the departure station (bus station).

The event graph of figure 1 works as follows:

- the firing of the transition x_1 removes one token in P_2 and puts it in P_1. It models a bus of the line L_p that leaves its departure station to reach the connection stop in τ_1 time units. Then x_2 is fired when the bus leaves this stop and a token is put in P_2, which models the bus of L_p on its return journey. Another token is also put in the place P which represents passengers waiting for the bus of the line L_s.
- the firing of x_3 means that a bus of L_s leaves its departure stop to join the connection stop. When the token associated with this bus arrives in P_3, the firing of the transition x_4 may occur after τ_3 time units, on condition that P contains one token at this time. Then this firing means that the bus of L_p picks up the passengers waiting at the connection stop.
- the firing of x_4 requires the availability of tokens in its upstream places P and P_3. The token arriving first at one of these two places has to wait for the arrival of one token at the other place. It means the waiting of either passengers (token in P) or a bus of the line L_s (token in P_3) at the connection stop. In fact we only consider the buses of line L_s which make the connection with line L_p.
- in order to avoid the accumulation of tokens in the place P, involved by several firings of x_2 between two consecutive firings of x_4, we propose a timetable that enables to fire the transition x_3 for the first m times. In other words: a timetable is given (respectively calculated) for the first m buses (resp. for each k^{th} round, with $k > m$) of the line L_s.

in ordre to analyse the behaviour analysis of the TEG model, we use the (max, plus) algebra. In the following paragraph we give the state model of our TEG model.

3.2 State model in dioid algebra

To ensure all the connections from the principal line, we assume the number of buses that circulate on the line L_s (m) is greater than the number of buses on L_p (n). In the following we suppose $n\in\angle^*$. From the TEG model we deduce a (max, +) linear model. The variable $x_i(k)$, called *dater*, is the time of the k^{th} firing of transition x_i ($1\leq i\leq 4$). The system is given by:

$\forall\ k > m$ and $m \geq n$,

$$\begin{cases} x_1(k) = \tau_2 \otimes x_2(k-n) \\ x_2(k) = \tau_1 \otimes x_1(k) \\ x_3(k) = u(k) \oplus \tau_4 \otimes x_4(k-m) \\ x_4(k) = x_2(k) \oplus \tau_3 \otimes x_3(k) \\ y(k) = x_4(k) \end{cases} \tag{1}$$

The laws $\oplus$ and $\otimes$ are defined by: $\forall$ (a, b)$\in(3_{max})^2=3\cup\{-\infty\}$ we have: $a\oplus b=\max(a,b)$ and $a\otimes b=a+b$. 3_{max} endowed with these two laws is called *dioid*. We denote respectively $\varepsilon=-\infty$ and $e=0$ the neutral elements of $\oplus$ and $\otimes$. ε is absorbing for $\otimes$ ($a\otimes\varepsilon=\varepsilon\otimes a=\varepsilon$). In what follows we drop $\otimes$ and we write $a\otimes b$ as ab.

In order to put this system in a matrix form, we define the following vectors:

- state vector: $X(k) = [x_1(k), x_2(k), x_3(k), x_4(k)]$;
- output system: $Y(k) = y(k)$;
- input system: u(k)

We can write (1) in the following implicit matrix form:

$\forall\ k > m$ and $m \geq n$,

$$\begin{cases} X(k) = A_0X(k) \oplus A_nX(k-n) \oplus A_mX(k-m) \oplus B\otimes u(k) \\ Y(k) = CX(k) \end{cases} \tag{2}$$

where A_0, A_n, A_m, B and C are the characteristic matrices of the system whose coefficients represent the network's data.

We calculate the Kleene star A_0^* (see [5]) whose the expression is : $A_0^*=E\oplus A_0\oplus\ldots\oplus A_0^{r-1}$ (where r is the order of matrix A_0 and E is the identity matrix). Then we rewrite the equations of our system (2) in the following explicit form:

$$\begin{cases} X(k)=A_0^* A_n X(k-n)\oplus A_0^* A_m X(k-m)\oplus u(k)A_0^* B \\ Y(k)=C\ \{A_0^* A_n X(k-n)\oplus A_0^* A_m X(k-m)\oplus u(k)A_0^* B \end{cases} \quad (3)$$

4 Resolution of the (max,+) linear system

In order to solve (3) and deduce the arrival and departure times of the buses at the network's stops during a period of work (for instance one day), a timetable of the buses moving on the line L_p is supposed given. For the line L_s a timetable is known just for the first m buses; otherwise we can consider the earliest possible departure times of the buses during one day (for instance: 6h a.m.). Then for these initial conditions we consider vectors X(1), …, X(m) whose we know the values of components. For the input vector U the m first times of departure on the line L_s (u(k), for $1\le k\le m$) are planned such that tokens do not accumulate in the place P of our TEG (each bus ensures one connection). Then the resolution of (3) enables us to deduce all the other arrival and departure times of the buses at each stop of the line L_s (for k>m).

