DYNAMIC MODELLING AND SIMULATION OF A FLEXIBLE RECTANGULAR ISOTROPIC PLATE STRUCTURE USING FINITE DIFFERENCE METHODS

I. Z. MAT DARUS and M. O. TOKHI
Automatic Control And Systems Engineering Department
University of Sheffield
Mappin Street, Sheffield, S1 3JD.
UNITED KINGDOM

Abstract: - An investigation into the dynamic characterisation of a rectangular flexible plate structure is presented and a simulation algorithm characterising the behaviour of the plate is developed using finite difference methods. A thin, rectangular plate, with all edges clamped, is considered. The investigation is accomplished by varying the width over the length ratio of the plate. The algorithm is implemented within the Matlab environment and allows application and sensing of a disturbance signal at any mesh point on the plate. Such a provision is desirable for the design and development of active vibration control techniques for the system. The performance of the developed algorithm shows that it converges faster than previously reported work. The simulation algorithm thus developed and validated forms a suitable test and verification platform for development of active vibration control strategies for flexible rectangular plate structures of various width over length ratios.

Keywords: Dynamic modelling, finite difference method, flexible rectangular plate, numerical simulation.

1 Introduction

Plates are elements of practical importance in many engineering applications. Study of the natural modes, frequencies and the dynamic behaviour of thin isotropic rectangular plates is a subject that has received considerable attention due to its technical importance, for the last decade. In addition to being a problem of academic interest, many applications of rectangular plates are found in industry. In recent years, circular and rectangular thin plates have been used as ultrasonic radiators in ultrasonic ranging, ultrasonic leviation, and ultrasonic drying. The reason is that radiators of this plate in flexural vibration can improve the acoustic impedance matching between the piezoelectric transducers and air medium [1]. The control of a vibrating plate is, however, a complex problem. This is due the highly non-linear dynamics of the system, which involve complex processes. Accordingly, there is a growing need for developing suitable modelling and control strategies for such systems.

It is crucial to obtain an accurate model of the plate structures in order to control the vibration of a plate efficiently. An accurate model will lead to the realisation of satisfactory control. Various approaches such as finite element (FE) method and boundary element (BE) formulation have previously been considered for modelling thin flat plates [2-3]. The computational complexity and consequent software coding involved in both the BE formulation and the FE method constitute major disadvantages of these techniques, especially in real-time applications. The relatively reduced amount of complexity in computation involved in the finite difference (FD) method makes the technique more suitable in real-time applications. Moreover, in applications involving a uniform structure, such as the plate system considered here, the FD method is found to be more appropriate.

2 The Dynamic Equation of a Plate

In this section the classical dynamic equations of motion of a thin rectangular plate are developed. The thin plate is assumed to undergo a small deflection, w. Considering all the forces and the effect of shear forces Q_x and Q_y, in terms of the moments M_x, M_y and M_{xy} on bending yield [4-5]

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{\rho}{D}\frac{\partial^2 w}{\partial t^2} = \frac{q}{D} \qquad (1)$$

where w is the lateral deflection in the z direction, ρ is the mass density per unit area, $q = q(x, y)$ is the transverse external force at point (x, y) and has dimensions of force per unit area, $\frac{\partial^2 w}{\partial t^2}$ is the acceleration in the z direction, $D = \frac{Eh^3}{12(1-\upsilon)}$ is the flextural rigidity, with υ representing the Poisson ratio, h the thickness of the plate and E the Young's modulus.

Equation (1) represents the dynamic equation characterising the behaviour of the flexible plate in lateral motion. Solution of this equation can be obtained by considering the corresponding boundary and initial conditions. It is noted that the model thus utilised incorporates no damping. To construct a suitable simulation environment characterising the behaviour of the plate as a platform for test and verification of controller designs, a numerical approach based on FD methods is considered.

