Delay-Dependent Robust Stability Tests for CSTR Systems with Recycle Including Input Nonlinearities

KOSTAS HRISSAGIS[1], JONATHAN D. ENGELSTAD[2], and OSCAR D. CRISALLE[2]
[1]Experian-Scorex, Athos Palace, 2 Rue de la Lujerneta, 98000 MONACO

[2]Chemical Engineering Department, University of Florida, Gainesville FL 32611, USA

Abstract: The robust stabilization of an integrated reactor/separator module with recycle is considered. Linear time-delay models are employed with system nonlinearities appearing in two different terms: the first term includes perturbations which are allowed to be nonlinear and/or time-varying, and the second term accounts for nonlinearities in the input channel. For a class of uncertain plants with delays in the state variables, sufficient robust stabilization conditions are derived. These conditions are given in terms of scalar inequalities that are easily calculated. Moreover, they do not require the solution of Lyapunov or Riccati equations. Instead, induced norms and corresponding matrix measures are used to yield stability criteria which are easy to evaluate. A key observation is that nonlinearities and plant uncertainty may destabilize the time-delay system. Finally, a design procedure to find a robustly stabilizing feedback matrix is given followed by an example that thoroughly illustrates the results. IMACS/IEEE CSCC'99 Proceedings, Pages:6661-6666

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1 Introduction

Many processes exhibit dynamic behavior that is significantly affected by time delays. The tranport of reactants across membranes or the transmission of signals by the circulation of hormones, are examples of events that can induce a delayed outcome on the regulation of reaction paths in biochemical processes [1]. Delays are also imposed by process design constraints as is the case of illuminated thermochemical reactions in the presence of delayed feedback, and reactors which recycle unreacted feed material.

Chemical process control systems are typically designed using the unit operations approach. That is, controllers are designed for each piece of equipment or unit in a plant, and then any conflicts between control loops are reconciled [2]. As Price and Georgakis [3] demonstrate in their plantwide modular control design, two of the candidate structures that frequently introduce time lags in the model are materials recycle and the chemical reactor/separator module. Both operations are used extensively in the chemical processing industry. In most of the literature on recycling, the process models assume that no time delays are present in the recycle stream [4]. This implies that the separation process and the return time to the reactor are instantaneous. While this assumption makes the analysis simpler, it is nevertheless unrealistic. The process of recycling requires a finite amount of time which introduces a delay into the system since both the concentration of the reactants and the temperature in the reactor depend on some past time.

The description of time-delay systems leads to differential-difference equations, the solutions of which require knowledge of past values of the system variables. The response of a system with a time delay can be quite complex. For example, studies of isothermal reactions indicate that delayed feedback may stabilize unstable stationary states, or destabilize an otherwise stable steady state [5]. It is then evident that the existence of time delays may cause major difficulties in the design and implementation of control and can cause significant performance deterioration. A variety of dead-time compensation techniques have been proposed. Much attention has been given to the Smith predictor which effectively removes the time delay from the characteristic equation if the process model is perfect. However, it is well known that this technique can give unacceptable closed-loop responses in the presence of plant/model mismatch [6].

Because the introduction of time delays makes the analysis much more complicated, convenient methods to determine stability have long been sought. Lyapunov theory has played a central role in the stability analysis of ordinary or time-delay dynamic systems [7]. However, general systematic procedures to construct appropriate Lyapunov functions are yet not available. Of the existing stabilizing approaches, techniques that make use of differential inequalities are highly appealing. These techniques have features useful for design, and have been used to analyze ordinary as well as time-delayed systems [8]. Two kinds of criteria have been developed: conditions that are independent of the size of time delay, and delay-dependent stability criteria. For an extensive list of references see [7].

Although linear control theory has a wide range of applicability, many systems of interest display nonlinear features that must be considered in practice. For example, actuators have physical limitations that may cause saturation during operation. If such nonlinearities are not taken into account during control system design, integral wind-up or limit cycles may occur [9]. The stability of linear systems with saturating actuators has been studied extensively [10]; however, there are few reports available on the robust stabilization of time-delay systems with input nonlinearities, let alone chemical reactors with modeling uncertainty. Hence the problem of designing robust controllers to stabilize uncertain CSTR models with time-delay and input nonlinearities in the presence of uncertainty is well motivated.

In this paper the stabilization of a linear model of an integrated reactor/separator module with recycle is studied. Nikolaou and Hanagandi [11] have shown that there exist nonlinear systems that are virtually linear for a nonvanishingly small range of inputs as well as nonlinear systems that can be approximated by linear models. In this work, linear time-delay models are employed with system nonlinearities appearing in two different terms: the first term includes perturbations which are allowed to be nonlinear and/or time-varying, and the second term represents nonlinearities in the input channel. The latter class includes input saturation as a special case.

