Explicit A-stable Runge-Kutta methods for linear Stiff-Equations

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Abstract

Some Runge-Kutta type methods of order 1 for solving Stiff-Differential Equations are present. Stability analyze of the methods for 2-dim and general linear system differential equations are given. It is shown that the methods presented are suitable for solving stiff-equations. Some numerical test justifying the results are presented.

1 Introduction

During the last years, there has been a considerable amount of research on the numerical integration of stiff systems of ODE's. A basic difficulty in the numerical solution of stiff systems is the satisfying of the requirement of stability. J.C Butcher [1] provided Implicit Runge-Kutta methods(abb;R-K methods) to overcome the stability problem. However for solving the stiff equations by using explicit methods, Lambert [2,3], Shaw[3] and many scholars have studied and proposed explicit type methods imposing stability conditions. Hairer [4] has proved instability of higher order(more than three) rational R-K methods proposed by Wambecq [9]. In the previous papers [5,6,7], the author proposed the rational coefficients explicit R-K methods for solving system of stiff equations. However, stability analysis in general differential systems equations was not given. The discussion of stability analysis of rational coefficients formulas for system equations is quite difficult. In this papers,using the same method for deriving rational coefficients methods which was proposed in [6,7,8] , the author propose the rational coefficients algorithms of order 1 and studies the stability analysis for 2-dim and general system differential equation;

The outline of this paper is as follows; In § 2, we consider first order two stage rational coefficients R-K methods for 2-dim linear differential equation and study stability of the methods, in § 3, we propose the numerical methods for general linear system of differential equation and study the stability of the proposed methods. In § 4 , some numerical tests justifying the results are present.

2 Derivation of Methods for 2-dim linear differential equation of the first order

We study the numerical solution of the following 2-dim linear differential equation,

$$\dot{Y} = AY,$$
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (a, b, c, d \in R) \tag{2.1}$$

where the matrix A has the eigenvalue with negative real.

We consider the following first order R-K formulas for solving (2.1),

$$^1y_{n+1} = {}^1y_n + h\,{}^1k_1 - h^2 \sum_{j=1}^{2} {}^1d_j,$$
$$^2y_{n+1} = {}^2y_n + h\,{}^2k_1 - h^2 \sum_{j=1}^{2} {}^2d_j,$$
$$^1k_1 = {}^1f(x_n, {}^1y_n, {}^2y_n),$$
$$^2k_1 = {}^2f(x_n, {}^1y_n, {}^2y_n), \tag{2.2}$$

where

$$^1f(x_n, {}^1y_n, {}^2y_n) = a\,{}^1y_n + b\,{}^2y_n,$$
$$^2f(x_n, {}^1y_n, {}^2y_n) = c\,{}^1y_n + d\,{}^2y_n,$$
$$^1d_1 = \frac{aL_1}{1 + hL_1}{}^1y_n, \quad {}^1d_2 = \frac{bL_1}{1 + hL_1}{}^2y_n,$$

$$^2d_1 = \frac{cL_1}{1+hL_1}{}^1y_n, \;\; {}^2d_2 = \frac{dL_1}{1+hL_1}{}^2y_n, \qquad (2.3)$$

and L_1 is the constant depending on the eigenvalue of the matrix A.

Introducing the vector notations,

$$Y_n = [{}^1y_n, {}^2y_n],$$

(2.2) becomes

$$Y_{n+1} = B_1 Y_n,$$

$$B_1 = (I + \frac{h}{1+hL_1}A). \qquad (2.4)$$

We study the conditions where the error of numerical processes (2.4) will grow with n or not. We define the following stability definition.

Definition; The numerical processes $\{Y_n\}$ of (2.4) is stable if the absolute value of the eigenvalue of B_1 are less than one.

We will see that, as n increases,the numerical process Y_n tends to zero if and only if the absolute value of the spectral radius of B_1 is less than one. We assume that the matrix A is normalizable,

$$TAT^{-1} = diag\{\lambda_1, \lambda_2\},$$

by means of a nonsingular matrix T.

Setting

$$Z_n = TY_n, \qquad (2.5)$$

(2.4) becomes

$$Z_{n+1} = (I + \frac{h}{1+hL_1}\Lambda)Z_n,$$

$$\Lambda = diag\{\lambda_1, \lambda_2\}, Z_n = [{}^1z_n, {}^2z_n],$$

or

$$^iz_{n+1} = (1 + \frac{h}{1+hL_1}\lambda)^i \; z_n, (i = 1, 2) \qquad (2.6)$$

and,so,the eigenvalues of B_1 are given by

$$\rho_i = 1 + \frac{h\lambda_i}{1+hL_1}. \; (i = 1, 2) \qquad (2.7)$$

Putting

$$\lambda_j = \alpha_j + \beta_j \, i, \; (\alpha_j < 0, \beta_j \in R). \; (j = 1, 2) \qquad (2.8)$$