3 The Finite Difference Simulation Algorithm

Unlike analytical solutions, the FD method computes solutions of the model only at discrete points. Therefore, an important step in this method is dividing the computation domain into small regions and assigning each region a reference index [6]. The x-axis is represented with the reference index i and the y-axis with the reference index j, where $x = i\Delta x$ and $y = j\Delta y$. In the case of a rectangular plate structure, a three dimensional coordinate system is considered. The additional dimension is time t, which is represented with a reference index k, where $t = k\Delta t$.

For each nodal point in the interior of the grid (x_i, y_j, t_k), (i = 0,1, …n; j = 0, …m; and k = 0, 1…p), a Taylor series expansion is used to generate the *central difference formulae* for the partial derivative terms of the response (deflection), $w(x, y, t) = w_{i,j,k}$ of the plate at point $x = i\Delta x$, $y = j\Delta y$ and $t = k\Delta t$ [6-7]. Thus, using first-order approximations at the mesh point inside the plate, on the boundary and as well as at the fictitious points outside the plate, a general solution of the partial differential equation (PDE) in equation (1) can be obtained in the discrete form as:

$$\begin{aligned} w_{i,j,k+1} = -\frac{D\,\Delta t^2}{\rho}\Big(& P\,w_{i,j,k} + Q\left(w_{i+1,j,k} + w_{i-1,j,k}\right) \\ & + R\left(w_{i,j+1,k} + w_{i,j-1,k}\right) \\ & + S\left(w_{i+1,j+1,k} + w_{i-1,j+1,k} + w_{i-1,j-1,k} + w_{i+1,j-1,k}\right) \\ & + T\left(w_{i+2,j,k} + w_{i-2,j,k}\right) + U\left(w_{i,j+2,k} + w_{i,j-2,k}\right)\Big) \\ & + 2\,w_{i,j,k} - w_{i,j,k-1} + \frac{\Delta t^2 q_{i,j}}{\rho} \end{aligned} \tag{2}$$

where

$$P = \frac{6}{\Delta x^4} + \frac{8}{\Delta x^2 \Delta y^2} + \frac{6}{\Delta y^4}$$

$$Q = -\frac{4}{\Delta x^4} - \frac{4}{\Delta x^2 \Delta y^2}$$

$$R = -\frac{4}{\Delta y^4} - \frac{4}{\Delta x^2 \Delta y^2}$$

$$S = \frac{2}{\Delta x^2 \Delta y^2}$$

$$T = \frac{1}{\Delta x^4} \text{ and } U = \frac{1}{\Delta y^4}$$

$w_{i,j,k+1}$ is the deflection of nodal point (x_i, y_j) of the plate at time step $k+1$.

For simulation purposes, it is natural to assume that initially the plate has no deflection. In other words the forces and moments of the plate due to its weight are neglected [5]. Therefore,

$$w_{i,j,k}\Big|_{t=0} = 0$$

The boundary condition along a clamped edge, say y=a, is

$$w\Big|_{y=a} = \frac{\partial w}{\partial y}\Big|_{y=a} = 0$$

4 The algorithm development

To realise the simulation algorithm based on the FD formulation in equation (2), a rectangular plate with width = a, length = b and thickness = h is considered. The plate is divided into n sections in the x direction and m sections in the y direction. The length of each section accordingly is, $\Delta x = b/n$ and the width is $\Delta y = a/m$.

Using equation (2), a difference equation corresponding to each nodal point is developed. It follows from this equation that, to obtain the deflection of certain points on the plate, the deflection of fictitious points outside the defined interval are required. These correspond to points outside the rectangular plate. The known boundary conditions related to the dynamic equation of the plate are utilized to eliminate these fictitious points. Using matrix notation, equation (2) can be written in a compact form as

$$\mathrm{W}_{i,j,k+1} = \mathrm{A}\mathrm{W}_{i,j,k} + 2\,\mathrm{W}_{i,j,k} - \mathrm{W}_{i,j,k-1} + \mathrm{B}\,\mathrm{F}_{i,j,k} \quad (3)$$

where, $\mathrm{W}_{i,j,k+1}$ is the deflection of grid points i = 1, 2,…, n and j = 1, 2,…, m at time step $k+1$. $\mathrm{W}_{i,j,k}$ and $\mathrm{W}_{i,j,k-1}$ are the corresponding deflections at time steps k and $k-1$ respectively. A is a constant n×n matrix whose entries depend on physical dimensions and characteristics of the plate and the number of sections the plate is divided into, F is an n×1 matrix known as the forcing matrix and B is a scalar constant related to the time step Δt and ρ is mass per unit area of the plate.

$$\mathrm{W}_{i,j,k+1} = \begin{bmatrix} w_{1,1,k+1} \\ w_{1,2,k+1} \\ \vdots \\ w_{1,m,k+1} \\ w_{2,1,k+1} \\ w_{2,2,k+1} \\ \vdots \\ w_{2,m,k+1} \\ w_{3,1,k+1} \\ w_{3,2,k+1} \\ \vdots \\ w_{n,m,k+1} \end{bmatrix}, \quad \mathrm{W}_{i,j,k} = \begin{bmatrix} w_{1,1,k} \\ w_{1,2,k} \\ \vdots \\ w_{1,m,k} \\ w_{2,1,k} \\ w_{2,2,k} \\ \vdots \\ w_{2,m,k} \\ w_{3,1,k} \\ w_{3,2,k} \\ \vdots \\ w_{n,m,k} \end{bmatrix} \quad \mathrm{W}_{i,j,k-1} = \begin{bmatrix} w_{1,1,k-1} \\ w_{1,2,k-1} \\ \vdots \\ w_{1,m,k-1} \\ w_{2,1,k-1} \\ w_{2,2,k-1} \\ \vdots \\ w_{2,m,k-1} \\ w_{3,1,k-1} \\ w_{3,2,k-1} \\ \vdots \\ w_{n,m,k-1} \end{bmatrix}$$

$$\mathrm{A} = \begin{bmatrix}
a_1 & a_2 & a_3 & 0 & \cdots & 0 & a_4 & a_5 & a_4 & 0 & \cdots & 0 & a_6 & 0 & \cdots & 0 & 0 \\
a_2 & a_1 & a_2 & a_3 & 0 & \cdots & 0 & a_4 & a_5 & a_4 & 0 & \cdots & 0 & a_6 & 0 & \cdots & 0 \\
a_3 & a_2 & a_1 & a_2 & a_3 & 0 & \cdots & 0 & a_4 & a_5 & a_4 & 0 & \cdots & 0 & a_6 & \ddots & \vdots \\
0 & a_3 & a_2 & a_1 & a_2 & a_3 & 0 & \cdots & 0 & a_4 & a_5 & a_4 & 0 & \cdots & \ddots & \ddots & 0 \\
\vdots & 0 & a_3 & a_2 & a_1 & a_2 & a_3 & 0 & \cdots & 0 & a_4 & a_5 & a_4 & \ddots & \ddots & 0 & a_6 \\
0 & \vdots & 0 & a_3 & a_2 & a_1 & a_2 & a_3 & 0 & \cdots & 0 & a_4 & \ddots & \ddots & 0 & \vdots & 0 \\
a_4 & 0 & \vdots & 0 & a_3 & a_2 & a_1 & a_2 & a_3 & 0 & \cdots & \ddots & \ddots & a_5 & a_4 & 0 & \vdots \\
a_5 & a_4 & 0 & \vdots & 0 & a_3 & a_2 & a_1 & a_2 & a_3 & \ddots & \ddots & 0 & a_4 & a_5 & a_4 & 0 \\
a_4 & a_5 & a_4 & 0 & \vdots & 0 & a_3 & a_2 & a_1 & \ddots & \ddots & 0 & \vdots & 0 & a_4 & a_5 & a_4 \\
0 & a_4 & a_5 & a_4 & 0 & \vdots & 0 & a_3 & \ddots & \ddots & a_2 & a_3 & 0 & \vdots & 0 & a_4 & a_5 \\
\vdots & 0 & a_4 & a_5 & a_4 & 0 & \vdots & \ddots & \ddots & a_2 & a_1 & a_2 & a_3 & 0 & \vdots & 0 & a_4 \\
0 & \vdots & 0 & a_4 & a_5 & a_4 & \ddots & \ddots & 0 & a_3 & a_2 & a_1 & a_2 & a_3 & 0 & \vdots & 0 \\
a_6 & 0 & \vdots & 0 & a_4 & \ddots & \ddots & 0 & \cdots & 0 & a_3 & a_2 & a_1 & a_2 & a_3 & 0 & \vdots \\
0 & a_6 & 0 & \vdots & \ddots & \ddots & a_5 & a_4 & 0 & \cdots & 0 & a_3 & a_2 & a_1 & a_2 & a_3 & 0 \\
\vdots & 0 & a_6 & \ddots & \ddots & 0 & a_4 & a_5 & a_4 & 0 & \cdots & 0 & a_3 & a_2 & a_1 & a_2 & a_3 \\
0 & \vdots & \ddots & \ddots & 0 & \cdots & 0 & a_4 & a_5 & a_4 & 0 & \cdots & 0 & a_3 & a_2 & a_1 & a_2 \\
0 & 0 & \cdots & 0 & a_6 & 0 & \cdots & 0 & a_4 & a_5 & a_4 & 0 & \cdots & 0 & a_3 & a_2 & a_1
\end{bmatrix}$$