4.1 Recurring equation

To obtain recurring equations of order 1 from (3), we define a new vectors: $\forall\ \ k\ge m$ and $m\ge n$

$\tilde{X}(k)=[X(k), X(k-1), .., X(k-n), .., X(k-m+1)]$ in $3^{4m}_{\max}$;

$\tilde{u}(k) = u(k)$ and $\tilde{Y}(k) = Y(k)$.

Then we deduce the following system:

$$\begin{cases} \tilde{X}(k) = \tilde{A}\otimes\tilde{X}(k-1)\oplus\tilde{B}\otimes\tilde{u}(k) \\ \tilde{Y}(k) = \tilde{C}\otimes\tilde{X}(k) \end{cases} \quad (4)$$

where $\tilde{A}\in 3^{4m\times 4m}_{\max}$ (the matrix $\tilde{A}$ is represented by blocks and the order of each element of $\tilde{A}$ is (4x4)), $\tilde{B}\in 3^{4m}_{\max}$ and $\tilde{C}\in 3^{4m}_{\max}$.

The solution of the system (4) is given by:

$$\tilde{X}(k)=\tilde{A}^{k-m}\tilde{X}(m)\oplus\tilde{B}\,\tilde{u}(k)\oplus\ldots\oplus\tilde{A}^{k-m}\tilde{B}\otimes\tilde{u}(m)$$

$$= \tilde{A}^{k-m}\otimes\tilde{X}(m)\oplus\bullet\otimes\tilde{U}(k) \quad (5)$$

Consequently we deduce:

$$\tilde{Y}(k) = \tilde{C}\ \tilde{A}^{k-m}\ \tilde{X}(m)\oplus\tilde{C}\,(\bigoplus_{j=0}^{k-m}\tilde{A}^{j}\tilde{B}\,\tilde{u}(k-j))$$

$$= \tilde{C}\otimes\tilde{A}^{k-m}\otimes\tilde{X}(m)\oplus\tilde{C}\otimes\bullet\otimes\tilde{U}(k) \quad (6)$$

with $\bullet = [\tilde{B}, \tilde{A}\,\tilde{B}, \ldots, \tilde{A}^{k-m}\tilde{B}]$,

and $\tilde{U}(k)=[\tilde{u}(k), \tilde{u}(k-1), \ldots, \tilde{u}(m)]^T$.

4.2 Control to minimise the connection time

In this study the concept of improvement of the service's quality is considered under two aspects corresponding to the requirements of both users and operators of a bus fleet:

- reduction of the waiting times for the passengers at the connection stops. For that we determine departure times of the buses on L_s from the bus station. These times must guarantee that each connection time remains smaller than a fixed upper bound M (M: maximal waiting time accepted by passengers).
- reduction of the number of buses that ensure the connections with the line L_p while guaranteeing the requested service on the line L_s (waiting times limited by M). If this minimal service is ensured on line L_s, it enables to save some round trips on this line (cost saving) and to use some buses to reinforce exceptionally other lines if necessary.
- Given desired output transition firing times $Y_d=\{Y_d(k)\}_{k\in\angle}$ which represent the desired arrival times of buses of the line L_s at the connection stop, we can find the latest input transition firing times $u = \{u(k)\}_{k\in\angle}$ such that the associated output transition firing times ($y=\{y(k)\}_{k\in\angle}$) are *the same* as the desired ones. In other words the following relation must be checked:

$$y(k)=\tilde{C}\ \tilde{A}^{k-m}\tilde{X}(m)\oplus\tilde{C}\ \tilde{B}\ \tilde{u}(k)$$

$$\oplus\tilde{C}\,(\bigoplus_{j=1}^{k-m}\tilde{A}^{j}\tilde{B}\,\tilde{u}(k-j)) = Y_d(k)$$

