Attractor Reconstruction for Chaotic Epileptic Signals

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Abstract: - Some techniques for reconstructing attractors from time series are shown in this paper. First, the time delay for obtaining the extra coordinates used for the reconstruction is selected using the Average Mutual Information (AMI); second, the embedding dimension of the attractor is obtained determining the False Nearest Neighbours (FNN). An important feature of this reconstruction algorithm is that it only needs one state variable measurement for reconstructing the attractor. Finally, some attractor reconstructions are shown for two different cases: Chua's circuit and certain epileptic data measured during a seizure, which shows random-like behaviour.

Key-Words: - Chaos, Biomedical Processing, Attractor Reconstruction, Embedding Dimension, Mutual Information

1 Introduction

Chaotic dynamics have been widely studied in several disciplines during the last decades. A chaotic signal is generated by a deterministic dynamical system, but because of its sensitivity to initial conditions, it is long-term unpredictable. Some methods have been developed for situations where the system dynamics are known, given by a mathematical model. However, in a real situation one normally has only a time series, obtained from the measurement of a typical system variable. In this case, obtaining fundamental invariants of the system, like the local dimension of the steady state dynamics and the reconstruction of the corresponding attractor, is very important but it is not an easy problem. There are some techniques developed in [1], [2], [4] and [6] for the reconstruction of attractors. These techniques are used to choose the parameters of the reconstruction.

In this paper we have applied some of these techniques to study the behaviour from Chua's circuit and some epileptic signals in steady-state. The main objective is to analyse the complex dynamics of these two cases to help in the selection of a suitable control strategy. This analysis relies on the availability of only a time series, obtained from a measured variable of the system. So, given a time series obtained from some measured variables of Chua's circuit or an electroencephalograph, possibly showing an irregular behaviour, the objective is to determine the stochastic or deterministic nature of the system dynamics, as well as some fundamental parameters. In particular, we are interested in calculating the global and local dimension of the system dynamics, and to reconstruct the corresponding attractor.

The paper is organized as follows. In section 2 some basic concepts on dynamical systems are reviewed. Section 3 gives a description of the technique used for the attractor reconstruction. The Chua's Circuit is described and a reconstructed attractor for this system is shown in Section 4. In section 5 some basic concepts of epilepsy are reviewed and a reconstructed attractor for an epileptic signal is shown. Finally some concluding remarks are given.

2 Some Concepts on Dynamical Systems

Consider a system given by

$$\dot{x} = f(x,t,\mu) \qquad (1)$$

where $x \in \Re^n$ is the state, $f : \Re^n \rightarrow \Re^n$ is a smooth vector field, and μ denotes the system parameters. The solution of (1) is some vector function $x = x(t)$ that describes the trajectories in the state space constructed with its coordinates. Depending on the parameter values the system may display different steady states, ranging from equilibrium points to chaotic attractors.

Definition 1 (Chaotic Attractor) [9]. Consider a C^r ($r \geq 1$) autonomous vector field on $\Re^n$, defining a system like (1). Denote the flow generated by (1) as $\phi(t,x)$, and assume that $\Lambda \subset \Re^n$ is a compact set, invariant under $\phi(t,x)$. Then Λ is said to be chaotic if

i. Sensitive dependence on initial conditions. There exists $\varepsilon > 0$ such that, for any $x \in \Lambda$ and any neighbourhood U of x, there exist $y \in U$ and $t > 0$ such that $|\phi(t,x) - \phi(t,y)| > \varepsilon$.

ii. Topological transitivity. For any two open sets $U,V \subset \Lambda$, there exists $t \in \Re$ such that $\phi(t,U) \cap V \neq \varnothing$.

Systems showing this behaviour are called chaotic. In recent years, many techniques have been developed for the analysis of the dynamics of this kind of systems, in the next section some of them are described, but before we will give some definitions and a useful theorem.

