On Ranking Fuzzy Numbers Using Valuations

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ABSTRACT

The importance and difficulty of the problem of ranking fuzzy numbers is pointed out. Here we consider approaches to the ranking of fuzzy numbers based upon the idea of associating with a fuzzy number a scalar value, its valuation, and using this valuation to compare and order fuzzy numbers. Specifically we focus on expected value type valuations which are based upon the transformation of a fuzzy subset into an associated probability distribution. We develop a number of families of parameterized valuation functions.

1. Introduction

In many applications of fuzzy set theory, particularly in decision making, we often obtain a measure of a course of action expressed as a fuzzy number, a fuzzy subset of the real line. For example the profit obtained by using the new XYZ process may be *about $300,000*. Essentially here we have some uncertainty as to the exact value of the profit. As noted in the literature [1] this is a kind of possibilistic uncertainty. Often in these decision making environments we are faced with the problem of selecting one from among a collection of alternative actions. This selection process may then require that we rank, order, fuzzy numbers. While it is clear when considering two pure numbers which is bigger or smaller the situation with respect to fuzzy numbers is not always obvious. It was early [2] in the development of the fuzzy set theory that the problem of comparing fuzzy subsets of the real line was seen to be an important and difficult problem, see Bortolan and Degani [3] for a review of some methods suggested to address this problem. The recent literature has also addressed this problem [4]. What seems to be clear is that there exists no uniquely best method for comparing fuzzy numbers, the different methods satisfy different desirable criteria. While certain properties are necessary for any methodology that orders fuzzy numbers user preferences account for a significant part of the performance of a preferred approach. Our focus here is to try to understand and suggest some methodologies for comparing fuzzy numbers. Here we should note that parameterized classes of methods for ordering fuzzy numbers are particularly useful in that they allow uses to train a methodology to satisfy the user. Parameterized classes are often suggested by a process in which we try to unify and connect already existing approaches. In this work we look at some valuation based methods for comparing fuzzy numbers.

2. Valuation Methods for Comparing Fuzzy Numbers

One general approach to the problem of comparison of fuzzy numbers is to associate with a fuzzy number F some representative single value, Val(F), and compare the fuzzy subsets using these single representative values. An example of an approach in this spirit, obtaining a unique value as the proxy of a fuzzy subset, is the one introduced by Yager in [5] where he suggested using

$$\text{Val}(F) = \int_0^1 \text{Ave}\,(F_\alpha)\, d\alpha.$$

Here $F_\alpha = \{x \mid F(x) \geq \alpha\}$ is the α–level set of and $\text{Ave}(F_\alpha)$ is the average of the elements in the α–level set. We note that if the underlying support, those elements having non-zero membership grades is finite then

$$\text{Ave}(F_\alpha) = \frac{1}{\text{Card}(F_\alpha)} \sum_{x \in F_\alpha} x.$$

If F is a convex fuzzy subset then F_α is a closed interval $F_\alpha = [a_\alpha, b_\alpha]$.and in this case

$$\text{Ave}(F_\alpha) = \frac{b_\alpha + a_\alpha}{2}.$$

If F is non-convex and non discrete then we can express F_α as a union of distinct intervals

$$F_\alpha = \bigcup_{i=1}^{n_\alpha} [a_i^\alpha, b_i^\alpha]$$

and here

$$\text{Ave}\,(F_\alpha) = \frac{\sum_{i=1}^{n_\alpha} (b_\alpha^i - a_\alpha^i)\left(\frac{b_\alpha^i + a_\alpha^i}{2}\right)}{\sum_{i=1}^{n_\alpha} (b_\alpha^i - a_\alpha^i)}$$

In the preceding in using $\int_0^1$ Ave (F_α) dα it is assumed that F is a normal subset, there exists at least one element with membership grade one. If F is non-normal, but not empty, then Yager [5] suggests using

$$Val(F) = \frac{1}{\alpha_{max}} \int_0^{\alpha_{max}} Ave\ (F_\alpha)\ d\alpha$$

where α_{max} is the maximal membership grade in F. In [4] Yager discusses a number of properties associated with this approach.

In general any procedure used to obtain a unique characterizing value for a fuzzy subset F should be based on a valuation, Val(F), that is a reflection of the information carried by the fuzzy subset. One primary consideration in this process is the fact that the membership grade, F(x), associated with the quantity x is a measure of the weight or strength associated with the belief that x is the actual value of the variable whose value is expressed by the fuzzy set F. This effectively means that increasing the value of $F(x^*)$ should make Val(F) move closer to x^*. If ρ is a distance measure (metric) than

$$\frac{\partial \rho\ (Val(F), x^*)}{\partial F(x^*)} \leq 0.$$

Another primary consideration is that each x should be handled in the same manner, that is the formulation Val(F) should be symmetric regarding the elements of the underlying space.

While the choice of the actual function Val(F) is subjective, it depends upon the metric chosen, it should satisfy a number of reasonable conditions.

If F is a fuzzy subset of the real line and Val(F) is a unique value representative of this fuzzy subset then the following properties should be satisfied by Val(F):

1) **Boundedness**: Let b be the largest number for which F(x) > 0 and let a be the smallest number for what F(x) > 0 then boundedness requires a ≤ Val(F) ≤ b. We note a special case of this occurs if F is such that F(x) = 0 for all x except x* in this case we require Val(F) = x*.

2) **Monotonicity**: Assume F and E are two fuzzy subsets such that F(x) = E(x) for all x except a and b where b > a. If F(b) > F(a) then Val(F) ≥ Val(E). An alternative representation of

this condition is the following. Assume Val(E) = a. Let F be fuzzy subset such that F(x) = E(x) for all $x \neq b$ where F(b) > E(b)

1) if $b \geq a$ then $\mathrm{Val}(F) \geq \mathrm{Val}(E)$

2) if $b \leq a$ then $\mathrm{Val}(F) \leq \mathrm{Val}(E)$

In essence symmetry, boundedness and monotonicity implies that Val(F) is a kind averaging operation.

An operation that is closely related to that of trying to evaluate fuzzy numbers is that of defuzzification [6], a process that is used in the modeling of fuzzy control systems [7]. In a fuzzy controller, where we use a collection of fuzzy rules to represent a complex relationship between input and output variables, we end up with the output variable expressed as a fuzzy subset. In order to provide an input for the system being controlled we must defuzzify this output fuzzy subset to obtain a unique crisp value as the input to the controller. Here we see that we are also presented with the problem of obtaining a representative value for a fuzzy subset. In [6] Yager and Filev investigated the issue of defuzzification in a very general way, in the following we shall draw upon some ideas suggested there to investigate the problem to evaluating fuzzy numbers.