For every $k\in\angle$, if the objective $y(k)=Y_d(k)$ cannot be reached, we fix the new objective $y(k)\le Y_d(k)$. Then we seek out the latest departure times such that the output transition firings occur *before* the desired ones. Let us note that this study is based on the Resudiation theory in the dioid algebra. Indeed we search a control such that the output of the system behaves as desired. In other words we search $u=\{u(k)\}_{k\in\angle}$ such that $y(k)=H\otimes u(k)\le Y_d(k)$ with H is the transfer matrix of system, then the expression of a such control is given by: $u(k) = Y_d(k)/H$. In our case the expression of this control is given, by: $\forall$ k>m and $m\ge n$,

$$u_{opt}(k) = Y_d(k)\ /\ \tilde{C}\ \tilde{B}$$

The sign '' / '' appearing in this expression represents the subtraction in the dioid algebra.

We remark that for every k>m, the expression of the control $u_{opt}(k)$ depends on the desired output. Then to deduce the control of the system we have to know the values of $Y_d(k)$. In the following we precise, for every k, how to determine the sequence of desired output $\{Y_d(k)\}_{k\in\angle}$, how to calculate the system control and the connection time.

4.1.1 Algorithm of control

The Initial conditions: We note N_{max}, the number of rounds to carry out on the line L_p during a working period. Then we can deduce the N_{max} (resp. m) arrival times of the buses of L_p.(resp. L_s) at the connection stop. For $k\geq m+1$ we must choose the desired output $Y_d(k)$ according to the availability of one bus of the line L_s, while respecting several constraints. The first one is related to the tolerance M. The second constraint is related to the working of TEG model. It must prevent tokens from accumulating in the place P; it is formulated by: $x_2(k)\leq Y_d(k)<x_2(k+1)$. Then we define $Y_d^i(k)=Z(i)\lambda_2$ for every $i \in \{1, \ldots, m\}$, where Z(i) is the last firing time of the transition x_4 by a token that represents the bus i. We calculate the associated connection times expressed as follows:

For $k\geq m+1$, $T^i(k)=Y_d^i(k)/x_2(k) \quad ,\forall\ i \in\{1,\ldots, m\}$

By analysing these connection times we define: $[1, m]=I\cup J\cup K$ such that $\forall i\in I$, $T^i(k)<0$, $\forall i\in J$, $0\leq T^i(k)\leq M$ and $\forall\ i\in K$, $T^i(k)>M$, we consider the two following cases:

- $J \neq \emptyset$: there is at least one bus available for the connection with the line L_p within the definite time limit. The connection time $T_{min}(k)$ is chosen with: $T_{min}^{i_0}(k)=Sup_{i\in J}\{T^i(k)\}$, where we consider the greatest connection time, what is less interesting for passengers but may allow a more efficient management of the buses. Indeed several rounds of one or more buses of the line L_s may be saved; then during the associated unoccupied time those buses may make a pause or be used in other lines of the network. That's why we search for the longest pause periods for those buses. So we have: $Y_d(k)=Z(i_0)\lambda_2$, consequently the control associated with this desired output will be :
  $u_{opt}(k)=Z(i_0)\lambda_2 / \widetilde{C}\,\widetilde{B}$,

- $J = \emptyset$, among the buses being in circulation (not in pause), no bus can join the connection stop within the definite time limit. Then for every $i\in I\cup K$ we have $T^i(k)>M$ or $T^i(k)<0$. However we choose the desired output according to the time when the k^{th} bus of the line L_p arrives at the connection stop: $Y_d(k) = x_2(k)$ It means that $T(k) = 0$ and the control $u_{opt}(k)$ has the following form: $u_{opt}(k)=x_2(k) / \widetilde{C}\,\widetilde{B}$.

Let us note that two situations are possible:

♦ $I\neq\emptyset$: $\exists\ i\in I$, $T^i(k)<0$ it means that there is at least one unoccupied bus of the line L_s, therefore we send one of these unoccupied buses at the connection stop by applying the same heuristic rule as for the case $J \neq \emptyset$.

♦ $I = \emptyset$: all the buses of the line L_s are circulating and the upper bound M for the connection time cannot be respected. In this case we add a new bus on the line L_s to ensure the connection and we increment the number of buses by 1.

