Wavelet-Based Solution to Elliptic Two-Point Boundary Value Problems with Non-Periodic Boundary Conditions*

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Abstract. The Wavelet solution for boundary-value problems is relatively new and has been mainly restricted to the solutions in data compression, image processing and recently to the solution of differential equations with periodic boundary conditions. This paper is concerned with the wavelet-based Galerkin solution to two-point boundary-value problems involving elliptical problems with non-periodic boundary conditions. The wavelet method can offer several advantages in solving the boundary-value problems than the traditional methods such as Fourier series, Finite Differences and Finite Elements by reducing the computational time near singularities because of its multi-resolution character. In order to demonstrate the wavelet technique to non-periodic boundary value problems, we extended our prior research of solution of parabolic problems to two elliptical problems, the one-dimensional Helmholtz Equation and a two-dimensional elliptical equation. The results of the wavelet solutions are examined and they are found to compare favorably to the exact solutions. This paper on the whole indicates that the wavelet technique is a strong contender for solving two point boundary value problems with non-periodic conditions involving elliptical problems.

0. Introduction.

The term "**wavelet**" denotes a function, defined on domain R, which, when subjected to fundamental operations of shifts (i.e., translation by integers) and dyadic (two fold) dilation (act of expanding), yields an orthogonal basis of L^2R. That is, the functions

$$\phi_{m,k}(x) = 2^{m/2}\phi(2^m x - k)$$

form a complete orthogonal system for L^2R with the usual inner product and also has compact support, where ϕ is the fundamental scale function. The wavelet expansion for a function f takes the form

$$f(x) = \sum_k C_{m,k}\phi_{m,k}(x)$$

The coefficients are defined as

$$C_{m,k} = \int f(x)\phi_{m,k}(x)dx$$

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*Research was supported by the National Science Foundation under grant # 963375.
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Now, from the definition of wavelets we need to know what a scale function is? The scale function is given as the solution of the (recursive) dilation or scale equation:

$$\phi(x) = \sum_{k=0}^{N-1} a_k \phi(2x-k)$$

The '**N**' here represents the number of filter coefficients and is named after the inventor, Ingrid Daubechies. Throughout we will use the Daubechies scale functions D6 (N=6). The filter coefficients are the ones, which define the scaling function. For more about Daubechies wavelets and their properties consult Drake [7].

With the brief introduction to basics of the wavelets, we now briefly describe their application areas. In the areas of time series analysis, matrix compression, and approximation theory, wavelets have carved out a practical niche. In the solution of differential equations, however, wavelets have not, thus far, been able to replace other more traditional techniques such as Fourier analysis and Finite differences. When we talk about PDEs, wavelet basis functions have many properties that make them desirable as a basis for a Galerkin approach: they are orthonormal, with compact support, and their connection coefficients (that is, integrals of products of basis functions, with or without derivatives) can be computed [1][2]. Even though some work has been done on applying Wavelet-Galerkin method for the solution of time-independent differential equations with periodic conditions, little if any, work has been done to solve the differential equations with non-periodic boundary conditions. We had applied this technique successfully to parabolic equations [9] with non-periodic boundary conditions. We now extend that work to one-dimensional Helmholtz Equation and a two-dimensional two-point boundary value elliptical equation, for investigation. In the first case we solve a problem with the Helmholtz Equation and in the other case we investigate the solution to a two-point boundary value problem involving elliptic equations. This research serves as a basis for future solution of non-linear and singular problems where obtaining the exact solution is not possible.