Definition 2. (Capacity Dimension) [8]. Let A be a bounded subset of $\hat{A}^n$. Let $N_\delta(A)$ the smallest number of sets of maximum diameter δ that cover A. Then, the capacity dimension is defined, if it exists, by:

$$dim_K(A) = \lim_{\delta \to 0} \frac{\log N_\delta(A)}{|\log(1/\delta)|} \qquad (2)$$

Typically, this quantity is not an integer number for a chaotic attractor A. When this situation occurs it is said that A is a *fractal* set.

Theorem 1. (Embedding Theorem) [3]. Let A be a compact and E a subspace of finite dimension such that

$$dimE > 2\, dim_K(A) + 1 \qquad (3)$$

where dim_K is the capacity dimension. Then the set of projections π: $A \rightarrow E$, such that π is injective, is dense among all projections with respect to the norm operator topology.

Definition 3. (Embedding Dimension) [5]. The dimension $dimE = \boldsymbol{d_E}$ in (3) is called the Embedding dimension and it is the dimension for which the attractor is fully unfolded, i.e. the dimension in which two points far away each other in the original space are not projected near each other in the observation space.

Due to this theorem, it is possible to reconstruct the attractor in some previously determined embedding dimension. The problem here is to find this dimension from a time series. In the next sections some prescriptions for finding this dimension, and some other necessary parameters for the attractor reconstruction will be given.

3 Attractor Reconstruction

There are no analytical solutions to equations describing chaotic phenomena, even an approximate solution is not easy to find. Some analysis techniques for this kind of systems involve perturbation methods [8] for setting approximate solutions of (1). An important point here is that usually, it is possible to measure at least one of the variables involved in the time evolution of the system. There are some methods for analysing the chaotic phenomena by using time series. These methods are based on the embedding theorem for the reconstruction of the attractor, and some prescriptions have been proposed to calculate some important system parameters [4], [6]. Due to this theorem, it is possible to reconstruct the attractor if the embedding dimension is previously determined. Two problems arise here, the first one is how to find this dimension from a time series and the second one is how to determine the time delay factor (multiple of the sampling period). In what follows some prescriptions for solving this problems, and some other necessary parameters for the attractor reconstruction, will be given.

There are several procedures to reconstruct a chaotic attractor from discrete time measurements [2]. In general, the solution relies on choosing a suitable sampling period for the signal such that topological characteristics of the attractor can be reproduced. The attractor reconstruction is then accomplished by using time delay versions of a scalar quantity *s(t)*, observed from time t_0 to some final time, as coordinates for the state space. Let us define $x(n) = s(t_0 + n\Delta t)$, $n = 1,2,...$, for some initial time t_0 and a sampling interval Δt. From the observations, *d*-dimensional vectors

$$y(n) = [x(n), x(n + T), \dots , x(n + (d-1)T)] \qquad (4)$$

are used to trace out the orbit of the system. Thus, the problem arising here is what values of time delay factor T and the embedding dimension $d = d_E$ to choose. The next two subsections deal with those problems.

3.1 Average Mutual Information (AMI)

Before formally describing the idea of mutual information, we have to consider some restrictions. First, if the value of T is too short, coordinates $x(n)$ and $x(n+T)$ would not be independent enough. And second, if T is too large, every connection between these coordinates would be numerically subject to be random like one with respect to the other. In [4] it is suggested to base the selection of T in a fundamental aspect of chaos: the information generation. The average mutual information concept is based on the Shannon's idea for information. Let us consider two measurements a_i and b_j from sets $A = \{a_i\} = x(n)$ and $B= \{b_j\} = x(n+T)$ respectively. The mutual information between measurement a_i and measurement b_j is the quantity learned by measurement a_i about measurement b_j. In bits, this is given as follows:

$$log_2\left[\frac{P_{AB}(a_i,b_j)}{P_A(a_i)P_B(b_j)}\right] \qquad (5)$$

where $P_{AB}(a, b)$ is the joint probabilistic density for measurements in A and B. $P_A(a)$ and $P_B(b)$ are the individual probability densities for the measurements in A and B, respectively. The average of all these statistic information is called Average Mutual Information between A and B and it may be written as [2]:

$$I_{AB} = \sum_{a_i b_j} P_{AB}(a_i,b_j) log_2\left[\frac{P_{AB}(a_i,b_j)}{P_A(a_i)P_B(b_j)}\right] \qquad (6)$$