3. Probability-Expected Value Valuations

One view of the defuzzification process, discussed in [6], which can form the basis of the valuation of fuzzy numbers involves a process in which the fuzzy subset is used to generate a probability distribution. This probability distribution is then used to obtain an expected value, which can be used as the evaluation of the fuzzy subset.

Assume F is a fuzzy subset of the discrete finite subspace X of the real line. In order to obtain the valuation, Val(F), it is suggested that we proceed as follows:

1. From F obtain a probability distribution **P** on X where p_j is the probability of x_j.

2. Evaluate Val(F) as

$$\mathrm{Val}(F) = \sum_{j=1}^{n} p_j \, x_j$$

In order to use this method for evaluating a fuzzy subset we essentially become faced with the problem of converting the fuzzy subset F into a probability distribution **P**. In obtaining **P** from F we would expect the following criteria to be satisfied:

P - 1: If $F(x) > F(y)$ then $P(x) \geq P(y)$

P - 2: If $F(x) = F(y)$ then $P(x) = P(y)$

P - 3: If $F(x) = 0$ then $P(x) = 0$

An additional condition that may be desirable is that if $\sum_{x} F(x) = 1$ then $F(x) = P(x)$, we shall this fidelity..

Condition **P-2** is a reflection of the fact that no property other than the membership grade be used to generate probability. Condition **P-3** is a reflection of interpreting $F(x) = 0$ as one in which x is impossible. This precludes simple translation of type transformations such as

$$p(x_i) = F(x_i) + \frac{1}{n}(1 - \sum_j F(x_j))$$

Condition **P - 1** is a reflection of the basic relationship between p(x) and F(x), the larger F(x) the larger p(x).

We see that the satisfaction of conditions **P-1** to **P-3** along with the use of expected value guarantees the required boundedness described earlier. Let a and b be the smallest and largest values for which $F(x) \neq 0$. From **P-2**, this implies $p(x) = 0$ for $x < a$ and $x > b$. Since $\text{Val}(F) = \sum_{j=1}^{n} p_j x_j$ then $a \leq \text{Val}(F) \leq b$.

Monotonicity is also guaranteed by these three conditions. Consider two fuzzy subsets A and B on the spaces $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ respectively. Assume $A(x_i) = B(y_i) = u_i$ and without loss of generality let $u_i \geq u_j$ if $i > j$. Both these fuzzy sets generate the same probability distribution p_i, for i =1 to n, and from property **P-1**, if $i > j$.then $p_i \geq p_j$. Hence

$$\text{Val}(A) = \sum_{j=1}^{n} p_j x_j \text{ and } \text{Val}(B) = \sum_{j=1}^{n} p_j y_j.$$

Let X and Y be such that

$x_j = y_j$ for j = 1 to n - 2

$x_{n-1} = a$ and $x_n = b$

$$y_{n-1} = b \text{ and } y_n = a$$

where a > b. In this case

$$\text{Val}(A) = \sum_{j=1}^{n-2} x_j p_j + a\, p_{n-1} + b\, p_n$$

$$\text{Val}(B) = \sum_{j=1}^{n-2} y_j p_j + b\, p_{n-1} + a\, p_n$$

$$\text{Val}(B) - \text{Val}(A) = (a - b)\, p_n - (a-b)\, p_{n-1} = (a - b)\,(p_n - p_{n-1})$$

Since a > b and $p_n > p_{n-1}$ we see Val(B) ≥ Val(A), imposing monotonicity.

One approach to the generation of a probability distribution from a fuzzy subset involves a simple normalization. If F is a fuzzy subset over the finite space X we obtain a probability distribution over X by calculating

$$P(x_j) = \frac{F(x_j)}{\sum_i F(x_i)}$$

We note that this satisfies the three desired conditions. This approach doesn't require F to be a normal subset, that is $\text{Max}_X[F(x)]$ can be less than one. In addition this approach can be easily extended to the case in which X is a continuous subinterval of the real line rather than a finite subset. In this case instead of generating a probability distribution P over X we obtain a probability density function f over X such that $f(x) = \frac{F(x)}{\int_X F(x)\, dx}$ from this we see that $\int_X^{?} f(x) = 1$

Using this formulation for the probability associated with a fuzzy subset, we get as the expected value, i.e. the valuation of the fuzzy subset,

$$\text{Val}(F) = \sum_{j=1}^{n} x_j p_j = \frac{\sum_{j=1}^{n} x_j F(x_j)}{\sum_{j=1}^{n} F(x_j)}$$

in the discrete case and in the continuous case

$$\text{Val}(F) = \int_{x \in X} x\, f(x)\, dx = \frac{\int_{x \in X} x\, F(x)\, dx}{\int_{x \in X} F(x)\, dx}$$

Using some algebraic manipulations the formula $P(x_j) = \frac{F(x_j)}{\sum_i F(x_i)}$ can be expressed in a slightly different way which provides an interesting view of this normalization process. In particular we see that $F(x_j)$ can be expressed as

$$P(x_j) = \frac{F(x_j)}{\sum_i F(x_i)} = F(x_j) + \frac{F(x_j)}{\sum_i F(x_i)} \left(1 - \sum_i F(x_i)\right).$$

From this expression we see that we that we obtain $P(x_j)$ by initially assigning the membership grade as the probability, $P(x_j) = F(x_j)$, and then proportionally modifying this based upon how close the sum of the membership grades are to one.

Another approach to the association of a probability distribution with a fuzzy subset is based upon the connection between a fuzzy subset, representing some imprecisely known value, and a possibility distribution [8] and in turn the representation of the possibility distribution by a Dempster-Shafer belief structure [9-11].

