CRAZY ROLLERS

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Abstract: The cars we are familiar with have circular wheels. Here, we consider cars whose wheels are not circular and nevertheless these cars move smoothly.
A mathematical development characterizing the condition for a smooth motion of exotic wheels is presented. This mathematical development has been tested on different simulation environments. Finally, we introduce a specific virtual environment developed to support an easy presentation of mathematical concepts related with this situation.

Key-Words: rollers, instantaneous center of rotation, cycloid, double cone, simulation, virtual environment, OpenGL

1 Introduction

Usually, cars have circular wheels and run on flat roads. Is it possible for cars to have wheels with a more or less exotic geometrical shape and provide a smooth motion? Does smooth motion depend only on the geometry of the wheel or also on the geometry of the track? To obtain a smooth motion, can the geometries of the wheel and the track be independent?
Can you imagine how the motion of a circular wheel would be along one of these tracks? Would the center still move along a straight line? Would the motion be smooth or jerky?
If we observe attentively the motion of a common wheel, we conclude that its center describes a rectilinear trajectory. Is this the secret of motion without jerks?
If the track is wavy or saw-edged or has any other exotic shape, the shape of the wheel must adapt itself to the shape of the track for the axial center of the wheel to move along a straight line and for the motion to be smooth. This is what happens with these small cars.
Given a specific track, the problem is: How to find the convenient shape of the wheels that are adapted to it? Is it possible to have several wheels for a specific track?
Based on the mathematical background, we developed a virtual environment to show how we could obtain a smooth motion with several geometries for the wheel. As it will be seen, the geometries of the wheel and the track are not independent.

2 The wheel, a great invention

The wheel is an outstanding invention in the history of Humanity. It occurred in the fourth millennium before Christ and is contemporary to the discovery of writing and of metals. It is an interesting coincidence that the discoveries of writing and of vehicles with wheels, although having nothing in common, are nevertheless coeval.
It is difficult to imagine our planet without the wheel. As Martin Garden states, it is hard to conceive an advanced civilization without wheels and avows his

surprise for the evolution of species not having chosen the wheel as a means of animal locomotion. After all, who has not heard of the fabulous snake in arc, which bites its tail and rolls away in a coil? Nothing prevents us from imagining that in other planets, rolling creatures do exist and roll wildly through the cosmos.

In the books of *Oz*, G. L. Frank gives life to the "rollers" which have four wheels instead of four legs and to the bird Ork which flies with the help of a propeller in its tail. H. G. Wells, in "*The war of the worlds*", creates a fiction over a developed civilization that does not utilize the wheel in its machinery.

Traditionally, it is admitted that the wheel was invented in the Mesopotamia on the account of the existence in this site of pictures of mechanisms with wheels dated from 3 000 a. C. and archeological vestiges of massive wheels from 2 700 a. C.. In the middle of the XX century, Russian archeologists found in the Caucasus mechanical models of cars with wheels, suggesting that the wheel may have appeared in southern Russia earlier than in Mesopotamia. The wheel may have been invented, independently, in several geographical places or may have been disseminated from a single center by cultural diffusion. It is hard to make a definitive judgment on this matter.

A moving wheel has paradoxal properties. The upper points have higher speed than those closer to the ground. A point on the wheel attains its maximum speed precisely when it is on the top, and minimum speed (zero) when it touches the ground. In the wheels of the train, whose borders come slightly below the rails, there are points moving backwards. One of the most subtle paradoxes of the wheel is known as *Aristotle's wheel*.

3 Aristotle's Wheel

Aristotle's wheel is the name under which a paradox of the wheel, originally presented in his *Mecanica*, is known. Galileo, Descartes, Fermat, among many others, contributed to clarify this question, which consists in the following.

Fig. 1 - Two solidly bound wheels

Lets us consider 2 solidly bound wheels (Fig. 1). When the larger wheel rolls from *A* to *B*, the outer ring of the smaller wheel rolls from *C* to *D*, along a parallel line. In each instant, a single point of the outer ring of the larger wheel touches the line *AB* and a single point of the smaller wheel touches *CD*. No point is excluded from any one of the circles so their length must be the same.