In terms of $x(n)$ and $x(n+T)$:

$$I(T) = \sum_{x(n),x(n+T)} P(x(n),x(n+T)) log_2\left[\frac{P(x(n),x(n+T))}{P(x(n))P(x(n+T))}\right] \qquad (7)$$

The prescription for determining if the values of $x(n)$ and $x(n+T)$ are independent enough such that we can use them to construct the vector ***y(n)*** is to take T where the first minimum of the ***I(T)*** occurs.

3.2 Global False Nearest Neighbours

Theorem 1 tells us that if the attractor dimension defined by the orbits associated to (1) is $dim_K(A)$, then the attractor will unfold in an integer embedding dimension $d_E > 2dim_K(A)+1$ as a maximum value. In an embedding dimension that is too small to unfold the attractor, not all the points that lie close to one another will be neighbours because of the dynamics, some of them will actually be far from each other and appear as neighbours, because the geometric structure of the attractor has been projected down onto a smaller space. In a *d*-dimensional space and denoting the *r*th nearest neighbour of $\boldsymbol{y}(n)$ by $\boldsymbol{y}_r(n)$, the square of the Euclidean distance between these two points is given by:

$$R_d^2(n,r) = \sum_{k=0}^{d-1}\left[x(n+kT) - x_r(n+kT)\right]^2 \qquad (8)$$

In a $(d+1)$-dimensional space we add $x(n+dT)$ as a coordinate to each of the vectors ***y(n)***. Again, the squared Euclidean distance in this dimension between both points is:

$$R_{d+1}^2(n,r) = R_d^2(n,r) + \left[x(n+dT) - x_r(n+dT)\right]^2 \qquad (9)$$

A criterion to find false neighbours may be the increase in distance between ***y(n)*** and $\boldsymbol{y}_r(n)$ when going from dimension d to d+1. The increase of distance can be stated as:

$$\sqrt{\frac{R_{d+1}^2(n,r)-R_d^2(n,r)}{R_d^2(n,r)}}=\frac{|x(n+dT)-x_r(n+dT)|}{R_d(n,r)}>R_{TH} \quad (10)$$

Where R_{TH} is some threshold. For our case we took $R_{TH} \geq 15$, this result was founded experimentally.

4 Reconstruction of Chua´s Attractor

Chua's circuit is a well-known oscillator that exhibits bifurcations and chaotic behaviour. We have chosen this system in order to validate our reconstructions before applying the method to other signals. The circuit contains three linear energy-storage elements (an inductor and two capacitors), a linear resistor and a nonlinear resistor N_R [7].

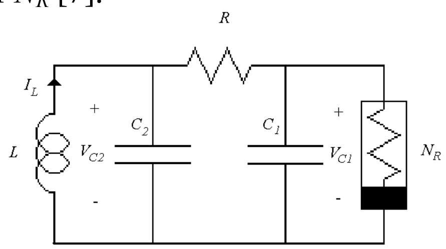

Fig. 1 Chua´s Circuit

$$\frac{dv_{C1}}{dt}=\frac{1}{C_1}\left[G(v_{C2}-v_{C1})-f(v_{C1})\right]$$
$$\frac{dv_{C2}}{dt}=\frac{1}{C_2}\left[G(v_{C1}-v_{C2})+i_3\right] \quad (11)$$
$$\frac{di_3}{dt}=-\frac{1}{L}\left[v_{C2}\right]$$

where $G = 1 / R$ and $f(v_{C1})$ is a piecewise-linear function defined by

$$f(v_{c_1})=G_b v_{c_1}+\frac{1}{2}(G_a-G_b)\left[\left|v_{c_1}+B_p\right|-\left|v_{c_1}-B_p\right|\right] \quad (12)$$