This algorithm is schematised by the following flow chart:

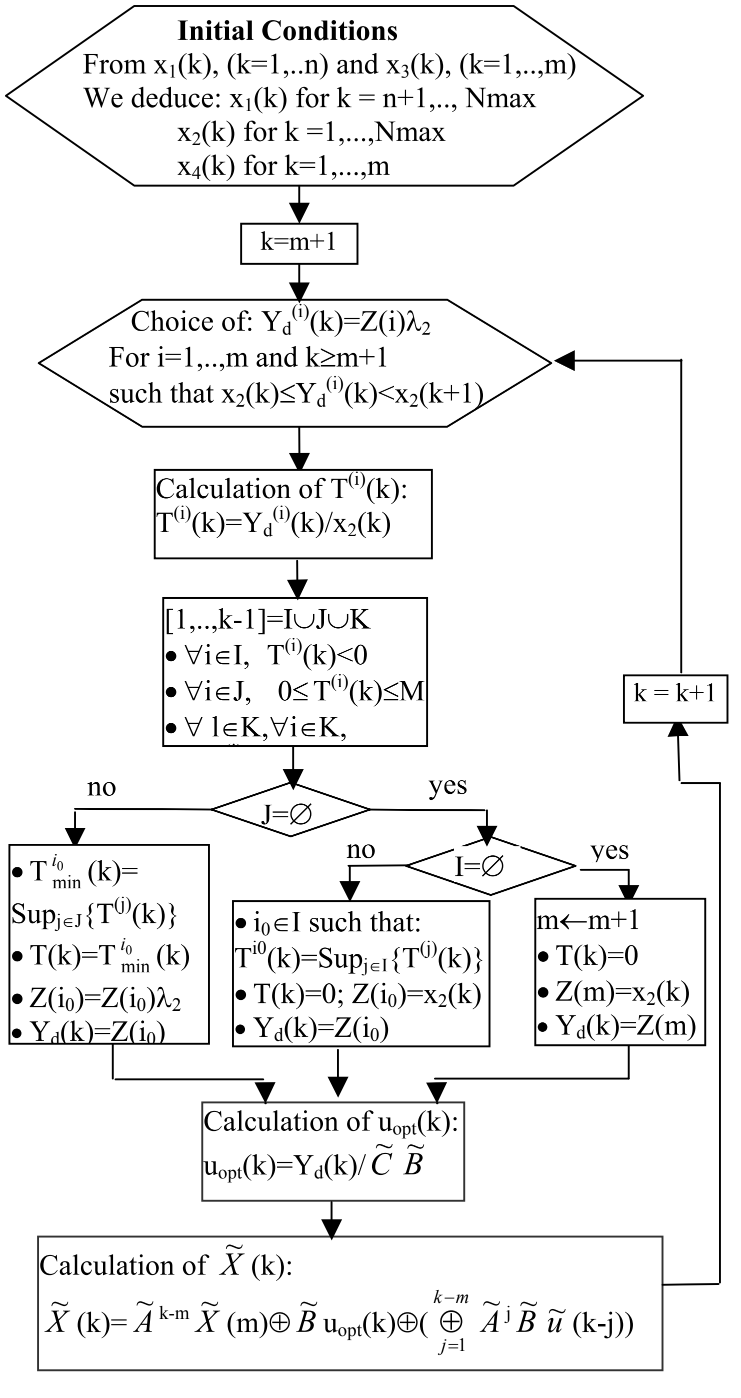

Fig.2 Calculation of the system control

4.2.2 Remark:

If J=∅ and K≠∅ to hold the service's quality we stop the pause of a bus (if I≠∅) or we add a new bus on the considered line. But the acceptance of a greater upper bound M could avoid this last solution while favouring the use of the buses which are already in circulation.