To solve the paradox, Galileo imagined what would happen if the two wheels were replaced by regular polygons, for instance two squares (Fig. 2). While the larger square performs a complete turn along *AB*, the sides of the smaller square coincide with *CD* on four line segments separated by three gaps. For pentagonal wheels, the smaller wheel would jump four gaps, and so on for polygons with larger number of sides, with the number of gaps increasing, but their length decreasing. In the limiting case, we would have a circular polygon with an infinite number of sides and infinitely many gaps, each of which infinitely small.

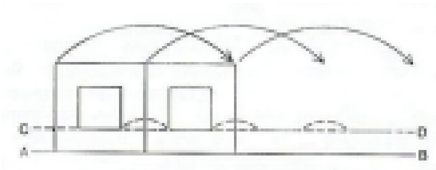
Fig. 2 – The Galileo’s gaps

But how to explain that, having infinitely many gaps, infinitely small, the smaller wheel travels a finite distance while the larger wheel moves uniformly? Galileo’s "gaps" are the problematic *infinitesimals* that so many problems would raise to the development of calculus.

4 Instantaneous center of rotation

The concept of *instantaneous center of rotation* will be useful. If a plane figure moves turning around a fixed point *O* called *center of rotation*, any point of the moving figure will describe an arc of circle. At each instant, the velocity *v* of the point *A* is tangent to the circle described by *A*, that is, it is perpendicular to the segment *OA* (Fig. 3).


Fig. 3 – Rotation of a figure around a fixed point

It may be mathematically shown that the plane motion of a rigid figure is, instantaneously, either a translation or a rotation around a point called *instantaneous center of rotation*. The point *O*, being

the *instantaneous center of rotation*, will remain instantaneously motionless (center of instantaneous standstill), its velocity vanishes, while the velocity of any other point A is perpendicular to the line segment *OA*.
The concept arises in a memory of *Chasles* of 1830, although this terminology only appears in later works of the author.
Let us consider a rigid figure *F* that moves in the plane rolling without sliding over a curve *L*, Fig. (4). We say that *F* rolls over *L* if, at each instant, there is a single point of contact of F with L and we say that it rolls without sliding if the point of contact has zero velocity, that is, if this point is the *instantaneous center of rotation* of *F*. Summarizing, we may say that if the figure *F* rolls without sliding over a fixed curve *L*, the point of contact is the *instantaneous center of rotation*. Denoting by *O* this point, an arbitrary point *A* of *F* possesses an instantaneous velocity perpendicular to the line segment *OA*.

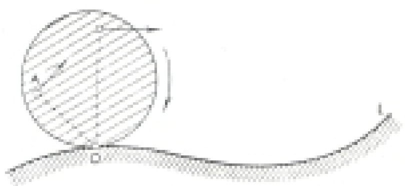

Fig. 4 – Rigid figure moving in the plane, rolling without sliding over a curve.

5 The smooth motion of exotic wheels: mathematical development

The solution to the problem of the exotic wheels requires solving a *differential equation*. Let us consider non-circular wheels which roll without sliding over a surface with non-rectilinear profile (Fig. 5). It is required that the trajectory of the wheel axis is a straight line, in order to ensure a smooth displacement.

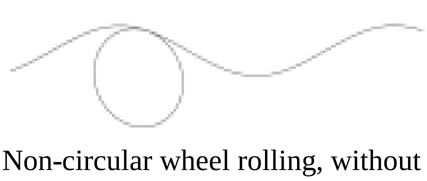

Fig. 5 – Non-circular wheel rolling, without sliding, over a surface with non-rectilinear profile

The point *C* of contact of the wheel with the road is the instantaneous center of rotation of the wheel. Instantaneously, the velocity of any point of the wheel is perpendicular to the line segment connecting it to *C*. Therefore, the velocity of *O* is perpendicular to *OC*. Thus, *OC* is, at each instant, perpendicular to the trajectory of *O*.