Assume that V is some allowable variable taking its value on the real line. Assume our knowledge of the value of this variable is expressed in terms of a fuzzy subset, V is *large* or V is *about 27*. Fundamentally in this situation there exists some uncertainty with respect to value of this variable. As suggested by Zadeh, this uncertainty induces a **possibility distribution** $\prod$ over the domain X of V such that for $x \in X$, $\prod(x) = A(x)$ indicates the possibility that V is equal to the value x. That is, the possibility of x equals its membership grade. In addition this possibility distribution induces a **possibility measure** π where

$$\pi : 2^X \rightarrow [0,1]$$

such that

1. $\pi(\{x\}) = \prod(x)$.
2. $\pi(D \cup E) = \text{Max}\,[\pi(D), \pi(E)]$

In [9-11] it is shown how a possibility measure can be obtained from a Dempster-Shafer belief structure; it is the upper probability or plausibility measure. Since the Dempster-Shafer belief structure can be interpreted as the performance of a random experiment, the generation of random sets, this framework can then be used to associate a probability distribution with a fuzzy set.

We recall that a Dempster-Shafer belief structure defined on a set X consists of a collection of subsets of X, B_i for i = 1 to n, called the focal elements, and a mapping *m*

$$m : 2^X \rightarrow [0,1]$$

called a basic assignment function such that

1. $m(E) = 0$ for $E \neq B_i$ $\quad (m(B_i) \geq 0)$
2. $\sum_{i=1}^{n} m(B_i) = 1$

One interpretation associated with this D-S belief structure involves the performance of a random experiment to determine the value of some variable V. This experiment is a compound experiment. In the first part of the experiment we select a subset of X by using a biased random experiment in which $m(B_i)$ is the probability that we select B_i. In the second part of the experiment we select as the value of V some element from the chosen subset, B_q. It should be emphasized that the exact mechanism used to select the element from B_q is not specified or known. It has this additional uncertainty. One view of this second selection mechanism is as the performance of a random experiment on the set B_q in which the probability distribution is unknown.

While for any $x \in X$ the determination of its exact probability, p(x), is not possible in this environment what is possible is the determination of its upper probability $p^*(x)$, $p(x) \leq p^*(x)$, where

$$p^*(x) = \sum_{\substack{B_i \\ \text{s.t.} \\ x \in B_i}} m(B_i).$$

Furthermore the associated *upper probability measure*, $P^*: 2^X \rightarrow [0,1]$,

$$P^*(D) = \sum_{\substack{B_i \\ \text{s.t.} \\ B_i \cap D \neq \emptyset}} m(B_i),$$

is a *possibility measure*. As discussed in [12] this upper probability, possibility measure, can be directly related to an inducing fuzzy subset. Let A be a fuzzy subset of the space $X = \{x_1, ..., x_n\}$. We shall assume, without loss of generality, that the elements in X have been indexed such that $A(x_1) \leq A(x_2) \leq A(x_3) .... \leq A(x_n) \leq 1$.

Consider now a Dempster-Shafer belief structure with n focal elements, B_i for i = 1 to n, in which $B_i = \{x_j | j = i \text{ to } n\}$ and where

$$m(B_1) = A(x_1)$$

$$m(B_i) = A(x_i) - A(x_{i-1}) \quad \text{for } i = 2 \text{ to } n.$$

Note: If we introduce a pseudo element x_0 such that $A(x_0) = 0$, then we can more succinctly express this as $m(B_i) = A(x_i) - A(x_{i-1})$ for i = 1 to n.

Using this particular belief structure we can find a unique correspondence between a belief structure and a fuzzy subset. In particular, under this structure $p^*(x) = A(x)$, the upper probability is equal to the membership function of the associated fuzzy subset. Thus the upper probability measure is the possibility measure associated with the underlying fuzzy subset. This connection between a fuzzy subset and a Dempster-Shafer belief structure along with the random experiment interpretation of the D-S belief structure suggests that the belief structure provides a fertile ground for looking for a probability distribution to associate with a fuzzy set.

Again, consider the belief structure. Here we have a compound experiment:. first we select a subset B_i of X where $m(B_i)$ is the probability of selecting B_i and then we choose an element from the selected subset. Our inability to have a unique probability for the elements in X generated by the belief structure, all we have are upper probabilities, is due to the fact that we have not prescribed the method of selecting an element from the set obtained in the first part of the experiment. By introducing different methods of selecting the element from the chosen set we get different uncertainty distributions on X. One potential method for selecting the element is to perform a probabilistic experiment on the set B_i. In this case the probability $p(x_j)$ associated with the element x_j is

$$p(x_j) = \sum_{i=1}^{n} P(x_j|B_i)\, P(B_i).$$

Since $P(B_i) = m(B_i)$, which is known, all we need is to decide upon $P(x_j|B_i)$. Many possibilities exist for this, each of which will lead to a different value for $p(x_j)$. One possibility is to purely randomly select the element from B_i. In this case

$$P(x_j|B_i) = 0 \quad \text{if } x_j \notin B_i$$

$$P(x_j|B_i) = \frac{1}{n_i} \quad \text{if } x_j \in B_i,$$

where n_i is the number of elements in B_i. Since $B_i = \{x_j | j= i \text{ to } n\}$ we get $n_i = n + 1 - i$. This choice of $P(x_j|B_i)$ can be seen as an application of the principle of maximal entropy, we use the most neutral distribution.

Using this approach it can be seen that

$$p(x_j) = \sum_{i=1}^{n} P(x_j|B_i)\, P(B_i) = \sum_{i=1}^{j} \frac{P(B_i)}{n_i}.$$

Since $x_j \notin B_i$ for $i > j$, then $p(x_j) - \sum_{i=1}^{j} \frac{P(B_i)}{n + 1 - i}$. Furthermore, since $P(B_i) = m(B_i) = A(x_i) - A(x_{i-1})$ we get

$$p(x_j) = \sum_{i=1}^{j} \frac{A(x_i) - A(x_{i-1})}{n + 1 - i}.$$

This formulation is discussed in [12]. A recursive formulation of this is

$$p(x_1) = \frac{A(x_1)}{n}$$

$$p(x_j) = p(x_{j-1}) + \frac{A(x_j) - A(x_{j-1})}{n + 1 - j} \qquad \text{for } j = 2 \text{ to } n$$

This probability distribution then provides one possible formulation for the probability distribution associated with a a fuzzy set. If we now calculate the expected value associated with this probability distribution this leads to one possible valuation of the fuzzy set:

$$\text{Val}(A) = \sum_{j=1}^{n} x_j\, p(x_j) = \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{j} \frac{A(x_i) - A(x_{i-1})}{n + 1 - i}\right)$$