1. Problem 1.

In the first case, we will consider a problem involving an elliptic 1-D partial differential equation, Helmholtz Equation. The equation is given by

$$U_{xx} + \alpha U = f \quad , \qquad 0 \le x \le 1 \qquad \ldots\ldots\ldots\ldots 1.1$$

Considering $f = 1$, we get

$$U_{xx} + \alpha U = 1 \qquad \ldots\ldots\ldots\ldots 1.2$$

<u>**BOUNDARY CONDITIONS:**</u>

$$U(0) = 0 \qquad \ldots\ldots\ldots\ldots 1.3a$$

$$U(1) = 1 \qquad \ldots\ldots\ldots\ldots 1.3b$$

Solution:

Although exact solution exists in literature, our purpose is to find the approximate solution with wavelets in order to establish the wavelet theory for the solution of the elliptic boundary value problems with non-periodic boundary conditions. Once established this will pave the way for solving non-linear and singular problems.

Wavelet Based Galerkin's Solution

We attempt to apply the Galerkin's approach to the wavelet solution of the problem, which involves assuming an approximate solution in terms of orthogonal basis and then making the differential and boundary residuals zero. Here we shall transform the problem in wavelet space as follows:

Changing of Variables:

To transform back and forth from the physical space to the wavelet space we make the following substitution:

Let $y=2^m x$ when $x = 0, y = 0; x = 1, y = 2^m = N\text{-}1$

$$\Rightarrow \frac{\partial z}{\partial x} y = 2^m$$

$$\Rightarrow \frac{\partial U}{\partial x} = \frac{\partial U}{\partial y}\cdot\frac{\partial z}{\partial x} = \frac{\partial U}{\partial y}2^m$$

$$\Rightarrow \frac{\partial^2 U}{\partial x^2} = 2^{2m}\frac{\partial^2 U}{\partial y^2}$$

where N is the Daubechies number[1].

Substituting these into our main equation (1.2) results in,

$$2^{2m}U_{yy} + \alpha U = 1 \qquad \text{. . . . . 1.4}$$

Let $U(y) = \sum_k W_k \phi_k(y), \qquad \forall k \in [-(N-1),(N-1)],$ 1.5

be the wavelet approximate solution.

Substituting the above assumption (1.5) in equation 1.4, we get the differential residual as:

$$\Rightarrow Q\sum_k W_k\phi_k(y) + \sum_k W_k\phi_k^{"}(y) - 1 \neq 0 \qquad \text{where } Q = \frac{\alpha}{2^{2m}} \text{ . . . . . . 1.6}$$

and the boundary residuals as,

$$\sum_k W_k\phi_k(0) \neq 0, \qquad \text{. . . . . 1.7a}$$

$$\sum_k W_k\phi_k(N-1) \neq 1 \qquad \forall k \in [-(N-1),(N-1)] \quad \text{. . . . 1.7b}$$

As the Galerkin method tries to make the differential residuals zero, we orthogonalize residuals with basis functions, i.e., we multiply (1.6) with $\phi_j(y)$ and integrate from [0, N-1], we get the following differential equations:

$$\Rightarrow Q\sum_k W_k\Omega_{jk}^{00} + \sum_k W_k\Omega_{jk}^{02} - 1 = 0 \qquad \text{. . . . . 1.8}$$

where $Q = \frac{\alpha}{2^{2m}}$, $\forall j,k \in [-(N-1),(N-1)]$ and

Connection coefficients [2], $\Omega_{ij}^{d_1 d_2} = \int_0^{N-1} \phi_i^{d_1}(y)\phi_j^{d_2}(y)dy$

Now, in a similar fashion, Galerkin approximation attempts to find boundary equations by orthogonalizing the respective boundary residuals with respect to basis functions. Thus the resulting boundary equations are:

$$\sum_k W_k \phi_k(0) = 0, \qquad \dots\dots\dots\dots 1.9a$$

$$\sum_k W_k \phi_k(N-1) = 1 \qquad \forall k \in [-(N-1),(N-1)] \qquad \dots\dots\dots 1.9b$$