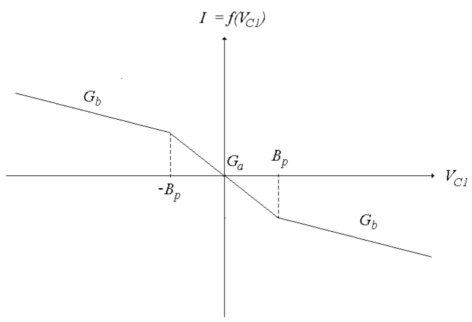

Fig. 2 Three-segment piecewise linear *v-i* characteristic of the nonlinear resistor

For the so-called Double Scroll Attractor, the circuit was constructed using the following values for the parameters: C_1 = 56 nF, C_2 = 5.6 nF, L = 10 mH, G_a = -409.0909 µs, G_b = -757.5757 µs, B_p = 1 V, R = 1820 Ω resistor. The following results are the computations of the Global Embedding Dimension using the Global False Nearest Neighbours and the time delay T using the Average Mutual Information. We have measured the voltage in C1 using an Analog Devices RTI-820 acquisition board on a PC, obtaining the time series for V_{C1}. The signals have been sampled every 0.001 seconds, and we have used only the first 10,000 samples for each reconstruction, without transients.

The AMI is calculated between the original signal and its delay version for each *time delay factor T* (multiples of the sampling period), the T for the double scroll attractor is set in the first minimum (Figure 3) which turns out to be T=8.

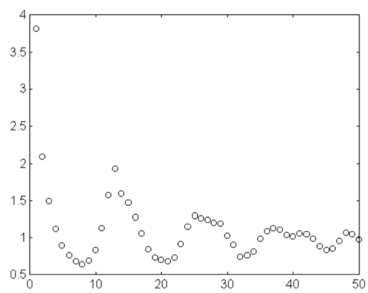

Fig. 3 Average Mutual Information for a Chaotic Attractor

Now the percentage of FNN is calculated, from figure 4 it is observed that in d=3 there are no false neighbours so the embedding dimension is fixed to d_E=3.

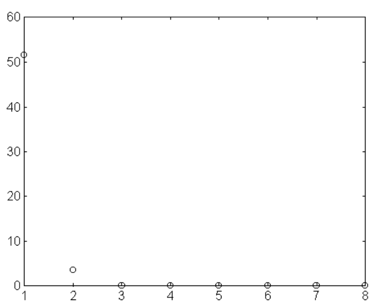

Fig. 4 Percentage of False Nearest Neighbours

Figure 5 shows the original attractor, and finally figure 6 shows the reconstructed attractor for the obtained parameters.

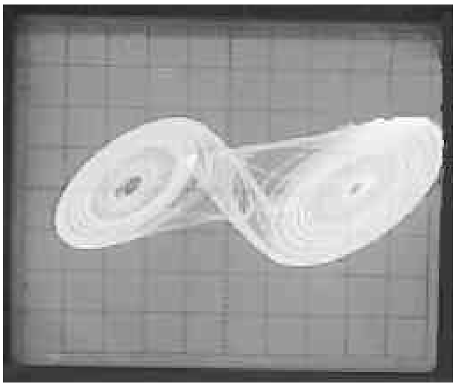

Fig. 5 Two-dimensional projection of the attractor

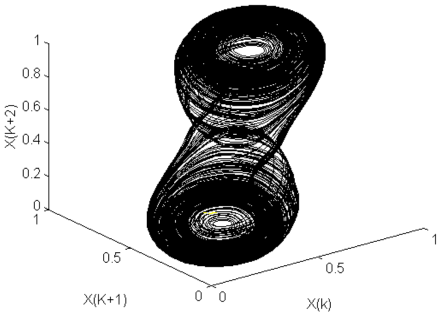

Fig. 6 Chaotic attractor reconstructed using v_{C1}

5 Reconstruction of Epileptic Attractors

Epilepsy is a generic term that involves a group of illness characterized by crises. This is a clinical condition, which is revealed by recurrence of spontaneous crises related with an abnormal electrical discharge in the brain.