Rearranging terms we see that

$$\text{Val}(A) = \sum_{i=1}^{n} \left(\frac{(A(x_i) - A(x_{i-1}))}{n + 1 - i} \sum_{j -= i}^{n} x_j\right)$$

Recalling that $B_i = \{x_j \mid j = i \text{ to } n\}$ we see that $\sum_{j -= i}^{n} x_j$ is the sum of all the elements in B_i. Furthermore since $\text{Card}(B_i) = n + 1 - i$, then $\frac{1}{n + 1 - i} \sum_{j -= i}^{n} x_j$ is the average value of the elements in B_i. Denoting this as $\text{Ave}(B_i)$ we see that $\text{Val}(A) = \sum_{i=1}^{n} (A(x_i) - A(x_{i-1})) \,.\, \text{Ave}(B_i)$. Since $(A(x_i) - A(x_{i-1})) = P(B_i)$ we see that

$$\text{Val}(A) = \sum_{i=1}^{n} P(B_i) \,.\, \text{Ave}(B_i).$$

This formulation is reassuring in that the process we are using to obtain an element is to select one of the B_i, with probability $P(B_i)$, and then purely randomly select an element from this B_i Given a subset B_i, the expected value associated with the random selection of an element from B_i is the average of elements in B_i, thus the above formulation is clearly a reflection of this process, it is the expected value of these values.

An additional view of this formula is possible which is very interesting. We recall that the α–level set of A, A_α, is defined as $A_\alpha = \{x \mid A(x) \le \alpha\}$. With the elements in X indexed such that $A(x_i) \ge A(x_j)$ for i > j, then we see that

$$A_\alpha = \{x_j \mid j = i \text{ to } n\} \text{ for } A(x_{i-1}) \le \alpha \le A(x_i).$$

That is, for $\alpha \in [A(x_{i-1}), A(x_i)]$ we have $A_\alpha = B_i$.

Consider now the formulation suggested by Yager in [5] for evaluating fuzzy subsets, $\text{Val}(A) = \int_0^1 \text{Ave}(A_\alpha)\, d\alpha$. Since as we have just shown $A_\alpha = B_i$ for $\alpha \in [A(x_{i-1}), A(x_{i-1})]$ and $A(x_0) = 0$ and $A(x_n) = 1$ we can express this as

$$\text{Val}(A) = \sum_{i=1}^{n} \left(\int_{A(x_{i-1})}^{A(x_i)} \text{Ave}(B_i)\, d\alpha\right) = \sum_{i=1}^{n} \text{Ave}(B_i)\,(A(x_i) - A(x_{i-1})).$$

Thus the approach suggested by Yager in [5] can be seen as being the same as using the expected value of the elements in X where the probabilities are obtained by using the method above, that is

$$\int_0^1 \text{Ave}(A_\alpha)\, d\alpha = \sum_{j=1}^{n} p(x_j)\, x_j$$

where $p(x_j) = \sum_{i=1}^{j} \frac{A(x_i) - A(x_{i-1})}{n+1-i}$.

We recall that this result was obtained by considering a compound experiment, based upon the Dempster - Shafer belief structure, in which we selected a subset B_q of X using a probabilistic experiment in which $P(B_i) = A(x_i) - A(x_{i-1})$ where $B_i = \{x_j \mid j = i \text{ to } n\}$ and then choose the element from B_q purely randomly.

As we indicated, in the D-S model, the process used to select the element from the chosen focal set B_q is not specified. The decision to use a pure random selection is an arbitrary one. In [13, 14] Yager suggested, also based upon the D-S model, a different approach for the selection of the element from B_q, one which was shown to better capture the spirit of a possibility measure. Yager suggests the following compound experiment for obtaining the probabilities of the elements associated with a fuzzy subset. The first part of this compound experiment is the same as in the preceding, we introduce the n focal elements $B_i = \{x_j \mid j = i \text{ to } n\}$ and using chance we select one of these where $P(B_i) = A(x_i) - A(x_{i-1})$. Once having obtained a subset B_q, having cardinality $n_q = n + 1 - q$, we perform n_q random experiments **without** replacement on B_q. It should be noted that the selection of B_q leads to n_q outcomes, each focal element leads to a different number of outcomes. The output of these random experiments are then used to determine the probability associated with the elements, $p(x_j) = \frac{\#\text{ times } x_j \text{ appears}}{\#\text{ outcomes that appear}}$. Using this approach Yager [13] showed that the probability distribution is the simple normalization of the fuzzy subset A

$$p(x_j) = \frac{A(x_j)}{\sum_{i=1}^{n} A(x_i)}$$

Thus we see that the simple normalization corresponds to a Dempster - Shafer experiment with a different mechanism for selecting the elements from the chosen focal element.

The valuation function obtained under the simple normalization is

$$\text{Val}(A) = \sum_{j=1}^{n} \frac{A(x_j)\, x_j}{\sum_{i=1}^{n} A(x_i)}$$

The following theorem provides an interesting alternative representation for this valuation.

Theorem: $$\sum_{j=1}^{n} \frac{A(x_j)\, x_j}{\sum_{i=1}^{n} A(x_i)} = \frac{\sum_{j=1}^{n} \text{Aver}(B_j)\, n_j\, P(B_j)}{\sum_{i=1}^{n} n_i\, P(B_i)}$$

Proof: 1. First we consider the denominator; $n_i = n + 1 - i$ (the number of elements in B_i) and $p(B_i) = A(x_i) - A(x_{i-1})$. If we consider the term $\sum_{i=1}^{n} P(B_i)\, n_i = \sum_{i=1}^{n} (A(x_i) - A(x_{i-1}))\, n_i$ we see this can be expressed as $n_n F(x_n) + \sum_{i=1}^{n-1} (n_i - n_{i+1})\, A(x_i) - n_1 A(x_0)$. Since $n_n = n + 1 - n = 1$, $A(x_o) = 0$ and $n_i - n_{i+1} = 1$ we get that $\sum_{i=1}^{n} P(B_i)\, n_i = \sum_{i=1}^{n} A(x_i)$.