where $a_1 = P\,c$, $a_2 = Q\,c$, $a_3 = Tc$, $a_4 = S\,c$, $a_5 = Rc$, $a_6 = Uc$ and $c = -\dfrac{D\Delta t^2}{\rho}$,

$$\mathrm{F} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ q_{i,j} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ and } \mathrm{B} = \left(\frac{\Delta t^2}{\rho}\right)$$

The algorithm is developed using an iterative scheme based on equation (3) within the Matlab environment, and it allows application and sensing of a disturbance signal at any mesh point on the plate. Such a provision is desirable for the design and development of active vibration control techniques for the system.

5 The algorithm stability

The stability of the algorithm can be examined by ensuring that the iterative scheme described in equation (3) always converges to a solution. Therefore, for the simulation algorithm to be stable at all the time, a well-known Von Neumann stability condition is realised. Accordingly, it is found that a necessary and sufficient condition for stability of the simulation algorithm is

$$0 \le \left(\frac{\mathrm{D}\Delta t^2}{\rho \Delta x^2 \Delta y^2}\right)\left(32 + 16\frac{\Delta y^2}{\Delta x^2} + 16\frac{\Delta x^2}{\Delta y^2}\right) \le 4 \qquad (4)$$

For the algorithm developed, by setting the parameters Δx, Δy and ρ to certain values, the sampling time, Δt, is varied so that it satisfies the requirement in equation (4) and hence stability is achieved. The sampling time is chosen to be the biggest value possible within the stability region. This is because, in the simulation environment, as the sampling time is smaller, for the algorithm to complete the computation within certain time, it is required to undergo more iteration rather than having a bigger sampling time.

6 The Algorithm Implementation

In the simulation results presented here, an aluminium type thin flat rectangular plate is considered. The plate is clamped at all edges. The parameters of the plate are listed in Table 1.