So, here, we have got a set of nine differential equations (1.8) and two boundary equations (1.9) forming a differential-algebraic system. Substituting the two algebraic equations in place of top and bottom differential equations, we get a 9x9 differential algebraic system, which can be represented as,

$$AY + BY = D \qquad \dots\dots\dots\dots 1.10$$

where $A = \begin{bmatrix} zeros(1,9) \\ Q\Omega_{ij}^{00} \\ zeros(1,9) \end{bmatrix} [-(N-2) \le i, j \le (N-2)]$

$$B = \begin{bmatrix} BoundaryCondition1 from 1.9a \\ \Omega_{ij}^{02} \\ BoundaryCondition2 from 1.9b \end{bmatrix} [-(N-2) \le i, j \le (N-2)]$$

$D = [0;\ 0;\ 0;\ 0;\ 0;\ 0;\ 0;\ 0;\ 1]$ is the constant matrix involving the right-hand terms in boundary equations,

Y: is the 9x1 vector of unknowns, i.e. W_k, $\forall k \in [-(N-2),(N-2)]$

Once the above equation (1.10) is solved, we get a set of W_k's, which when substituted in equation (1.5), gives us the approximate solution. The approximate solution for Equation (1.1) is computed independently and plotted using MATLAB, a technical computing software package.

Exact Solution:

The exact solution of Equation 1.1 is:

$$U = C_1 e^{i\sqrt{\alpha}x} + C_2 e^{-i\sqrt{\alpha}x} + \frac{1}{\alpha} \qquad \dots\dots\dots\dots 1.11$$

where $$C_1 = -\frac{1}{\alpha} - \frac{\alpha - 1 + e^{i\sqrt{\alpha}}}{\alpha\left[e^{-i\sqrt{\alpha}} - e^{i\sqrt{\alpha}}\right]}$$

and $$C_2 = \frac{\alpha - 1 + e^{i\sqrt{\alpha}}}{\alpha\left[e^{-i\sqrt{\alpha}} - e^{i\sqrt{\alpha}}\right]}$$

Results and Discussion:

The plot for the approximate solution is given in Figure 1.1a. The exact solution for Helmholtz equation has been plotted (Figure 1.1b). The resulting matches between the graphs indicate that the wavelet method is a viable alternative for solving elliptical two-point boundary value problems with non-periodic boundary conditions.

2. Problem 2.

In the second case, we consider a two-point two-dimensional elliptical boundary value problem:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+fU=g \qquad 0\le x\le 1, 0\le y\le 1 \qquad \text{. . . . . 2.1}$$

where

f = -2 and g = 0

BOUNDARY CONDITIONS:

U (x, 0) = 0 *U (x, 1) = 1;*

U (0, y) = 0 *U (0, y) = 1;* 2.2

Solution:

Although exact solution exists in literature, our purpose is to find the approximate solution with wavelets in order to establish the wavelet theory for the solution of the boundary value problems with non-periodic boundary conditions. Once established this will pave the way for solving non-linear and singular problems.

Wavelet Based Galerkin's Solution

We attempt to apply the Galerkin's approach to the wavelet solution of the problem, which involves assuming an approximate solution in terms of orthogonal basis and then making the differential and boundary residuals zero. Here we shall transform the problem in wavelet space as follows:

Changing of Variables:

To transform back and forth from the physical space to the wavelet space we make the following substitution:

Let $z=2^m x$ when $x = 0$, $z = 0$; $x = 1$, $z = 2^m = N-1$

$$\Rightarrow \frac{\partial z}{\partial x}=2^m$$

$$\Rightarrow \frac{\partial U}{\partial x}=\frac{\partial U}{\partial z}\cdot\frac{\partial z}{\partial x}=\frac{\partial U}{\partial z}2^m$$

$$\Rightarrow \frac{\partial^2 U}{\partial x^2}=2^{2m}\frac{\partial^2 U}{\partial z^2}$$

Substituting these into our main equations (2.1) results in

$$2^{2m}\frac{\partial^2 U}{\partial z^2}+\frac{\partial^2 U}{\partial y^2}-2U=0 \qquad \text{. . . . . 2.3}$$

Let $U(z,y)=\sum_k W_k(y)\phi_k(z), \forall k\in[-(N-1),(N-1)]$ be the wavelet approximate solution.