2. Let us now consider the numerator. Consider the term $\text{Aver}(B_j)\ n_j\ P(B_j)$ since $\text{Aver}(B_j) = \frac{1}{n_j}\ \text{Total}(B_j)$ where $\text{Total}(B_j)$ is the sum of the elements in B_j, $\text{Total}(B_j) = \sum_{i=j}^{n} x_j$, we have $\text{Aver}(B_j)\ n_j\ P(B_j) = \text{Total}(B_j)\ P(B_j)$ and $\sum_{i=1}^{n} \text{Aver}(B_j)\ n_j\ P(B_j) = \sum_{i=1}^{n} \text{Total}(B_j)\ P(B_j)$ and therefore the numerator is $\sum_{i=1}^{n} (\sum_{i=j}^{n} x_j)\ P(B_j)$. Doing some algebraic manipulations we see that any x_k it gets multiplied by $\sum_{j=1}^{k} P\ (B_j)$ thus numerator is

$$\sum_{k=1}^{n} x_k \left(\sum_{j=1}^{k} P\ (B_j)\right) = \sum_{k=1}^{n} x_k\ A\ (x_k)$$

4. A Generalized Approach

Assume A is a normal fuzzy subset of $X = \{x_1, x_2, ... , x_n\}$ where the elements have indexed such that $A(x_i) \geq A(x_j)$ if $i > j$. Let B_j be a crisp subset of X defined by

$$B_j = \{x_i \mid j \leq i \leq n\}.$$

We shall call B_j a *focal element* of A and we note that B_j contains the n + 1 - j elements with the highest membership grade in A. In the preceding we have discussed two related approaches to the valuation of the fuzzy subset A. These approaches are based upon the calculation of expected values using probabilities inspired by the Dempster-Shafer representation. In this representation we have assumed $P(B_j) = A(x_j) - A(x_{j-1})$. We shall denote these two valuation methods as Val_1 and Val_2. In Val_1, the one originally introduced by Yager, we have

$$\text{Val}_1(A) = \sum_{i=1}^{n} \text{Aver}(B_j) \cdot P(B_j)$$

If we denote $B = \{B_1, B_2, ..., B_n\}$, the set of the B_j, then $\text{Val}_1(A)$ is the expected value of the average of the sets in B under the preceding probability distribution.

On the other hand $\text{Val}_2(A) = \dfrac{\sum_{j=1}^{n} \text{Aver}(B_j)\ n_j\ P(B_j)}{\sum_{j=1}^{n} n_j\ P(B_j)}$, where $n_j = n + 1 - j$ is the cardinality of B_j. It is interesting to note that since $\text{Aver}(B_j) = \frac{\text{Sum}(B_j)}{n_j}$, where $\text{Sum}(B_j)$ is the sum of the values of the elements in B_j, then

$$\text{Val}_2(A) = \frac{\sum_{j=1}^{n} \text{Sum}(B_j) \cdot P(B_j)}{\sum_{j=1}^{n} n_j \; P(B_j)}$$

thus

$$\text{Val}_2(A) = \frac{\text{Expected Sum of the } B_j}{\text{Expected number of elements in the } B_j}$$

whereas $\text{Val}_1(A)$ is the expected average of B_j.

Alternatively for a comparison with $\text{Val}_1(A) = \sum_{i=1}^{n} \text{Aver}(B_j) \cdot P(B_j)$ we can express $\text{Val}_2(A) = \sum_{i=1}^{n} \text{Aver}(B_j) \cdot \hat{P}(B_j)$ where $\hat{P}(B_j) = \frac{n_j \; P(B_j)}{\sum_{j=1}^{n} n_j \; P(B_i)}$. Here we see both these approaches can be seen as one in which we randomly select a focal element B_j from *B* and then use the average of the selected set. However, in each case the probability of selecting a given B_j is different.

We notice that the difference between the two probabilities is based upon a weighted average of the cardinalities of the focal elements, $\hat{P}(B_j) = \frac{n_j \; P(B_j)}{\sum_{j=1}^{n} n_j \; P(B_i)}$.

We can view the difference between Val_1 and Val_2 as one in which Val_2 puts more emphasis on the focal elements which have more members. Here it should be noted that since $n_j = n + 1 - j$ we have $n_j > n_i$ for $j < i$, thus Val_2 puts more emphasis on the elements with lower index.

Since the difference between the two approaches is essentially based upon the number of elements in the associated focal element we can provide a more general expression of the valuation function associated with a fuzzy subset.

Let B_j be the focal elements associated with the fuzzy subset A, B_j is the set consisting of the $n + 1 - j$ elements with the highest membership grade. Then a family of valuation functions associated with A is

$$Val_F(A) = \sum_{j=1}^{n} \text{Aver}(B_j)\, \widetilde{P}(B_j)$$

where

$$\widetilde{P}(B_j) = \frac{F(n_j)}{\sum_{j=1}^{n} F(n_j)\, P(B_j)} \quad P(B_j)$$

where $P(B_j) = A(x_j) - A(x_{j-1})$, the difference between the j and j - 1 largest membership grade in A, $n_j = n + 1 - j$ is the cardinality of B_j and F is some non-negative function.

We see that if $F(n_j) = 1$ for all j then $\widetilde{P}(B_j) = P(B_j)$ and we recover Val_1. If $F(n_j) = n_j$ then $\widetilde{P}(B_j) = \widehat{P}(B_j)$ and we recover Val_2. It appears natural to consider three classes of F:

1) $F(n_j) \geq F(n_i)$ if $n_j > n_i$

2) $F(n_j) \leq F(n_i)$ if $n_j > n_i$

3) $F(n_j) = F(n_i)$ for all n_i and n_j

In the first class we emphasize the focal elements with more members while in the second class we emphasize those with less elements. In class three no distinction is made based upon the number of elements.

As we have already noted for Val_2 we have $F(n_j) = n_j$ and hence this valuation puts more emphasis on those focal elements with more members. Another example of a transformation in this first class, one that puts more emphasis on the focal elements with more members is $F(n_j) = \text{Min}[K, n_j]$ where $K \in [1, n]$. Actually we see that if $K = n$ then $F(n_j) = n_j$ and if $K = 1$ we get $F(n_j) = 1$, $Val_1(A)$. Thus this gives us a family which includes our two valuations as special cases. Using this formulation for F we get $Val(A) = \sum_{j=1}^{n} \text{Aver}(B_j)\, \widetilde{P}(B_j)$ with

$$\widetilde{P}(B_j) = \frac{\text{Min}[K, n_j]}{\sum_{j=1}^{n} \text{Min}[K, n_j]\; P(B_j)} \quad P(B_j)$$

An example of a transformation of the second class, one that puts more emphasis on focal elements with less elements is $F(n_j) = n - n_j + 1$. In this case since $n_j = n + 1 - j$ we get $F(n_j) = j$. Thus in this case

$$\tilde{P}(B_j) = \frac{j}{\sum_{j=1}^{n} j \; P(B_j)} \; P(B_j) = \frac{j \; P(B_j)}{(n+1) - \sum_{j=1}^{n} A(x_j)} = \frac{j \; P(B_j)}{1 + \sum_{j=1}^{n} \bar{A}(x_j)}$$

It is interesting to note that this case is equivalent to the calculation $Val_F(A) = \sum_{j=1}^{n} x_j \, P(x_j)$ where

$$P(x_j) = \frac{1}{Q} \sum_{i=1}^{j} \frac{i}{n+1-i} (A(x_i) - A(x_{i-1}))$$

with $Q = (n + 1) - \sum_{j=1}^{n} A(x_j)$.

An extreme example of a formulation that puts more emphasis on those sets with less elements is one in which

$$F(n_j) = 0 \quad \text{if } j < n$$

$$F(n_n) = 1$$

Here we see that

$$\tilde{P}(B_j) = 0 \quad \text{for } j < n$$

$$\tilde{P}(B_n) = 1$$

which results in $Val_F(A) = Ave(B_n)$. This this selection gives as its valuation the element with the largest membership grade.

5. Level Set View

As we have indicated earlier the focal elements, the B_i, are closely related to the level sets of the fuzzy subset A. In particular the α-level sets of a fuzzy subset A, $A_\alpha = \{x \mid A(x) \leq \alpha\}$, are related to the B_i in the following way

$$A_\alpha = B_i \quad \text{for } A(x_{i-1}) \leq \alpha \leq A(x_i),$$

the B_i are essentially the level sets of A. From this relationship we showed that

$$Val_1(A) = \int_0^1 Aver(A_\alpha) \, d\alpha$$

In addition we can also express Val_2 using the level set representation

$$\mathrm{Val}_2(A) = \frac{\int_0^1 \mathrm{Aver}(A_\alpha)\,\mathrm{Card}(A_\alpha)\,d\alpha}{\int_0^1 \mathrm{Card}(A_\alpha)\,d\alpha}$$

These formulations lead us to consider a generalization in which we have

$$\mathrm{Val}_F(A) = \frac{\int_0^1 \mathrm{Aver}(A_\alpha)\,F(\mathrm{Card}(A_\alpha))\,d\alpha}{\int_0^1 F(\mathrm{Card}(A_\alpha))\,d\alpha}$$

We note that for Val_1, $F(\mathrm{card}(A_\alpha)) = 1$ while for Val_2 $F(\mathrm{card}(A_\alpha)) = \mathrm{Card}(A_\alpha)$.

Again F can be classified into three classes

$$\frac{\partial F(x)}{\partial x} \geq 0 \text{ for all } x \quad \text{(monotone)}$$

$$\frac{\partial F(x)}{\partial x} \leq 0 \text{ for all } x \quad \text{(anti-monotone)}$$

$$\frac{\partial F(x)}{\partial x} = 0 \text{ for all } x \quad \text{(neutral)}$$

Since the level sets are such that $A_{\alpha_1} \subset A_{\alpha_2}$ for $\alpha_1 > \alpha_2$ then $\mathrm{Card}(A_{\alpha 1}) \leq \mathrm{Card}(A_\alpha)$ for $\alpha_1 > \alpha_2$. If F is monotone it tends to put additional emphasis on the elements with high cardinality, the lower level sets, while F anti-monotone puts more emphasis on the high level sets. Here we see that since for Val_2, F is monotone it puts more emphasis on the lower level sets. That is Val_2 tends to move the valuation toward average of lower level sets. Figure #1 helps illustrate the situation. Here we see that if b is the value obtained using Val_1 then that using Val_2, a, will be to the left as the lower α–level sets are skewed in this direction.

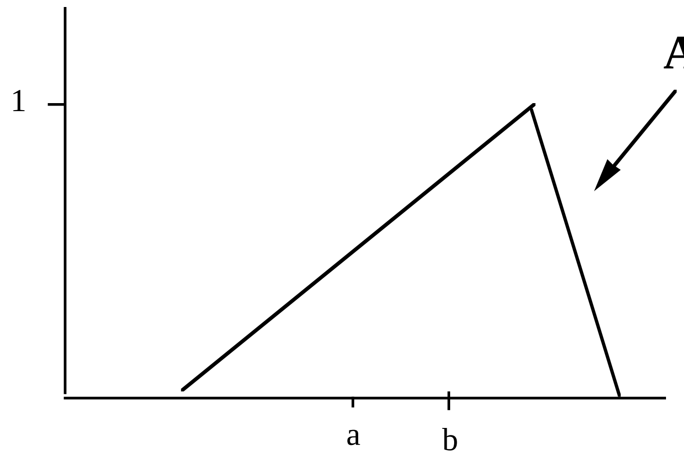

Figure 1. Illustration of effect of F

Since the elements with higher membership grade those in the higher α–level sets, are more strongly in A it may be desirable to use a non-monotone F, for example $F(n_j) = \frac{1}{n_j}$ or $F(n_j) = n + 1 - n_j$

Note: If A_α is continuous then we use the measure of A_α, $\mu(A_\alpha)$, instead of the cardinality of A_α.

Throughout the preceding we have explicitly assumed that the fuzzy subsets being evaluated are normal, there exists at least one x such that(A(x)) = 1. If A is not normal, $Max_x(A(x)) = \alpha_{max} < 1$, then we are faced with a problem, we can't calculate $Aver(A_\alpha)$ for $\alpha > \alpha_{max}$. However, this general formulation allows us to handle this problem in the same manner as was done in [5]. In particular we define

$$Val_F(A) = \frac{\int_0^{\alpha_{max}} Aver(A_\alpha)\, F(Card(A_\alpha))\, d\alpha}{\int_0^{\alpha_{max}} F(Card(A_\alpha))\, d\alpha}$$

where α_{max} is the maximum membership grade in A_α.