Let $y = f(x)$ be the equation of the profile of the road. If the motion of the vehicle is smooth, without jerks, then the motion of the point *O*, in turn of which the wheel rotates, relative to the vehicle, is a straight line. Assume the trajectory of *O* coincides with the *x* axis and is perpendicular to the *y* axis. The origin is arbitrary.
Beyond Cartesian coordinates we will use *polar coordinates*. In this system, the position of a point *P* is determined by the distance *r* of *P* to a fixed point *O* called *pole* and by the angle θ between the *radius vector OP* and the polar axis.
Let $g(\theta)$ be the equation of the wheel in polar coordinates. The pole is the point *O*. In each instant, the two curves are tangent at the point of contact. It may be shown that the differential equation

$$\frac{dy}{dx} = \frac{1}{r}\frac{dr}{d\theta}$$

contains the solution of the problem.
In the sequel, we will solve the problem of the exotic wheels in some concrete situations. We start with rectangular wheels turning around the center of the rectangle. We show that, in this case, the profile of the road is a sequence of catenary arcs.

5.1 Rectangular wheels and catenaries

What is the shape taken by a flexible and inextensible rope hanging from two fixed points? This problem, posed by Jacques Bernoulli around 1690 in *Acta Eruditorum*, was solved by the author, by his brother Johan, by Huygens and by Leibnitz. The answer is the *catenary*. The curve has a parallel to the *y* axis as symmetry axis and, at the point with *0* abscissa, is tangent to a parallel to the x axis.
The *catenary* is the plane curve whose Carthesian equation is

$$y = c/2\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) \quad \text{or} \quad y = c \cosh(x/c)$$

Let us consider the case of rectangular wheels, that is, the union of 4 line segments such that any two of them are either parallel or perpendicular. The catenary arises as the solution of the differential

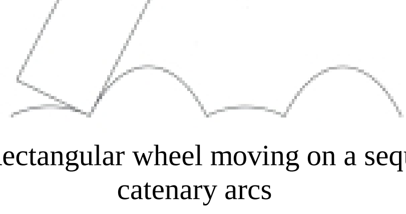

Fig. 6 – Rectangular wheel moving on a sequence of catenary arcs

equation giving the profile of the track. A series of repeated catenary arcs is the track of a regular polygonal wheel. If the wheel is a convex irregular polygon, the track is composed by differently shaped catenary arcs, one for each side.

5.2 Spiral wheels, saw-edged track

Let us consider a saw-edged track, or composed of line segments. Let us take the line segment whose Cartesian equation is

$$y = x, \quad a \leq x \leq y.$$

Then $dy/dx = 1$. In polar coordinates, the equation of the wheel adequate to this track is

$$\frac{1}{r}\frac{\mathrm{d}r}{\mathrm{d}\theta} = 1.$$

The general solution to this equation is

$$r = C\, e^{\theta},$$

where C is a constant. The constant may be determined, for instance, by the value taken by y when x takes the value a. This curve is called *logarithmic spiral*. The first references to this curve are found in two letters of Descartes to the Father Mersenne in 1638.

5.3 Ellipsoidal wheels, sinusoidal tracks

Let us consider the case of ellipsoidal wheels, turning round one of the foci. The polar equation of the ellipse, referred to the focus, is

$$r = \frac{p}{1 + \varepsilon \cos\theta}$$

where p denotes the parameter and $0 < \varepsilon < 1$ is the eccentricity. The solution to the differential equation for the profile of the track leads to a function involving the sinus. It is a sinusoidal track whose amplitude and period are determined by the eccentricity and parameter of the ellipse.

6 The simulation of wheels of different "geometry"

The development of the mathematical models was the first step. One may easily see that a wheel with a non-circular shape can move smoothly on a track. For young students and people with low scientific background, the visual simulation is an excellent way to understand phenomena and models. Also, for an interactive virtual museum, the only effective way to present this topic is by developing an interactive simulation.

From a pedagogic point of view, we consider another issue: the mathematical models are introduced in such a way that the user feels attracted by an appealing simulation of the phenomena.

After the visual understanding of the problem, the environment should provide information about the mathematical background.

6.1 2D simulations

From the mathematical point of view, 2D simulations are enough to prove the models. In fact, in the development of the mathematical background, we have used 2D simulations.