Epileptic crises are sudden involuntary alterations, which are related with changes in the motor, sensitive and conscious activities with a stereotyped pattern. They are not caused by alcohol, stress, fever or any other acute problem.

Epileptic crises can be originated in neurons capable of producing electrical discharges like the ones in the hippocampus or in the neocortex. The epileptic discharge follows a fail in the inhibitory mechanisms, particularly, the Gamma-Aminobutiric Acid (GABA). The electroencephalograms (EEG) obtained during an epileptic seizure show an apparently random-like behaviour.

We use some measurements from an EEG during an epileptic seizure as an example of using these techniques without having prior knowledge about the dynamics of the system we are dealing with.

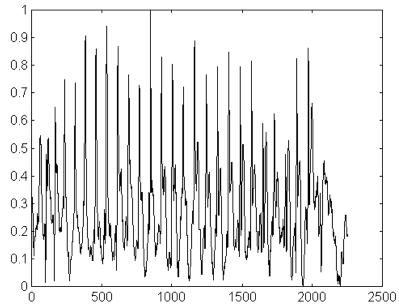

Fig. 7 Measured signal during an epileptic seizure

Figure 7 shows a measured signal during an epileptic seizure, the AMI is shown in figure 8.

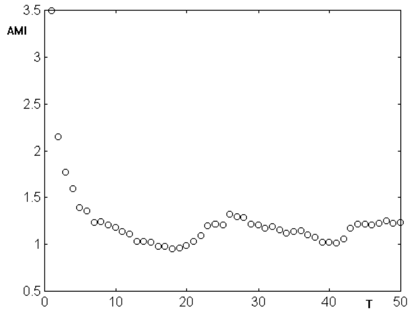

Fig. 8 Average Mutual Information for an epileptic signal

Using the value T=7 from the computed AMI, the reconstructed attractor in d_E=3 is shown in figure 9.

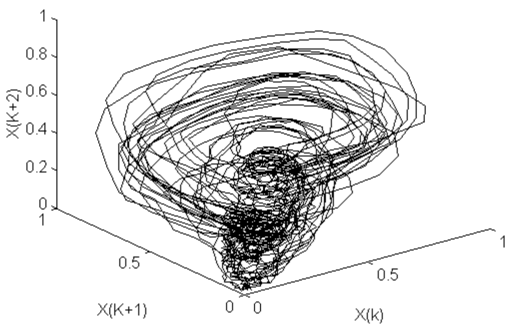

Fig. 9 Attractor reconstruction from an epileptic measured signal

6 Concluding Remarks

In this work some reconstructions for the different attractors have been shown. An important feature of these algorithms is that we only need one state variable time series for reconstructing the attractor. The computation of the FNN and the AMI for determining the embedding dimension and the time delay is an important tool for knowing some dynamical properties. An advantage of this kind of techniques is that we do not need to have the system's mathematical model, it is enough to have a time series for the reconstruction.

The stochastic or deterministic nature of the measured signals is determined using both, AMI and FNN techniques. For a stochastic signal the AMI plot does not have a strict minima and the FNN plot never reaches the 0% value.

These analysis techniques are helpful for better understanding of different nonlinear phenomena, using the algorithms presented in this paper it is possible to know some important dynamic properties of this phenomena by reconstructing the attractor.

The preliminary results for analysing epileptic signals will be used by our group, with several further analysis, for analytically classifying different kind of seizures.

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