The general formulation introduced in the preceding can be used to inspire another class of valuation functions. Since the cardinality of the level sets have a unique relationship with respect to the value of the level, $Card(A_{\alpha_1}) \geq Card(A_{\alpha_2})$ for $\alpha_1 < \alpha_2$, it would appear natural to try to bypass the use of cardinality and express a class of valuation functions directly in terms of α. Here we introduce a class of valuation functions, denoted $\text{Valpha}_F(A)$, and defined

$$\text{Valpha}_f(A) = \frac{\int_0^1 \text{Aver}(A_\alpha)\, f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha}$$

where f: [0,1] → [0,1].

If A is symmetric, where $\text{Aver}(A_\alpha) = a$ then $\text{Valpha}_f(A) = a$. A number of special cases of f are worth introducing. If $f(\alpha) = k \neq 0$ for all α, then we get

$$\text{Valpha}_{f(}A) = \int_0^1 \text{Aver}(A_\alpha)\, d\alpha = \text{Val}_1(A)$$

Another interesting class of these evaluations occurs when $f(\alpha) = \alpha^q$ for $q \geq 0$. In this case we get

$$\text{Valpha}_{f(}A) = \frac{\int_0^1 \text{Aver}(A_\alpha)\, \alpha^q\, d\alpha}{\int_0^1 \alpha^q\, d\alpha}$$

Here we notice that for q > 0 we are placing more emphasis on higher α level sets, the larger the q the more emphasis. We see that if q = 1 we get

$$\text{Valpha}_{f(}A) = \frac{\int_0^1 \text{Aver}(A_\alpha)\, \alpha\, d\alpha}{\int_0^1 \alpha\, d\alpha} = 2\int_0^1 \text{Aver}(A_\alpha)\, \alpha\, d\alpha$$

If q -> ∞, we see that this gives us the mean of maxima as the expected value

$$\text{Valpha}_f(A) = \text{Aver}(A_1)$$

the average of elements is the one level set.

Complementary to this is the case when $f(\alpha) = (1 - \alpha)^q$, again we assume $q \geq 0$, here

$$\text{Valpha}_f(A) = \frac{\int_0^1 \text{Aver}(A_\alpha)\,(1-\alpha)^q\, d\alpha}{\int_0^1 (1-\alpha)^q\, d\alpha}$$

Here we place more emphasis on the lower α-level sets with smaller α, the larger the q the more the emphasis. If q = 1 we get

$$\text{Valpha}_f(A) = \frac{\int_0^1 \text{Aver}(A_\alpha)\,(1-\alpha)\, d\alpha}{\int_0^1 (1-\alpha)\, d\alpha} = \frac{\int_0^1 \text{Aver}(A_\alpha)\,(1-\alpha)\, d\alpha}{\frac{1}{2}}$$

$$\text{Valpha}_f(A) = 2\Big[\int_0^1 \text{Aver}(A_\alpha)\; d\alpha - \int_0^1 \text{Aver}(A_\alpha)\,\alpha\, d\alpha\Big]$$

$$\text{Valpha}_f(A) = 2\,\text{Val}_1(A) - 2\int_0^1 \text{Aver}(A_\alpha)\,\alpha\, d\alpha$$

If $q \to \infty$, then $\text{Valpha}_f(A) \to \text{Aver}(A_0)$, the average of the zero level set, the average of the space underlying A. Here we have discarded all information contained in A in evaluating it.

It should be noted that the valuation, $\text{Valpha}_f(A)$ can be generalized to work in situations in which the fuzzy subset A being evaluated is subnormal. In particular if $\text{Max}_x[A(x)] = \alpha_{max}$ then we define

$$\text{Valpha}_f(A) = \frac{\int_0^{\alpha_{max}} \text{Aver}(A_\alpha)\, f(\alpha)\, d\alpha}{\int_0^{\alpha_{max}} f(\alpha)\, d\alpha}.$$

In the following we shall look at the form of $\text{Valpha}_f(A)$ for the special case when A is a trapezoidal type fuzzy set (see fig. 2)

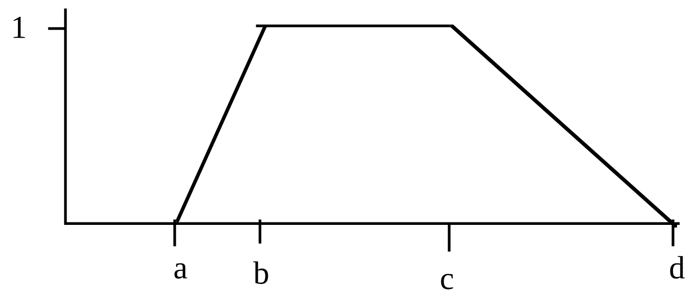

Figure 2. Trapezoidal Fuzzy Sets

For the trapezoidal fuzzy subset A the membership function is

$A(x) = 0$	for $x<a$
$A(x) = \frac{x - a}{b - a}$	for $a \le x \le b$
$A(x) = 1$	for $b \le x< c$
$A(x) = \frac{d - x}{d - c}$	for $c \le x < d$
$A(x) = 0$	for $x \ge d$

Using $Valpha_f$ we get $Valpha_f(A) = \dfrac{\int_0^1 Aver(A_\alpha)\, f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha}$.

The trapezoidal type of fuzzy sets have a very special form for the level sets, $A_\alpha = [u_\alpha, v_\alpha]$, the level sets are closed intervals. From this we easily get that $Aver(A_\alpha) = \frac{u_\alpha + v_\alpha}{2}$. Thus in this case

$$Valpha_f(A) = \frac{\frac{1}{2}\int_0^1 (u_\alpha + v_\alpha)\, f(\alpha)\, d\alpha}{\int_0^1 f(\alpha)\, d\alpha}$$

The end points of the level sets, the u_α and v_α can be easily obtained from the definition of of the fuzzy subset A. In particular from $\alpha = \frac{u_\alpha - a}{b - a}$ we get $u_\alpha = (b - a)\, \alpha + a$ and from $\alpha = \frac{d - v_\alpha}{d - c}$ we get $v_\alpha = d - (d - c)\, \alpha$. Using this we see

$$\text{Valpha}_{f(}(A) = \frac{\frac{1}{2}\int_0^1 [(b - a)\ \alpha - (d - c)\ \alpha + a + d]\ f(\alpha)\ d\alpha}{\int_0^1 f(\alpha\ )d\alpha}$$

This can be simplified to

$$\text{Valpha}_f(A) = \frac{\frac{1}{2}\int_0^1 [(b + c)\ \alpha + (1 - \alpha)\ \ (a + d)]\ f(\alpha)\ d\alpha}{\int_0^1 f(\alpha\ )d\alpha}$$

Let us first consider the case in which $f(\alpha) = 1$, this is $Val_1(A)$. In this case

$$\text{Valpha}_f(A) = Val_1(A) = \frac{1}{2}\int_0^1 [(b + c)\ \alpha + (1 - \alpha)\ \ (a + d)]\ \ d\alpha$$

$$Val_1 \ = \frac{1}{2}(\frac{1}{2}(b + c)\ \alpha^2 - \frac{1}{2}\ (1 - \alpha)^2(a + d)\ \Big|_0^1$$

$$Val_1 = \frac{1}{4}\ (a+ b + c + d),$$

here we see that it is the average of four crucial points.