Fig. 7 – 2D simulations of two wheel-tracks

The 2D simulations are enough to demonstrate the smooth movement of different wheels on different tracks and to explain the mathematical models. Our experience shows that experienced users, and users already aware of the problem, could easily use the 2D simulations. The problem is how to capture the attention of eventual users and young students. The 2D simulations do not stimulate people to think about the problem and to understand why the smooth movement is possible. Although the nature of the problem and the heart of its solution are two dimensional, 3D simulations have the important merit of catching the attention and arising the interest of young users. They are also an important means for popularising science. Therefore, 3D is the natural improvement.

6.1 From 2D to 3D simulations

Switching from 2D to 3D implies the modelling of 3D wheels and 3D tracks. To allow an easy and fast visualization, we used polygonal surface models: set of planar polygons in the three dimensional Euclidean space R^3. Without loss of generality, we can assume that the model consists entirely of triangular faces.

Formally, we can define a polygonal surface model as follows: a polygonal surface model $M = (V, F)$ is a pair containing a list of vertices V and a list of triangular faces F. The vertex list $V = (v_1, v_2, \ldots, v_r)$ is an ordered sequence; each vertex may be identified by a unique integer i. The face list $F = (f_1, f_2, \ldots, f_n)$ is also ordered, assigning a unique integer to each face. Every vertex $v_i = [x_i\ y_i\ z_i]^T$ is a column vector in the Euclidean space R^3 or, extending the previous definition, each vertex v_i could be embedded in the *projective* space P^3, so that each vertex $v_i = [x_i\ y_i\ z_i\ 1]^T$ is a column vector in P^3. Each triangle $f_i = (j, k, l)$ is an ordered list of three indices identifying the corners (v_j, v_k, v_l) of f_i.

Sweeping is used to generate the wheels. A vertex $v_i = [x_i\ y_i\ z_i\ 1]^T$ swept along the path represented by

the sweep transformation T is given by $Q = T v_i$. The transformation T determines the shape of the curve. The simplest sweep surface is obtained by traversing a line segment along a path. If the sweep transformation contains only translations and/or local or overall scalings, the resulting surface is planar. If the sweep transformation contains rotations, the resulting surface is non-planar.
The curves describing the wheels were generated on the *xy* plane and were swept parallel to the *OZ-axis*. In sweeping a planar polygon or closed curve along an arbitrary path, some difficulties may arise that can stress the viewing system. For instance, which point in the polygon or closed curve lies continuously on the path? In general, any point in a polygon or on a closed curve can lie continuously on the path. Another difficulty is: what is the direction of the normal to the polygon or closed curve as the path is swept out? One can take two approaches here: the normal to the polygon or closed curve is in the direction of the instantaneous tangent to the path or, alternatively, the normal direction is specified independent of the path. The former alternative was taken, being much more flexible.
To provide viewing information, the models have surface properties beyond simple geometry as, for example, surface normals, colors and texture coordinates.
On developing the 3D simulation it turned out to be important to build a car to show not only the smooth movement of an isolated wheel but also the same effect applied to a set of 4 wheels moving together. One more condition has been considered in our model: the distance between the two axes of the car. This distance is a function of the wheel and the track. For the first approach to the 3D simulation, we assume that the most important was to control the smooth movement depending from the main variables of the model: geometry of the wheel and the track and the distance between the axes. The model of the car was not important to simulate the smooth movement.

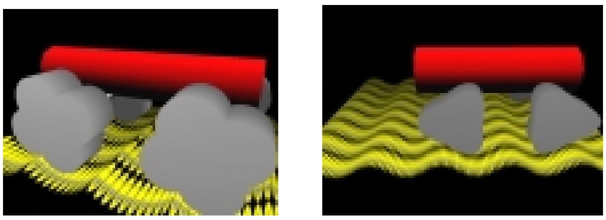

Fig. 8 – 3D simulations of two wheels-tracks

The experience with this simulation environment shows that it tends to stimulate users more strongly to these problems than a 2D simulation. The use of a very basic illumination model was enough for the scientific objective, but not enough to attract users with some experience on using interactive graphics environments. Specially for those with experience on using computer games, this is not the final solution.