In section 2, a mathematical model of a chemical reactor with recycle is developed as an example of systems that include delays and uncertainty. Robust stabilization conditions are derived in Section 3 for any given plant that contains delays in the state variables and plant uncertainty. The key observation is that nonlinearities and plant uncertainty may destabilize the time-delay system. Finally, a design procedure to find a robustly stabilizing feedback matrix is given in Section 4 followed by an example that illustrates the findings.

2 Motivation and Problem Formulation

Consider a continuous stirred tank reactor (CSTR) in which a first order reaction $A \rightarrow B$ occurs. The dimensionless equations describing the conservation of mass and energy in the CSTR with recycle stream are given by

$$\dot{x}_1(t) = f_1(x) := -x_1(t) + (1-\lambda)x_1(t-h) + D_a e^{\frac{x_2(t)}{1+x_2(t)/\zeta}}(1-x_1(t)) \quad (1)$$

$$\dot{x}_2(t) = f_2(x) := -x_2(t) + (1-\lambda)x_2(t-h) + BD_a e^{\frac{x_2(t)}{1+x_2(t)/\zeta}}(1-x_1(t)) - \beta(x_2(t) - x_{2c}(t)) \quad (2)$$

where the dimensionless variables x_1 and x_2 refer to the extent of conversion and the temperature in the reactor, respectively, and where x_{2c} is the temperature of thr cooling stream. The remaining dimensionless groups are defined in the Notation section. It is useful to remark that equations (1) and (2) require knowledge of past values of the variables x_1 and x_2. In the absence of time delay but with the recycle stream still operating, equations (1)-(2) reduce to

$$\dot{x}_1(t_n) = -x_1(t_n) + D e^{\frac{x_2(t_n)}{1+x_2(t_n)/\zeta}}(1-x_1(t_n)) \quad (3)$$

$$\dot{x}_2(t_n) = -x_2(t_n) + BD e^{\frac{x_2(t_n)}{1+x_2(t_n)/\zeta}}(1-x_1(t_n)) - \beta_n(x_2(t_n) - x_{2c}(t_n)) \quad (4)$$

where $t_n = \lambda t$, $D = D_a/\lambda$, and $\beta_n = \beta/\lambda$. Equations (3)-(4) are the well-known equations for a CSTR given by Uppal *et al*. [4].

Suppose that the control objective is that of regulating the extent of conversion of the reactant (x_1), by controlling the temperature of the cooling stream x_{2c}. Defining $u = x_{2c}$ as the input signal and then expanding (3)-(4) in a Taylor series and linearizing around a steady-state operating point, a linear delay-differential equation in matrix form $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{A}_d\boldsymbol{x}(t-h) + \boldsymbol{b}u(t)$ results, where $\boldsymbol{x} = [x_1 \quad x_2]^{\mathrm{T}}$, $\boldsymbol{f} = [f_1 \quad f_2]^{\mathrm{T}}$, $\boldsymbol{A} = [\partial \boldsymbol{f}/\partial \boldsymbol{x}]_{ss}$, $\boldsymbol{A}_d = [\partial \boldsymbol{f}/\partial \boldsymbol{x}(t-h)]_{ss}$, and $\boldsymbol{b} = [\partial \boldsymbol{f}/\partial u]_{ss}$.

When designing a control system, it is important to take into account modeling uncertainties related to the linearization process, or originating from various other sources: identification error, model reduction for design purposes, variations of the plant parameters during operation and other inaccuracies. Because the issue of robustness is part of the focus of this work, the model considered is given by the state equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{A}_d\boldsymbol{x}(t-h) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{g}(\boldsymbol{x}(t),t) + \boldsymbol{g}_d(\boldsymbol{x}(t-h),t) \quad (5)$$

which belongs to the class of uncertain time-delay systems where $\boldsymbol{g}(\boldsymbol{x}(t),t)$ and $\boldsymbol{g}_d(\boldsymbol{x}(t-h),t)$ represent nonlinear, possibly time-varying, modeling perturbations.

3 Theoretical Developments

A robust stability analysis is developed for time-delay systems that may be affected by nonlinearities in the input channel. Prior to the discussion of robust stability some useful concepts are presented. Let $\boldsymbol{u}(t) \quad ^m$ be the input vector and let $\boldsymbol{N}(\boldsymbol{u})$ be a nonlinear map of the imput.