(2.6) becomes

$$(1 + \frac{\alpha_i h}{1+hL_1})^2 + (\frac{\beta_i h}{1+hL_1})^2 < 1, \qquad (2.9)$$

or

$$\frac{h}{(1+hL_1)^2}\{2\alpha_i(1+hL_1) + (\alpha_i^2 + \beta_i^2)h\} < 0. \; (i = 1, 2)$$

If we set,

$$L_1 \geq -\frac{1}{2\alpha_i}({\alpha_i}^2 + {\beta_i}^2). \; (i = 1, 2) \qquad (2.10)$$

Then the stability condition (2.9) is satisfied. Using

$$\alpha_i = \frac{(a + d)}{2},$$

$${\alpha_i}^2 + {\beta_i}^2 = ad - bc,$$

(2.10) reduces to

$$L_1 \geq \frac{-(ad - bc)}{a + d}. \qquad (2.11)$$

Summing the results, we have the following results.

Theorem [1]. Let the matrix A in (2.1) be a normal matrix and the real part of eigenvalues of A is negative. If we set L_1 in the form (2.11). Then the numerical processes (2.2) is stable.

Let us consider the real case in (2.8), putting $\beta_i = 0$ leads to

$$L_1 \geq -\frac{1}{2}\alpha_i,$$

and using the relation,

$$\alpha_1 + \alpha_2 = -Tr(A),$$

we have the following corollary.

Corollary [1] If the matrix A have only real and negative eigenvalues. Let

$$L_1 = \frac{1}{2}Tr(A). \qquad (2.12)$$

Then numerical processes (2.4) is stable (A_0 stable).

3 Numerical integration for s-systems linear equations of the first order

In this section, we consider numerical integration for s-systems linear differential equation,

$$\dot{Y} = \tilde{A}Y, \qquad (3.1)$$

$$\tilde{Y} = ({}^1y, {}^2y, ..., {}^sy),$$

where $\tilde{A}$ is the constant $s \times s$ matrix which has the eigenvalue with negative real real.

We consider the following first order R-K methods for solving (3.1).

$$^iy_{n+1} = {}^iy_n + h{}^ik_1 - h^2 \sum_{j=1}^{s} {}^id_j,$$

$$^i k_1 = {}^i f(x_n, {}^1 y_n, {}^2 y_n, .., {}^s y_n),$$

$$^i d_j = \frac{a_{ij}\tilde{L}_1}{1 + h\tilde{L}_1}. \ (i = 1, 2, .., s) \qquad (3.2)$$

Introducing the vector notations,

$$\tilde{Y}_n = ({}^1 y_n, {}^2 y_n, ..., {}^s y_n),$$

(3.2) becomes

$$\tilde{Y}_{n+1} = \tilde{B}_1 \tilde{Y}_n,$$

$$\tilde{B}_1 = (I + \frac{h}{1 + h\tilde{L}_1}\tilde{A}). \qquad (3.3)$$

We assume that the matrix $\tilde{A} = (a_{i,j})$ is normalizable

$$\tilde{T}\tilde{A}\tilde{T}^{-1} = diag\{\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, .., \tilde{\lambda}_s\},$$

by means of a nonsingular matrix $\tilde{T}$.

Proceeding in a similar way as §2, we set $\tilde{Z}_n$ by

$$\tilde{Z}_n = \tilde{T}\tilde{Y}_n.$$

Then (3.3) becomes

$$\tilde{Z}_{n+1} = (I + \frac{h}{1 + h\tilde{L}_1}\tilde{\Lambda})\tilde{Z}_n, \qquad (3.4)$$

$$\tilde{\Lambda} = diag\{\tilde{\lambda}_1, \tilde{\lambda}_2, .., \tilde{\lambda}_s\},$$

$$\tilde{Z}_n = [{}^1\tilde{z}_n, {}^2\tilde{z}_n, .., {}^s\tilde{z}_n],$$

or

$$^i z_{n+1} = (1 + \frac{h}{1 + h\tilde{L}_1}\tilde{\lambda})^i \tilde{z}_n, \ (i = 1, 2, .., s) \qquad (3.5)$$

and so the eigenvalues of $\tilde{B}_1$ are given by

$$\tilde{\rho}_i = 1 + \frac{h\tilde{\lambda}_i}{1 + h\tilde{L}_1}. \ (i = 1, 2, .., s) \qquad (3.6)$$

Setting

$$\tilde{\lambda}_j = \tilde{\alpha}_j + \tilde{\beta}_j i. \ (\tilde{\alpha}_j, \beta_j \in R) \qquad (3.7)$$