7 A 3D virtual environment

Modern games are a very important application of the most actual computer graphics developments (HW and SW). The game metaphor has been used in the last years to show scientific concepts and to develop systems to be used by young students and users without specific computer experience. Therefore, this metaphor seems to be the key to develop an advanced application adequate to be used in different contexts: from the school to interactive museums.
To develop the virtual environment for our wheels we have to take care to different aspects: modeling a complete 3D environment, using an adequate rendering pipeline to obtain illumination effects and realism and designing an attractive environment, including the cars.
For rendering the pipeline we have adopted the OpenGL™ architecture, with its three main stages: *geometry processing, rasterization and per-fragment operations*. *Geometry processing* operations are responsible for geometrical transformations and lighting, during *rasterization* polygons are scan-converted and textured, and finally the *per-fragment operations* perform depth, stencil and alpha tests, as well as blending operations.
In our environment, several types of lights can be defined with their positions and directions stored in viewing coordinates. After lighting, a projective transformation and a perspective division are applied in order to transform the viewing frustum into the unit cube, followed by a viewport transformation. The texture coordinates are used to look up texture values in 1D, 2D, 3D, or 4D texture. While interpolation of texture coordinates is perspective correct, colors are only interpolated linearly along scan lines. The color resulting from the texture lookup is then combined with the fragment color according to one of several blending modes.
After texturing, the fragments have to pass several tests before being written to the frame-buffer. These tests include an *alpha test*, which allows a fragment to be accepted or rejected based on a comparison of its alpha channel and a reference value, a *stencil test*, which is based on a comparison between a reference value and the value of the stencil buffer at the pixel location corresponding to the fragment, and finally a depth test between the fragment z value and depth buffer. Colors of fragments passing all these tests are then combined with the current contents of the frame-buffer and stored there as new pixel values.

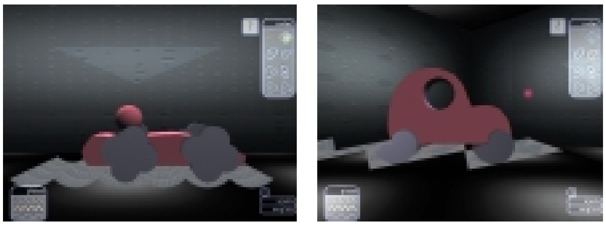

Fig. 9 – Virtual environment with two wheels-tracks

As may be seen in Fig. 9, a special design has been developed for this virtual environment, including cars, textures and menus to choose the wheels and the tracks. Since there is a well-defined relationship between the wheel and the track, if the user chooses a wheel, the environment shows the corresponding track and if the user chooses the track, the environment shows the possible wheels for this track.
The virtual environment is a 3D space with different sub-spaces organized according to the tracks and wheel types. The user can freely move in the virtual space, freely choose de viewpoint and zoom-in and zoom-out specific aspects. The cars move with spatial restrictions to maintain always the car on the track. To be able to have advanced on-line rendering, we use GeForce based graphics processors.
Since this virtual environment aims at combining effective attraction from the attractiveness on potential users with pedagogic aspects, the environment provides additional information about the problem, in two levels: the first level is easy to understand and descriptive (using informal descriptions) and the second level with the mathematical background (using precise mathematical formulations). When accessing the additional information, the user maintains the context of the virtual environment improving the perception of the problem.

Fig. 10 – Virtual environment with complementary information

This virtual environment has been tested with success with different types of users with success. The first reaction of the users is the interest in exploring the environment. After that, usually the users take a look into the additional information.
This environment has been presented at the program "2001" of the Portuguese TV and presently may be found at the Pavilhão do Conhecimento, in Lisbon, at the Exhibition Matemática Viva promoted by the "Associação ATRACTOR". In the same module of the exhibition, the user may also test physical models of different wheels.

8 Conclusion

The use of virtual environments in experimental sciences is one of our research themes. We believe that the actual developments in Computer Graphics allow the development of more attractive simulators and increase drastically the opportunities for new developments.
In the future, we would like to test other simulation environments, for example, other 3D tracks with different spatial geometries.
It is now time to use the more advanced technology in spreading science, from the basic sciences to the applied sciences. The costs are now compatible with the real use of this technology and the expectance from the users, specially the young students, are not compatible with hesitations from the scientific community. The key open issue is the analysis of the pedagogic effectiveness of the use of advanced applications. Informal conclusions are very positive, but we believe that psychologists and pedagogists have here an interesting area to explore. .

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