Definition 1. Given a continuous nonlinear mapping $\boldsymbol{N}: \quad ^m \rightarrow \quad ^m$, and two real numbers p and q such that $q > p$, $\boldsymbol{N}$ is said to lie inside a sector $[p, q]$ if $\boldsymbol{N}$ satisfies the following two properties: (*i*) $\boldsymbol{N}(\boldsymbol{0}) = \boldsymbol{0}$, and (*ii*)

$$\left\| \boldsymbol{N}(\boldsymbol{u}(t)) - \frac{p+q}{2}\boldsymbol{u}(t) \right\| \le \frac{q-p}{2}\|\boldsymbol{u}(t)\| \quad (6)$$

where $(p+q)/2$ is the center of the sector and $(q-p)/2$ is its radius.

Definition 2. The matrix measure is a function $\mu: \quad ^{nxn} \rightarrow \quad , \ \mu(\boldsymbol{A}) = \lim_{\varepsilon \to 0^+} \frac{\|\boldsymbol{I} + \varepsilon\boldsymbol{A}\| - 1}{\varepsilon}$ where $\| \ \|$ is an induced matrix-norm on $\quad ^{nxn}$ [12].

The following properties are of relevance: (i) $\mu(\ )$ is a convex function, (ii) $\mu(\delta \boldsymbol{A}) = \delta\mu(\boldsymbol{A})$, and (iii) $\mathbf{Re}\lambda(\boldsymbol{A}) \leq \mu(\boldsymbol{A})$, where λ is any eigenvalue of matrix $\boldsymbol{A}$.

In the presence of input nonlinearities the uncertain time delay system is represented by

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{A}_d\,\boldsymbol{x}(t-h) + \boldsymbol{B}\boldsymbol{N}(\boldsymbol{u}(t)) \\ &\quad + \boldsymbol{g}(\boldsymbol{x}(t),t) + \boldsymbol{g}_d(\boldsymbol{x}(t-h),t) \qquad (7) \\ \boldsymbol{x}(\theta) &= \phi(\theta) \quad , \quad \theta \in [-h,\, 0] \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) \qquad (8) \end{aligned}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector with initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$; $\boldsymbol{u}(t) \in \mathbb{R}^m$ is the input vector; $\boldsymbol{y}(t) \in \mathbb{R}^p$ is the output vector; $\boldsymbol{A}_i$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are constant matrices of appropriate dimensions; $\phi(t)$ is a continuous vector-valued initial function; and $h > 0$ is the time delay. The vector function $\boldsymbol{g}(\boldsymbol{x}(t),t) \in \mathbb{R}^n$ and $\boldsymbol{g}_d(\boldsymbol{x}(t-h),t) \in \mathbb{R}^n$ represent nonlinear modeling perturbations that depend on the current state $\boldsymbol{x}(t)$ and the delayed state $\boldsymbol{x}(t-h)$ of the system, respectively. No statistical information is required about the uncertainty vectors $\boldsymbol{g}$ and $\boldsymbol{g}_d$; it is only assumed that the modeling uncertainties satisfy the following norm-bounds:

$$\|\boldsymbol{g}(\boldsymbol{x}(t),t)\| \leq k\|\boldsymbol{x}(t)\| \qquad (9)$$

and

$$\|\boldsymbol{g}_d(\boldsymbol{x}(t-h),t)\| \leq k_d\|\boldsymbol{x}(t-h)\| \qquad (10)$$

where k and k_d are *a priori* known positive real constants, and the operator $\|\ \|$ may be any appropriate vector norm.

In this article a state feedback control law of the form

$$\boldsymbol{u}(t) = \boldsymbol{F}\boldsymbol{x}(t) \qquad (11)$$

where $\boldsymbol{F}$ is a matrix of appropriate dimensions, is used to derive the robust stability conditions. The objective can be stated as follows: *find conditions that* $\boldsymbol{F}$ *must satisfy in order to asymptotically stabilize the closed-loop* (7)-(8) *for all modeling perturbations that conform with the norm bounds* (9)-(10). Any matrix $\boldsymbol{F}$ that stabilizes the uncertain delayed system is said to be *robustly stabilizing*.

Theorem. *Suppose that the plant uncertainties satisfy conditions (9)-(10) and the following inequality holds:*

$$-\mu(\bar{\boldsymbol{A}} + \boldsymbol{A}_d) - \tfrac{q-p}{2}\,\|\boldsymbol{B}\boldsymbol{F}\| - k - k_d - hM > 0 \qquad (12)$$

where $\bar{\boldsymbol{A}} = \boldsymbol{A} + \frac{p+q}{2}\boldsymbol{B}\boldsymbol{F}$ *and*

$$M = \|\boldsymbol{A}_d\bar{\boldsymbol{A}}\| + \|\boldsymbol{A}_d\boldsymbol{A}_d\| + \|\boldsymbol{A}_d\|(\tfrac{q-p}{2}\|\boldsymbol{B}\boldsymbol{F}\| + k + k_d).$$

Then the uncertain delayed system (7)-(8) is asymptotically stable under the feedback law (11).