(3.7) reduces to

$$(1 + \frac{\tilde{\alpha}_i h}{1 + h\tilde{L}_1})^2 + (\frac{\tilde{\beta}_i h}{1 + h\tilde{L}_1})^2 < 1,$$

or

$$\frac{h}{(1 + h\tilde{L}_1)^2}\{2\tilde{\alpha}_i(1 + h\tilde{L}_1) + (\tilde{\alpha}_i^2 + \tilde{\beta}_i^2)h\} < 0.$$

$$(i = 1, 2, .., s) \qquad (3.8)$$

Therefore, let

$$\tilde{L}_1 \geq -\frac{1}{2\tilde{\alpha}_i}(\tilde{\alpha}_i^2 + \tilde{\beta}_i^2).(i = 1, 2, .., s) \qquad (3.9)$$

Then the numerical processes (3.4) is stable.

We study $\tilde{L}_1$ of (3.9) in more detail, firstly we consider the case $\tilde{\lambda}_i \in R(i = 1, 2, .., s)$.

Case (I). $\tilde{\beta}_i = 0 (i = 1, 2, .., s)$.

Putting $\tilde{\beta}_i = 0$ in (3.9), we find

$$\tilde{L}_1 \geq -\frac{1}{2}\tilde{\alpha}_i. \ (i = 1, 2, .., s) \qquad (3.10)$$

From the assumption $\tilde{\alpha}_i < 0$ $(i = 1, 2, .., s)$, we have

$$Tr(A) \leq Re(\tilde{\lambda}_i). \qquad (3.11)$$

Therefore, let

$$\tilde{L}_1 = -\frac{1}{2}Tr(A). \qquad (3.12)$$

Then $\tilde{L}_1$ satisfies the condition (3.10). Summing the results, we have the following results.

Theorem [2]. Let $\tilde{A}$ be normal matrix whose eigenvalue are real and negative. Let

$$\tilde{L}_1 = \frac{-1}{2}Tr(A).$$

Then the numerical processes (3.2) is stable.

Secondly we study the stability conditions when $\tilde{\lambda}_i (i = 1, 2, .., n)$ are complex .

We use the following Schure's inequality theorem.

Theorem (Schure's inequality). Let $\lambda_1, \lambda_2, ..., \lambda_n$ be eigenvalues of $n \times n$ matrix $\tilde{A}$, then

$$\sum_{j=1}^{s} |\lambda_j|^2 \leq \|\tilde{A}\|, \qquad (3.13)$$

where $\|.\|$ denotes the euclidean matrix norm.

We consider the case $Re(\tilde{\lambda}_i) < 0$.

Case (II) $Re(\tilde{\lambda}_i) \leq -d$ $(d > 0), \tilde{\beta}_i \neq 0$.

$$(i = 1, 2, .., s)(3.14)$$

We have

$$\frac{-1}{2\tilde{\alpha}_i}(\tilde{\alpha}_i^2 + \tilde{\beta}_i^2) \leq \frac{1}{2d}(\tilde{\alpha}_i^2 + \tilde{\beta}_i^2). \qquad (3.15)$$

From (3.13) and (3.15), we have the following theorem.

Theorem [3]. We assume that the matrix $\tilde{A}$ is normal and the eigenvalue of $\tilde{A}$ satisfy (3.14). Let

$$\tilde{L}_1 = \frac{1}{2d}\|\tilde{A}\|.$$

Then the numerical processes (3.2) is stable, where $\|, \|$ is the euclidean norm in R^n.

Lastly , we study stability when the eigenvalues of $\tilde{A}$ satisfy

$$|\pi - arg\tilde{\lambda}_i| \leq \alpha, \ (i = 1, 2, .., s) \tag{3.16}$$

which implies

$$\frac{Im(\tilde{\lambda}_i)}{Re(\tilde{\lambda}_i)} \leq tan\alpha, \tag{3.17}$$

and so,we have

$$\tilde{\alpha}_i^2 \ + \ \tilde{\beta}_i^2; \leq \ \tilde{\alpha}_i^2(1 \ + \ tan^2\alpha), \tag{3.18}$$

which leads to

$$\frac{-1}{2\tilde{\alpha}}(\tilde{\alpha}_i^2 \ + \ \tilde{\beta}_i^2) \ \leq \ -\frac{1}{2}(1 \ + \ tan^2\alpha)|\tilde{\alpha}_i|. \tag{3.19}$$

From (3.9) and (3.19), we have the following theorem.

Theorem [4]. We assume that the matrix $\tilde{A}$ is normal and the eigenvalue of $\tilde{A}$ satisfy (3.16). Let

$$\tilde{L}_1 \ = \ -\frac{1}{2}(1 \ + \ tan^2\alpha)Tr(A).$$

Then the numerical processes (3.2) is stable.