Substituting the above assumption in equation (2.3), we get the differential residual as:

$$2^{2m}\sum_k W_k(y)\phi_k^{''}(z)+\sum_k W_k^{''}(y)\phi_k(z)-2\sum_k W_k(y)\phi_k(z)\neq 0$$

To make the differential residual zero, we orthogonalize the above equation with basis functions, i.e., we multiply with $\phi_j(z)$ and integrate from [0, *N*-1], we obtain the differential equation as:

$$\Rightarrow 2^{2m}\sum_k W_k(y)\Omega_{jk}^{02}+\sum_k W_k^{''}(y)\Omega_{jk}^{00}-2\sum_k W_k(y)\Omega_{jk}^{00}=0 \qquad \text{. . . . . 2.4}$$

where $\forall j,k \in [-(N-1),(N-1)]$, Connection coefficients [2], $\Omega_{ij}^{d_1 d_2} = \int_0^{N-1} \phi_i^{d_1}(y)\phi_j^{d_2}(y)dy$

Now, in a similar fashion, orthogonalizing the respective boundary residuals yields us the following boundary equations,

$$\sum_k W_k(y)\phi_k(0) = 0, \quad \sum_k W_k(y)\phi_k(N-1) = 1 \qquad \text{2.5a}$$

$$\sum_k W_k(0)\phi_k(Z) = 0, \quad \sum_k W_k(N-1)\phi_k(Z) = 1 \qquad \text{2.5b}$$

So, here, we have got a set of nine differential equations (2.4) and two sets of algebraic equations (boundary conditions) (2.5) defining *x* and *y* at both the ends. Here, we replace the top and bottom differential equations with boundary equations defining *x*, thus forming a square differential algebraic system, which can be represented as:

$$\alpha Y + \beta Y'' - \gamma Y = D \qquad \text{2.6}$$

where $\alpha : \begin{bmatrix} BoundaryCondition1 from 2.5a \\ 2^{2m}\Omega_{ij}^{02} \\ BoundaryCondition2 from 2.5a \end{bmatrix} - (N-2) \le i,j \le (N-2)$

$\beta : \begin{bmatrix} zeros(1,9) \\ 2^{2m}\Omega_{ij}^{02} \\ zeros(1,9) \end{bmatrix} - (N-2) \le i,j \le (N-2)$

$\gamma : \begin{bmatrix} zeros(1,9) \\ 2\Omega_{ij}^{02} \\ zeros(1,9) \end{bmatrix} - (N-2) \le i,j \le (N-2)$

$D : [0;\ 0;\ 0;\ 0;\ 0;\ 0;\ 0;\ 0;\ 1]$ is the constant matrix involving the constants in the boundary equations.

Y is the 9x1 vector of the unknowns, i.e., W_k, $\forall k \in [-(N-2),(N-2)]$

Central Differencing Technique:

To solve the above-mentioned differential algebraic system, Equation 2.6, we apply Central difference technique.

$$\Rightarrow (\alpha - \gamma)Y + \beta Y'' = D$$

$$\Rightarrow \beta Y'' = D + (\gamma - \alpha)Y$$

$$\Rightarrow \beta\left[\frac{Y_{h+1}^k - 2Y_h^k + Y_{h-1}^k}{\Delta h^2}\right] = D + (\gamma - \alpha)Y_h^k$$

$$\Rightarrow Y_h^k\left[-2\beta - (\gamma - \alpha)\Delta h^2\right] = D\Delta h^2 - \beta Y_{h+1}^k - \beta Y_{h-1}^k \qquad \text{2.7}$$

$$\Rightarrow \beta Y_{h+1}^k - AY_h^k + \beta Y_{h-1}^k = D\Delta h^2$$

where

$A = (\gamma - \alpha)\Delta h^2 + 2\beta$ and

h is the uniform spacing Δy for y variable.