5 Extension to a non periodic planning

Now we consider that the timetable of the line L_p is not a periodic one. The new TEG associated with our system is represented in the following figure:

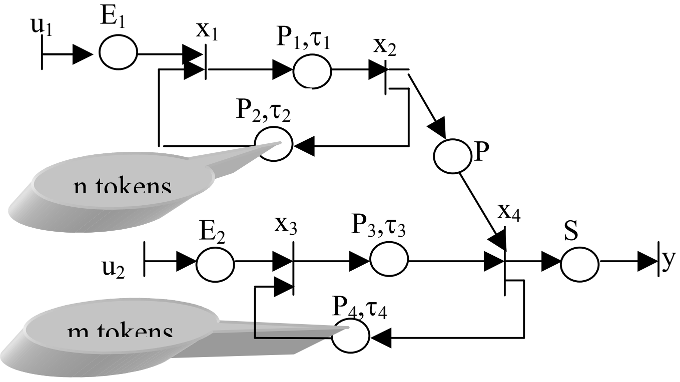

Fig. 3. TEG associated with a non periodic working

The transition u_1 models the authorization for a bus of the line L_p to start its journey. The transition u_2 has the same interpretation as the transition u defined in the previous model. The (max, +)-linear model deduced from this TEG is the following: $k > m$ and $\forall\ m \geq n$

$$\begin{cases} X(k)=A_0X(k)\oplus A_nX(k-n)\oplus A_mX(k-m)\oplus B_1u_1(k)\oplus B_2u_2(k) \\ Y(k) = C\,X(k) \end{cases} \tag{7}$$

Like the first case we put the system (7) in the form of a recurring system, we obtain then:

$$\begin{cases} \widetilde{X}(k)=\widetilde{A}\,\widetilde{X}(k-1)\oplus \widetilde{B}_1\,\widetilde{u}_1(k)\oplus \widetilde{B}_2\,\widetilde{u}_2(k) \\ \widetilde{Y}(k) = \widetilde{C}\otimes \widetilde{X}(k) \end{cases} \tag{8}$$

with $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{A}$ and $\widetilde{C}$ previously defined in paragraph 4.1. for the system (S4). We also take $\widetilde{u}_1=u_1$ and $\widetilde{u}_2=u_2$

The solution of the system (8) is:

$$\begin{cases} \widetilde{X}(k)=\widetilde{A}^{k-m}\,\widetilde{X}(m)\oplus \bigoplus_{j=0}^{k-m} \widetilde{A}^{j}\,\widetilde{B}_1\,\widetilde{u}_1(kj)\oplus \\ \quad \oplus \bigoplus_{j=0}^{k-m} \widetilde{A}^{j}\,\widetilde{B}_2\,\widetilde{u}_2(k-j) \\ \quad =\widetilde{A}^{k-m}\,\widetilde{X}(m)\oplus \bullet_1\widetilde{U}_1(k)\oplus \bullet_2\widetilde{U}_2(k) \\ \widetilde{Y}(k)=\widetilde{C}\,\widetilde{A}^{k-m}\,\widetilde{X}(m)\oplus \\ \quad \oplus\widetilde{C}\,(\bigoplus_{j=0}^{k-m} \widetilde{A}^{j}\,\widetilde{B}_1\,\widetilde{u}_1(k-j)\oplus \bigoplus_{j=0}^{k-m} \widetilde{A}^{j}\,\widetilde{B}_2\,\widetilde{u}_2(k-j)) \\ \quad =\widetilde{C}\,\widetilde{A}^{k-m}\,\widetilde{X}(m)\oplus \widetilde{C}\bullet_1\widetilde{U}_1(k)\oplus \widetilde{C}\bullet_2\widetilde{U}_2(k) \end{cases}$$

with $\bullet_i=[\widetilde{B}_i, \widetilde{A}\,\widetilde{B}_i,.., \widetilde{A}^{k-m}\widetilde{B}_i]$ and

$\widetilde{U}_i(k)=[\widetilde{u}_i(k), \widetilde{u}_i(k-1),.., \widetilde{u}_i(m)]^T$ for $1\leq i\leq 2$.

In this case the control is expressed by:

$$u_{opt}(k)=Y_d(k)\ /\ \widetilde{C}\,\widetilde{B}_2$$

Remark that the choice of a sequence of desired outputs $\{Y_d(k)\}_{k\in \angle}$ is the same as in the previous paragraph.