Let us now consider the class of valuations in which $f(\alpha) = \alpha^q$. Here we get

$$\text{Valpha}_f(A) = \frac{1}{2}\ \frac{\int_0^1 [(b + c)\ \alpha + (1 - \alpha)\ \ (a + d)]\ \alpha^q\ d\alpha}{\int_0^1 \alpha^q\ d\alpha}$$

In this case since

$$\int_0^1 \alpha^q\ d\alpha = \frac{1}{q + 1}$$

we get

$$\text{Valpha}_f(A) = \frac{q + 1}{2}\ \int_0^1 [(b + c)\ \alpha + (1 - \alpha)\ \ (a + d)]\ \alpha^q\ d\alpha$$

$$\text{Valpha}_f(A) = \frac{q+1}{2} \int_0^1 [(b+c)\,\alpha^{(q+1)} + (\alpha^q - \alpha^{(q+1)})\ (a+d)]\ d\alpha$$

$$\text{Valpha}_f(A) = \frac{1}{2} \left(\frac{q+1}{q+2}(b+c) + \frac{1}{q+2}(a+d)\right)$$

$$\text{Valpha}_f(A) = \frac{1}{2}\,\frac{1}{(q+2)}\,[(a+b+c+a) + q\,(b+c)\,].$$

Again some special cases are worth pointing out:

- If q = 0, we get as before, $\text{Valpha}_f(A) = \frac{1}{4}(a+b+c+d)$
- If q = 1, then $\text{Valpha}_f(A) = \frac{1}{3}(b+c) + \frac{1}{6}(a+d)$
- If $q \to \infty$, then $\text{Valpha}_f(A) = \frac{1}{2}(b+c)$

Let us now consider the class of valuations in which $f(x) = (1-\alpha)^q$. Here we get

$$\text{Valpha}_f(A) = \frac{1}{2}\,\frac{\int_0^1 [(b+c)\,\alpha + (1-\alpha)\ (a+d)]\,(1-\alpha)^q\, d\alpha}{\int_0^1 (1-\alpha)^q\, d\alpha}$$

In this case since $\int_0^1 (1-\alpha)^q\, d\alpha = \frac{1}{q+1}$ we get

$$\text{Valpha}_f(A) = \frac{q+1}{2} \int_0^1 [(b+c)\,\alpha\ (1-\alpha)^q + (1-\alpha)^{(q+1)}\ (a+d)]\ d\alpha$$

After some algebraic manipulations we

$$\text{Valpha}_f(A) = \frac{q+1}{2} \left(\frac{(a+d)}{(q+2)} + \frac{(b+c)}{(q+2)\,(q+1)}\right)$$

$$\text{Valpha}_f(A) = \frac{1}{2} \left(\frac{q+1}{q+2}(a+d) + \frac{1}{q+2}(b+c)\right)$$

$$\text{Valpha}_f(A) = \frac{1}{2}\,\frac{1}{(q+2)}\,((a+b+c+d) + q\,(a+d))$$

Some special cases are worth pointing out:

- If q = 0, then as in the preceding, $\text{Valpha}_f(A) = \frac{1}{4}(a+b+c+d)$
- If q = 1, then $\text{Valpha}_f(A) = \frac{1}{3}(a+d) + \frac{1}{6}(b+c)$
- If $q \to \infty$, then $\text{Valpha}_f(A) = \frac{1}{2}(a+d)$

The valuation functions for trapezoidal membership functions just introduced,

$$F_1 = \frac{1}{2} \left(\frac{q+1}{q+2}(b+c) + \frac{1}{q+2}(a+d)\right) \qquad q \in [0, \in \infty]$$

and

$$F_2 = \frac{1}{2}\left(\frac{q+1}{q+2}(a+d) + \frac{1}{q+2}(b+c)\right) \qquad q \in [0, \in \infty]$$

provide interesting classes of valuation functions. Let us denote

$$\frac{b+c}{2} = M \qquad \text{(mean of maxima)}$$

$$\frac{a+d}{2} = S \qquad \text{(mean of support).}$$

and then we can express these as

$$F_1 = \frac{q+1}{q+2} M + \frac{1}{q+2} S \qquad q \in [0, \in \infty]$$

$$F_2 = \frac{q+1}{q+2} M + \frac{1}{q+2} S \qquad q \in [0, \in \infty]$$

Note: If the the fuzzy number A is symmetric, M = S = K, then for all q $F_1 = F_2 = K$

Note: As q goes from zero to infinity F_1 goes from $\frac{1}{2}(M + S)$ to M.

Note: As q goes from zero to infinity F_2 goes from $\frac{1}{2}(M + S)$ to S.

A unification of these two valuation functions can be made. First we note that $\frac{q+1}{q+2} + \frac{1}{q+2} = 1$ and as q goes from zero to infinitely $\frac{q+1}{q+2}$ goes from 0.5 to 1. Let us denote

$$F = \beta M + (1 - \beta) S$$

where $\beta \in [0, 1]$. Then we see that for $\beta \in [0, 0.5]$ we are getting F_2 where $\beta = 0$ corresponds to $q = \infty$ and $\beta = 0.5$ corresponds to $q = 0$ while for $\beta \in [0.5, 1]$ we are getting F_1 where $\beta = 1$ corresponds to $q = \infty$ and $\beta = 0.5$ corresponds to $q = 0$ Thus we have essentially provided, in this case of trapezoidal fuzzy numbers, a valuation function which is a weighted average of the Mean of the Maxima and Mean of the Support.

Conclusion

We have suggested a number of approaches to be used for the problem of comparing and ordering. These approaches are based upon the use of valuation procedures which convert a fuzzy number into a scalar value.

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