Table 1: Parameters of the plate

Parameter	Value (in S.I unit)
Length, *b*	1.000 m
Width, *a*	0.500 m
Thickness, *h*	0.0032004 m
Mass density per area, ρ	2700 kg/m^2
Young's Modulus, E	7.11×10^{10}N/m^2
Second Moment of Inertia, I	5.1924×10^{-11}m^2
Poisson's Ratio, ν	0.3

For implementation of the simulation algorithm using the FD method, the plate is divided into 20 sections in the x and 10 sections in the y directions. The width over length ratio, *a/b* = 0.5. A sampling time of $\Delta t = 0.001$ seconds is chosen which satisfies the stability requirement, and the response of the plate is considered over 4 seconds. Throughout these simulation exercises, a finite duration step input force per unit area with an amplitude F=0.5N is applied to the centre-point of the plate from $t = 0.2$ second to $t = 0.5$ second. The maximum deflection occurs at the centre-point of the plate. The time domain response and the corresponding spectral density at x = 0.3 m and y=0.3 m of the plate are shown in Fig. 1 and Fig. 2, respectively. It is noted that the plate response is characterised by a set of resonance modes, among which the first few (1-3) are the dominant ones. Fig.3 and Fig. 4 show the corresponding plate deflection at $t = 0.327$ sec and $t = 0.641$ sec when maximum and minimum deflections occur, respectively.

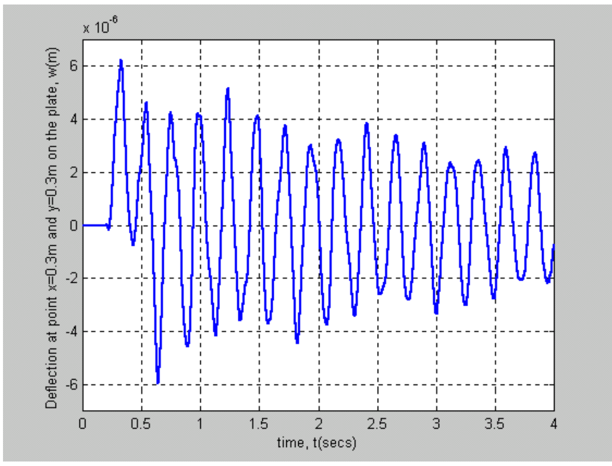

Fig.1: Finite difference simulated time-domain response of the plate at the point x=0.3 m and y=0.3m.

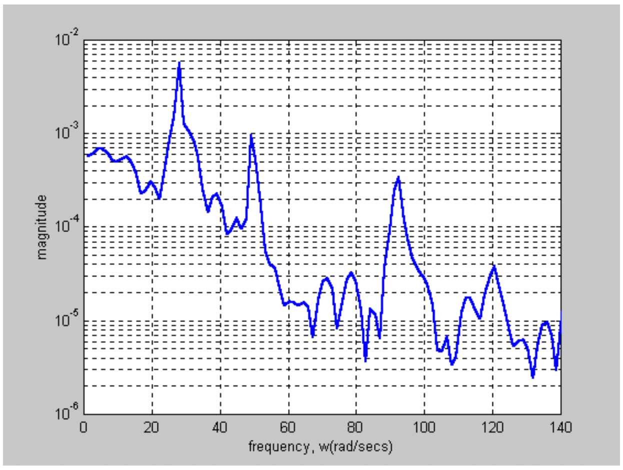

Fig.2: Finite difference simulated frequency-domain response of the plate at the point x=0.3 m and y=0.3m.