4 Numerical Example

Using the numerical process (3.2), we present some numerical tests to sure the results derived in this paper. We consider the simple hyperbolic initial boundary problem which is taken from Richtmyer and Morton [8],

$$u_t \ + \ u_x \ = \ 0, \ 0 \ < x \ \leq 1, \ t \ > \ 0,$$
$$u(0,t) \ = \ 0, \ \ t \ > \ 0,$$
$$u(x,0) \ = \ sin(\pi x), \ 0 \ \leq \ x \ \leq \ 1.$$

Here the interval [0,1] is divided into m equal subintervals of length $\triangle x \ = \ \frac{1}{m}$ and

$$\{x_j; \ x_j = \ j \triangle x, j \ = \ 1(1)m\}.$$

We approximate u_x by

$$u_x|_{x=x_j} \ = \ \frac{(u_j \ - \ u_{j-1})}{\triangle x},$$

where $u_j(t) \ \cong u(x_j,t)$ denotes the continuous time approximation evaluated at x_j. The continuous time grid function $U \ = \ [U_1, U_2, ..., U_m]$ satisfies the semidiscreate problem,

$$\dot{U} \ = \ \frac{AU}{\triangle x},$$

$$A = \begin{pmatrix} -1 & 0 & 0 & .. & 0 \\ & -1 & 1 & 0.. & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0.. & -1 & 1 & 0 \\ 0.. & .. & .. & -1 & 1 \end{pmatrix}.$$

All eigenvalues λ are equal to

$$\lambda_i[A] \ = \ -1.$$

Its theoretical solution is

$$u(x,t) \ = \ sin((x-t)\pi.$$

We use the scheme (3.2) with (3.12).

A calculation are shown in TABLE I, II,III,IV.

TABLE

The data are the absolute error of numerical solution with step size

$$\tau \ = \ \frac{1}{100}, \ \triangle x \ = \ \frac{1}{10}, \ \frac{\tau}{\triangle x} = \frac{1}{10},$$

x	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$
t 0.1	-0.603E+0	-0.211E+0	-0.221E-1
t 0.5	-0.953E+0	0.647E+0	0.279E+0
t 0.7	0.951E+0	-0.965E+0	-0.693E+0
t 0.9	-0.587E+0	-0.954E+0	-0.984E+0

x	$\frac{7}{10}$	$\frac{10}{10}$
t 0.1	0.938E-1	0.132E+0
t 0.5	0.147E-1	0.191E+0
t 0.7	-0.329E+0	0.101E+0
t 0.9	-0.735E+0	-0.182E+0

$\tau = \frac{1}{50}$, $\triangle x = \frac{1}{10}$, $\frac{\tau}{\triangle x} = \frac{1}{5}$

	x	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$
t	0.1	-0.103E+0	0.185E-1	0.374E-1
t	0.5	-0.953E+0	-0.656E+0	-0.315E+0
t	0.7	-0.951E+0	-0.969E+0	-0.716E+0
t	0.9	-0.587E+0	-0.955E+0	-0.995E+0

	x	$\frac{7}{10}$	$\frac{10}{10}$
t	0.1	0.391E-1	0.150E-1
t	0.5	-0.657E-1	0.182E+0
t	0.7	-0.384E+0	0.486E-1
t	0.9	-0.774E+0	-0.261E+0

$\tau = \frac{1}{10}$, $\triangle x = \frac{1}{10}$, $\frac{\tau}{\triangle x} = \frac{1}{10}$

	x	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$
t	0.1	0.154E+0	0.110E+0	0.244E-1
t	0.5	-0.818E+0	-0.522E+0	-0.355E+0
t	0.7	-0.100E+1	-0.898E+0	-0.746E+0
t	1.0	-0.588E+0	-0.971E+0	-0.112E+1

	x	$\frac{7}{10}$	$\frac{10}{10}$
t	0.1	-0.170E-1	-0.154E+0
t	0.5	-0.119E+0	0.286E+0
t	0.7	-0.507E+0	0.134E+0
t	1.0	-0.112E+1	-0.574E+0

$\tau = \frac{1}{5}$, $\triangle x = \frac{1}{10}$, $\frac{\tau}{\triangle x} = 2$

	x	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$
t	0.2	0.206E+0	0.147E+0	0.326E-1
t	0.4	-0.343E+0	-0.179E+0	-0.843E-1
t	0.6	-0.820E+0	-0.606E+0	-0.473E+0
t	1.0	-0.810E+0	-0.107E+1	-0.128E+1

	x	$\frac{7}{10}$	$\frac{10}{10}$
t	0.2	-0.946E-1	-0.206E+0
t	0.4	0.429E-1	0.189E+0
t	0.6	-0.158E+0	0.381E+0
t	1.0	-0.117E+1	-0.232E+0

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