On expanding equation (2.7), for different values of h, ($0<h<1$), at the interval of 0.1, we get nine equations in which we know the values of y_0 and y_{10}, which are nothing but the boundary conditions of y. So, when we represent all the nine equations in matrix form, we get,

$$MZ = d \qquad \qquad . . . . 2.8$$

where

M	is the coefficient matrix of y [Appendix-I].
Z	is the unknown vector [Appendix-I].
d	is the matrix containing the right hand side of the equations [Appendix-I].

Results and Discussion:

The approximate solution, Equation (2.8), is computed independently and plotted using MATLAB, a technical computing software package. The numerical solution is plotted in MATLAB using the PDE Tool box and is as shown in the Figure 2.1a, 2.2a. Figure 2.1a shows the flat color plot where the color scale indicates the height, while Figure 2.2a shows the 3-D plot. The plots for the approximate solution are shown in Figure 2.1b and 2.2b. The resulting matches between the graphs indicate that the wavelet method is a viable alternative for solving elliptical non-periodic two-point boundary value problems.

3. Conclusion.

From the above cases, the wavelet method has been shown to be a powerful tool for the study of solution of elliptic partial differential equations with non-periodic boundary conditions. The approximate solutions obtained using Daubechies Wavelet Coefficients (N = 6) have been compared with the available exact solutions and found to match very well. In some cases, the wavelet solutions have been found to converge much faster than the exact solution. Although wavelet approximate solutions in general require slightly more computational effort than the exact solutions, the gains in accuracy, particularly with the higher order wavelets, far outweigh the increase in cost. Furthermore wavelets have the capabilities of representing solutions at different levels of resolution, which makes then particularly useful for developing hierarchical solutions to engineering problems.

4. References.

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Appendix-I

```
M = [
   -A   beta zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9);
   beta -A  beta  zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9);
   zeros(9,9) beta -A  beta zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9);
   zeros(9,9) zeros(9,9) beta -A  beta zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9);
   zeros(9,9) zeros(9,9) zeros(9,9) beta -A  beta zeros(9,9) zeros(9,9) zeros(9,9);
   zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) beta -A  beta zeros(9,9) zeros(9,9);
   zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) beta -A  beta zeros(9,9);
   zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) beta -A  beta;
   zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) zeros(9,9) beta -A;
];
```

Z = [y_1; y_2; y_3; y_4; y_5; y_6; y_7; y_8; y_9], where y_i = [W_k] $\forall k \in [-(N-2),(N-2)]$

d = [D-(beta*y_0);D;D;D;D;D;D;D;D-(beta*y_{10})];