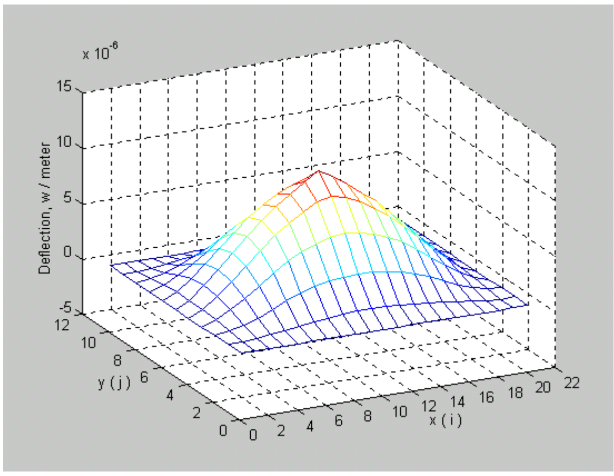

Fig. 3: Response of the plate to a finite duration step, t= 0.327seconds

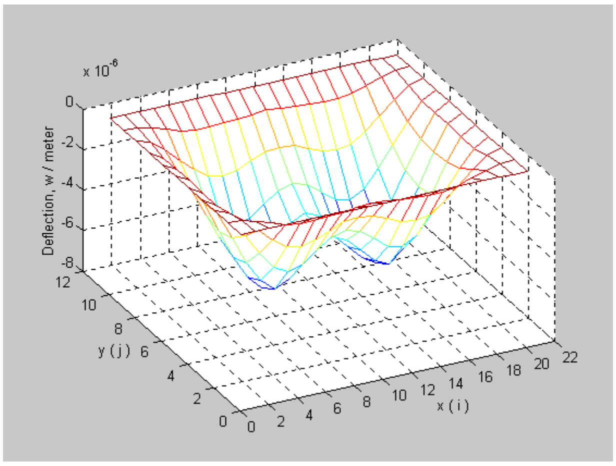

Fig. 4: Response of the plate to a finite duration step, t= 0.641 seconds

7 Algorithm Validation

The performance of the developed algorithm in characterising the dynamic behaviour of the system is assessed in relation to the frequency parameters, $\lambda = \omega a^2 \left(\sqrt{\rho / D}\right)$, where ω is the frequency in radian/sec, previously reported using other methods [4, 7]. The frequency parameters as a function of *a/b* ratio for the lowest three modes, as previously reported [7] are listed in Table 2 and these are treated as true values for comparison purposes.

The simulation was carried out by varying the width to length ratio *a/b* from 0.9 to 0.2. A comparison between the previously reported results with the simulation results is shown in Table 2.

It is noted that the modes obtained using the simulation algorithm are very close to the true value with small percentage error. Therefore, it can be concluded that the simulation algorithm characterises the behaviour of the rectangular plate with various width to length ratios reasonably accurate and to acceptable levels.

Table 2: Modes of vibration of the system with various *a/b* ratios.

a/b	Mode 1			Mode 2			Mode 3		
	True	Sim.	Error (%)	True	Sim.	Error (%)	True	Sim.	Error (%)
0.9	32.663	36.437	11.554	109.308	109.312	0.003	129.530	129.555	0.019
0.8	29.888	31.988	7.026	89.255	89.569	0.352	127.504	124.757	2.154
0.7	27.642	29.389	6.320	71.769	71.025	1.036	125.798	120.010	4.601
0.6	25.889	26.990	4.252	56.914	57.580	1.170	119.351	113.360	5.019
0.5	24.577	24.991	1.684	44.769	43.735	2.309	87.252	82.500	5.446
0.4	23.644	23.192	1.911	35.417	34.387	2.908	61.495	57.580	6.366
0.3	23.017	21.593	6.186	28.858	28.350	1.759	42.253	38.686	8.442
0.2	22.633	20.400	9.862	24.868	25.192	1.302	29.968	32.590	8.749

8 Conclusion

An investigation into dynamic modelling and simulation of a flexible rectangular isotropic plate structure using finite difference methods has been presented. The stability of the algorithm has been discussed to ensure the convergence of the algorithm. The algorithm is implemented in Matlab environment, and it allows application and sensing of a disturbance signal at any mesh point on the plate. The results obtained reveal that the algorithm provides a reasonably accurate characteristic behaviour of the rectangular plate.

The simulation algorithm thus developed and validated forms a suitable test and verification platform in subsequent investigations for development of active vibration control strategies for flexible plate structures.

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