6 An example of application

To illustrate our results we consider the given network in figure 3 with the following data:

- The line L_p: $n=1$, $\tau_1=11$, $\tau_2=9$, $\lambda_1=20$, $x_1(1)=0$, $x_2(1)=11$;
- The line L_s: $m=2$, $\tau_3=12$, $\tau_4=10$, $\lambda_2=22$, $x_3(1)=5$, $x_4(1)=17$, $x_3(2)=19$, $x_4(1)=31$;

The line L_p is served by only one bus (noted thereafter b). Then we can deduce the transition firing times $x_1(k)$ and $x_2(k)$ by considering the periodic mode at these stops. These times are given by: For $i\in\{1, 2\}$, for $k\geq 1$,

$$x_i(k+1)=20\ x_i(k),$$

The line L_s is served by two buses (noted b_2 and b'_2). Only the first departure times of b_2 and b'_2 are known. We consider that: $u_2(1)=x_3(1)=5$ and $u_2(2)=x_3(2)=19$. Then the initial condition of system is the following: $\widetilde{X}(2)=[X(2),X(1)]=[20,31,19,31,0,11,5,17]$.

We assume that the maximal waiting time accepted by any passenger is M=12 minutes (mn), and we consider that the working period studied corresponds to 20 journeys for the line L_p. The results are presented in the following table(Table.1). The word ''Bus'' means the bus of L_s that ensures the connection, and T_{wai} means the passenger waiting time.

k	x_2	x_4	Y_d	u_{opt}	T_{wait}	Bus
1	11	17	17	5	6	b_2
2	31	31	31	19	0	b'_2
3	51	53	53	41	2	b'_2
4	71	75	75	63	4	b'_2
5	91	97	97	85	6	b'_2
6	111	119	119	107	8	b'_2
7	131	141	141	129	10	b'_2
8	151	163	163	151	12	b'_2
9	171	171	171	159	0	b_2
10	191	193	193	181	2	b_2
11	211	215	215	203	4	b_2
12	231	237	237	225	6	b_2
13	251	259	259	247	8	b_2
14	271	281	281	269	10	b_2

15	291	303	303	291	12	b_2
16	311	311	311	299	0	b'_2
17	331	333	333	321	2	b'2
18	351	355	355	343	4	b'2
19	371	377	377	365	6	b'2
20	391	399	399	387	8	b'2

Table 1: Table of results (periodic case)

The times $x_4(k)$ ($\forall$ $k \geq 1$) represent the solution of the mathematical model; this solution is calculated from the control deduced by Residuation theory in the dioid algebra. The departure times of the buses which do not ensure the connection with the line L_p at the connection stop, are eliminated or can be added later to complete the service of the line L_S (normal working without obligation of connection).

Extension: In this case we keep the same system data except $x_3(2)=20$ and $x_4(2)=32$. Then we obtain the following results**:**

k	u_1	x_2	x_4	Y_d	u_{opt}	T_{wait}	Bus
1	0	11	17	17	5	6	b_2
2	21	32	32	32	20	0	b'_2
3	41	52	54	54	42	2	b'_2
4	63	74	76	76	64	2	b'_2
5	86	97	98	98	86	1	b'_2
6	106	117	120	120	108	3	b'_2
7	127	138	142	142	130	4	b'_2
8	147	158	164	164	152	6	b'_2
9	169	180	186	186	174	6	b'_2
10	190	201	208	208	196	7	b'_2
11	210	221	230	230	218	9	b'_2
12	230	241	252	252	240	11	b'_2
13	252	263	274	274	262	11	b'_2
14	273	284	296	296	284	12	b'_2
15	296	307	318	318	306	11	b'_2
16	318	329	340	340	328	11	b'_2
17	338	349	349	349	337	0	b_2
18	360	371	371	371	359	0	b_2
19	380	391	393	393	381	2	b_2
20	401	412	415	415	403	3	b_2

Table 2: Table of results for the extension case

We remark that in the periodic case, the connection time reaches a periodic mode after some bus rounds and the saved times are almost the same ones. In second case, the connection times never reach a periodic mode and the saved times are rather different. This is obviously due to the non-periodicity of the line L_p.

7 Conclusion

This paper dealt with the modelling and the control of a public transport network, in particular the management of its connections. The design of a network timetable was studied in order to reduce the passengers waiting times. These waiting times remain smaller than a given upper bound. This timetable is calculated via a control obtained by using the Residuation theory in the dioid algebra. A second important result of the control policy applied in this paper is the possibility to limit the number of buses on the network. This reduction enables to save some rounds on the secondary line or to serve other lines of the network if necessary. This may involve a cost reduction. One of the perspectives of this work is to extend the control to more concrete networks (more than two lines with various connection stops managed simultaneously). We also hope to free ourselves from the assumption $m \geq n$, which involves the management of some structural conflicts in the timed event graph

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