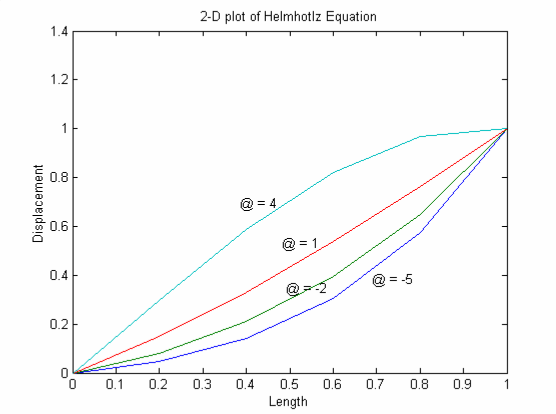

Figure 1.1a
Exact Solution for Helmholtz Equation

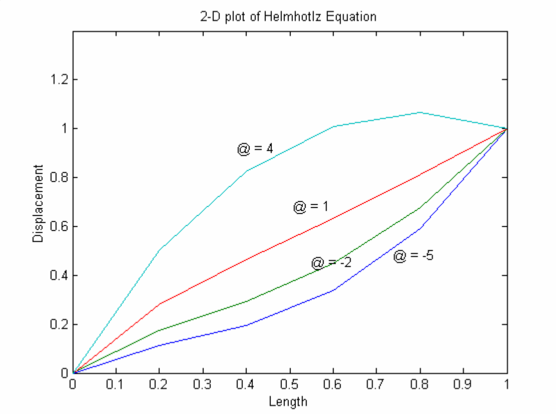

Figure 1.1b
Approximate Solution for Helmholtz Equation

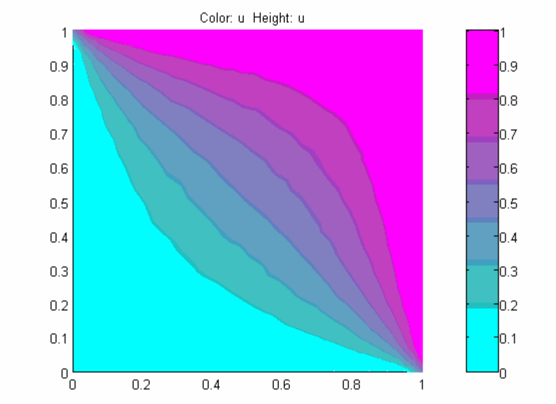

Figure 2.1a
Color plot
for Exact Solution of Elliptic problem

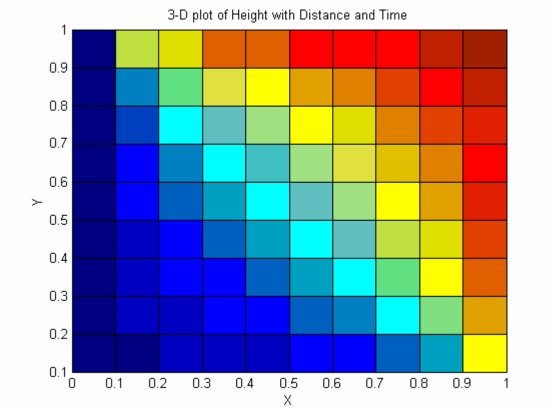

Figure 2.1b
Color plot
for Approximate Solution of Elliptic problem

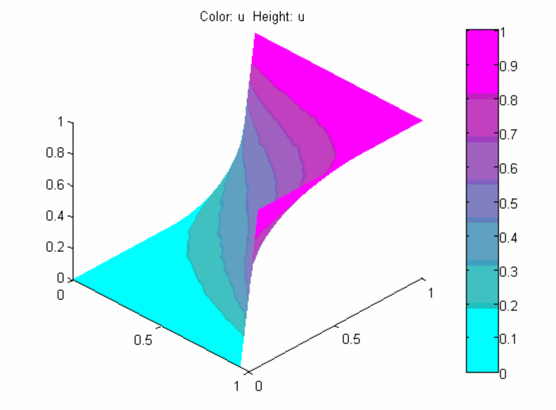

Figure 2.2a
3-D plot for Exact Solution of Elliptic problem

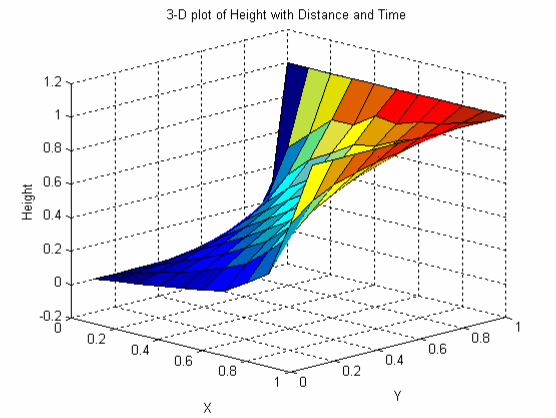

Figure 2.2b
3-D plot for Approximate Solution of Elliptic problem