The proof of the Theorem is given in the Appendix. When the time delay h is not exactly known, the Theorem can be alternatively stated in the following way which yields an upper bound on h: Let the feedback (11) be implemented where $\boldsymbol{F}$ is a known matrix, then the closed loop system (7)-(8) is asymptotically stable if the delay h is bounded by

$$0 < h < \bar{h} := \frac{-\mu(\bar{\boldsymbol{A}} + \boldsymbol{A}_d) - \frac{q-p}{2}\|\boldsymbol{B}\boldsymbol{F}\| - k - k_d}{M} \qquad (13)$$

Remarks

- The tightness of the bound in (12) or (13) varies with the chosen norm and the corresponding matrix measure [12]. In other words, it is possible to determine stability with a given norm and matrix measure while with other choices the stability condition may not hold. The largest bound computed for the 1, 2, or infinity norms should be selected.
- When checking the asymptotic stability of a given uncertain-delay system one should try the 1 or infinity vector norms first, avoiding the costlier eigenvalue computations associated with the 2-norm. The freedom in choosing a suitable norm and matrix measure to improve the stability condition resembles that of constructing an appropriate Lyapunov function candidate in the well-known and widely used Lyapunov approach for determining stability.
- For the nominal case where the uncertainty is negligible (*i.e.*, $\boldsymbol{g}$ and $\boldsymbol{g}_d$ are identically zero) and the delay $h = 0$, inequality (12) of the Theorem reduces to $\mu(\bar{\boldsymbol{A}} + \boldsymbol{A}_d) < 0$, which implies that $\bar{\boldsymbol{A}} + \boldsymbol{A}_d$ is asymptotically stable, since
$$\mathbf{Re}\lambda(\bar{\boldsymbol{A}} + \boldsymbol{A}_d) \leq \mu(\bar{\boldsymbol{A}} + \boldsymbol{A}_d) < 0.$$
When $h = 0$, condition (12) simply means that $\bar{\boldsymbol{A}} + \boldsymbol{A}_d$ should be stable enough to overcome the difficulty posed by the time delay in the system. It is thus evident that the time delay can destabilize an otherwise stable closed loop.
- Besides the well-known 1, 2, and infinity norms, other induced norms and matrix measures involving weighting parameters may be utilized in the stability conditions. As an example consider the following weighted matrix norm and corresponding matrix measure:
$$\|\boldsymbol{A}\|_w = \max_i \sum_j \tfrac{w_j}{w_i}|a_{ij}|,$$
$$\mu_w(\boldsymbol{A}) = \max_i \{a_{ii} + \sum_{j\neq i} \tfrac{w_j}{w_i}|a_{ij}|\} \qquad (14)$$

A simple optimization problem with respect to the arbitrary weighting factors is likely to yield less conservative bounds in (13). This is a topic that merits further investigation and is not pursued here.

The Theorem can also be specialized to the case of input saturation. For example the clasical saturation function $\boldsymbol{N}(\boldsymbol{u}(t)) = sat\,\boldsymbol{u}(t)$ is defined as follows (see Figure 1):

$$sat\,\boldsymbol{u}(t) = [sat u_1(t)\ \ sat u_2(t) \ldots\ \ sat u_m(t)]^{\mathrm{T}} \qquad (15)$$

where

$$sat u_i(t) = \begin{cases} \underline{u}_i, & u_i \leq \underline{u}_i \\ u_i, & \underline{u}_i < u_i < \bar{u}_i \\ \bar{u}_i, & u_i \geq \bar{u}_i \end{cases} \quad , \quad i = 1,2,\ldots,m \qquad (16)$$

and where $\underline{u}_i$ and $\bar{u}_i$ are real scalars representing lower and upper saturation limits, respectively. In this case, condition (12) is still applicable with $[p,q]=[0,1]$.

4 Design Procedure

The previous analysis is used as a paradigm to propose an iterative procedure for selecting a matrix $\boldsymbol{F}$ to satisfy the robust stability conditions.

Step 1: Given the norm bounds of the plant uncertainty, select distinct negative eigenvalues $\lambda_i,\ \ i=1,2,\ldots,n$ for the matrix $\bar{\boldsymbol{A}}$.

Step 2: When input nonlinearities are known to lie in sector $[p,q]$, find the control matrix $\boldsymbol{F}$ using a standard pole-placement technique. Next, check whether inequality (12) is satisfied. If so stop; a robust matrix $\boldsymbol{F}$ has been obtained. