A MATHEMATICAL MODELING AND A COMPUTER SIMULATION OF AN EPIDEMIC MODEL

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Abstract: In this paper we will discuss a mathematical modeling process and a computer simulation of an epidemic problem. The problem is modeled by control optimal, and then analyze by the maximal principle of Pontryagin with bang-bang control. The result will be simulated using MATLAB. As the result is the schedule of effective medical treatments can be determined provided that the values of parameters are known.

Key words: maximal principle of Pontryagin, bang-bang control., epidemiology

1 Introduction

Epidemiology is the study of the spread of diseases, in space and time. The objective is to trace casual factors responsible for, or contribute to, their occurrences. The spreads of the disease patterns are caused by several factors. The first factor is season. The data exhibit that places having four seasons have similar pattern: in the end of winter occurred mump and hepatitis A, in the end of summer occurred polio, in the early of winter occurred influenza and in the spring occurred mump. Meanwhile, Tuberculosis (TBC) doesn't depend on the season. The second factor is geographic distribution. The classical method to analyze the occurrence of an epidemic is by plotting every new case on the map. The distribution of the diseases shows the concentration on a geographic area. For instance, the spreads of polio disease in Chicago in 1956. The highest rate occurred in the city. This was caused by the concentration of the ethnic of Black American St. Louise Encephalitis in Houston in 1964. In 1854, John Snow demonstrated that cholera could be transmitted via drinking water. He matched incidence data from the surrounding Broad Street in London with a sketch of the location of water pump. Those are classic examples of how description and data analysis may lead to an explanation, which then can be used for prevention or prediction.

After we find parameters that are responsible of spreading the diseases, the next step is how to control the parameters. Furthermore, one of the problem that can be appeared from this epidemic phenomena is how to control the spreading of the diseases effectively. Until now, there are two control methods, i.e. quarantine (prevent the contact between infectious and susceptible) and decreasing the number of infectious by medical treatment. The first method is more difficult to implement, therefore in this paper we discuss the second method only. The goal is to optimize the medical treatments in order to control the epidemics.

The objectives of this paper are to show the modeling process, solve the model by maximal principle of Pontryagin with bang-bang control and computer simulation using MATLAB.

2 Mathematical Modeling

The mathematical modeling of epidemics in populations is a vast and important subject. Numerous opportunities are opened for mathematicians to work on biologically relevant and nontrivial mathematical problems. That field is about translating biological assumption into mathematics, mathematical analysis aided by interpretation and obtaining insight into epidemic phenomena when translating mathematical results into population biology.

Sethi (1970) and Sethi (1978) proposed models to optimize the limited medical resources related to an epidemic. At time t, suppose that the state x(t) represent the number of infectious (those the disease) in a population of N people, where $0 \leq x(t) \leq N$.

First case, we assume that there is no vaccination before and no medical effort. We also assume that the size of population, N, is limited. Therefore, the growth of the number of infectious can be explained in term of the logistic differential equation,

$$\dot{x}(t) = bx(t)[N - x(t)], \tag{1}$$

where b is a constant.

For the next case, we assume that u(t) represent the intensity of medical treatment, where 0 < u(t) < U; and U is some maximum practicable level. This medical treatment, u(t), then can decrease the growth of the infectious people by u(t)x(t), as shown by the following

$$\dot{x}(t) = bx(t)[N - x(t)] - x(t)u(t), \tag{2}$$

where $x(0)=x_0, x(T) = x_T,\ 0 \leq x(t) \leq N,\ 0 \leq u(t) \leq U.$

The equation (2) will be used to constrain the optimal control model we will discuss.

Some objective functions of such problem must be selected to measure the effectiveness of the medical effort. Consider the following cost function to be minimized:

$$\underset{u}{Min}\, J(u) = \int_0^T e^{-\delta t} [ku(t) + Kx(t)]dt, \tag{3}$$

where $e^{-\delta t}$ is the discount factor, and unit cost of k and K is attached respectively to use of medical resources and to the number of people affected.

We conclude that the control optimal model of the medical treatment can control the spread of the diseases (epidemic) such that it will satisfy:

$$\underset{u}{Min} J = \int_0^T e^{-\delta t} [ku(t) + Kx(t)]dt, \tag{4}$$

s.t.

$$\dot{x}(t) = bx(t)[N - x(t)] - x(t)u(t),$$

where $x(0) = x_0, x(T) = x_T,\ 0 \leq x(t) \leq N,\ 0 \leq u(t) \leq U$

In this model, u(t) is the control function and x(t) is the state function.

3 Exact Solution

In this section we will explore the solution of (4), the control function $\hat{u}$. Note that the control function in the objective and state function are linear. As the result, base on the maximal principle of Pontriagyn with bang-bang control, the control function attained 0 in the interval $(0,T^*)$ and it is equal to U in the interval (T^*,T), except on a singular point T^*,if the coefficient u(t) on Hamiltonian is zero.

Based on the maximal principle of Pontriagyn, the Hamiltonian of (4) is:

$$H = e^{-\delta t} [ku(t) + Kx(t)] + p_1(t)\{bx(t)[N - x(t)] - u(t)x(t)\} \tag{5}$$

To determine u(t) that minimize the objective function (4), we apply the maximal principle of Pontriagyn,

$$H[\hat{x}(t), \hat{u}(t), t, p(t)] \leq H[\hat{x}(t), u(t), t, p(t)] \tag{6}$$

or

$$e^{-\delta t} [k\hat{u}(t) + K\hat{x}(t)] + p_1(t)\{b\hat{x}(t)[N\hat{x}(t)]\hat{u}(t)\ \hat{x}(t)\}$$
$$\leq e^{-\delta t} [ku(t) + K\hat{x}(t)] + p_1(t)\{b\hat{x}(t)[N - \hat{x}(t)]u(t)\hat{x}(t)\} \tag{7}$$

By considering the terms of (7) in which contain the u(t) only, the problem can be expressed,

$$\min_u e^{-\delta t} ku(t) - p_1(t)x(t)u(t);\ 0 \leq u(t) \leq U \tag{8}$$

To simplify (8), we define $\mu(t) = e^{\delta t} p_1(t)$. Since the value of $e^{\delta t}$ is always positive, the problem cn be simplified as follow,

$$\min_u [k - \mu(t)x(t)]u(t);\ 0 \leq u(t) \leq U \tag{9}$$

The adjoin differential equation of (9) is:

$$-\dot{p}_1(t) = \frac{\partial H}{\partial x} = Ke^{-\delta t} + p_1(t)\{bN - 2bx(t) - u(t)\} \tag{10}$$

Hence

$$\dot{p}_1(t) + \{bN - 2bx(t) - u(t)\} p_1(t) = -K e^{-\delta t} \tag{11}$$

If both sides of the equation (11) are multiplied by $e^{\delta t}$,

$$e^{\delta t}\ \dot{p}_1(t) + \{bN\text{-}2bx(t)\text{-}u(t)\}e^{\delta t}\ p_1(t) = -K \quad (12)$$

and equation (12) can be represented as follows,

$$e^{\delta t}\ \dot{p}_1(t) + \delta\ e^{\delta t}\ p_1(t) + \{-\delta + bN\text{-}2bx(t)\text{-}u(t)\}e^{\delta t}\ p_1(t) = -K \quad (13)$$

Since $\mu(t) = e^{-\delta t}[\dot{p}_1(t) + \delta p_1(t)]$, then the adjoin differential equation becomes:

$$\dot{\mu}(t) + \{-\delta + bN\text{-}2bx(t)\text{-}u(t)\}\ \mu(t) = -K \quad (14)$$

From equation (9), the solutions of the problem (by bang-bang control) are,

$\hat{u}(t) = 0;\ if\ \mu(t)x(t) < k$

$\hat{u}(t) = U;\ if\ \mu(t)x(t) > k$

However, if x(t) hits the boundary, either 0 or N, then u(t) must be modified to prevent x(t) crossing the boundary.

Consider the given differential equation (1) for x(t), when u(t) = 0 on some interval of time t, $(0,T^*)$ and when u(t) = U on some interval of time t, (T^*,T).

When u(t) = 0 then from equation (2),

$\dot{x}(t) = bx(t)[N\text{-}x(t)]$.

Consequently,

$\hat{x}(t) = N/(1+\alpha e^{-Nbt})$,

where α is a constant of integration.

When $u(t) = U$,

$\dot{x}(t) = (bN\text{-}U)x(t)\text{-}\ b(x(t))^2$

If $b < U/N$, then

$\dot{x}(t) = -b(gx(t) + (x(t))^2);\ with\ g = (U/b)\text{-}N > 0.$

The solution of this differential equation is,

$\hat{x}(t) = g/(\beta e^{gbt} - 1)$,

where β is a constant of integration.

On the other hand, if $b > U/N$, then

$\dot{x}(t) = -bx(t)(-h + x(t));\ with\ h = -(U/b) + N > 0.$

The solution of this differential equation is,

$\hat{x}(t) = h/(1\text{-}\gamma e^{-hbt})$,

where γ is a constant of integration.

What the possible optimum value look like? The above theory leads to bang-bang control, except for a possible singular point, T^* say, in which u(t) is adjusted within (0,T) to fulfill the terminal constraint $x(T) = x_T$.

Consider a possible solution with just one switching time T^*. If $t < T^*$, then u(t) = 0 when $x(0) < x^*$ or then u(t) = U when $x(t) > x^*$. Let $x^* = x(T^*)$. When $t < T^*$, consider a possible singular interval solution, for which $x(t) = x^*$, $\mu(t) = \mu^*$; where $x^*\mu^* = k$ and $u(t) = u^*$. This solution would represent a steady state, during which the time derivatives $\dot{x}(t) = 0$ and $\dot{\mu}(t) = 0$. Since the derivatives are zero, then based on (14) and (2),

$-\delta + bN\text{-}bx^*\text{-}u^*)\ \mu^* = -K$ and $0 = bx^*[N\text{-}\ x^*]\text{-}u^*x^*$

Consequently

$-\delta + bN\text{-}bx^*\text{-}u^* = -K/\mu^* = -Kx^*/k$ and $0 = bN\text{-}u^*\text{-}bx^*$

These two equations solve

$x^* = \delta\ k/(K - bk)\ \ and\ \ u^* = b(N - x^*)$

In this singular condition, if $x^* > N$ then x^* must be changed by N and u^* by 0 to fulfill the terminal constraint.

From these above description, the solution of the control optimal problem is:

$$\hat{x}(t) = \begin{cases} N/[1+\alpha e^{-Nbt}]; \ \textit{if } \hat{u}\ (t) = 0 \\ \delta\ k/(K-bk); \ \textit{if } \hat{u}\ (t) = b(N-x^{*}) \\ g/[\beta e^{gbt}-1]; \ \textit{if } \hat{u}\ (t) = U \textit{ and } b<U/N \\ h/[1-\gamma e^{-hbt}]; \ \textit{if } \hat{u}\ (t) = U \textit{ and } b<U/N \end{cases}$$

4 Computer Simulations

The simulation is done by choosing the value of parameters, δ = 0.08, k = 0.003, K = 0.000004, N = 1000, x_0 = 20, b = 0.00003 and U = 1, then the length of interval t is 38 days for $\hat{u} = 0$ and singular condition at t = 39 where $\hat{u}$ (39) = 0.028 and $\hat{x}$(39) = 61.38 and for $\hat{u}$ =1 the length of the interval t is 8 days(see Fig.1).

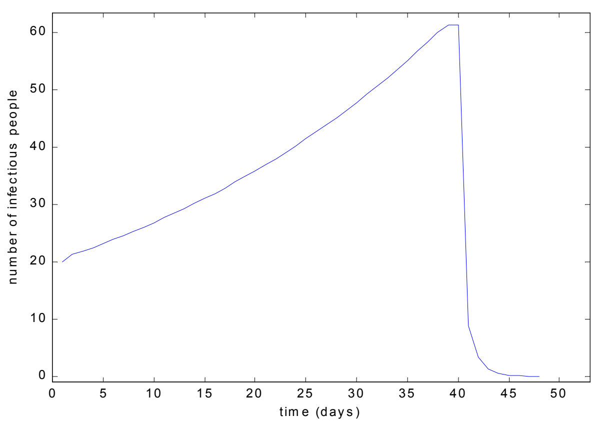

Fig.1 Dynamics of the number of infectious people

5 Conclusion

Controlling transmition of diseases is harder today than ever before, but some principles remain fundamental if control is to be achieved: political will (financial support, human resources), improvement of public health infrastructure and vector control programs, intersectional coordination (partnership among donors, the public sector, civil society, non governmental organizations and the private and commercial sectors), active community participation and reinforcement of health legislation. Ministries of health must direct control, and must establish epidemiological and entomological surveillance and education campaigns for the community. It is fundamental that the community recognizes its responsibility in spread of diseases control.

To refer the above discussion, we conclude mathematically that the schedule of effective medical treatments can be determined provided that the values of parameters are known.

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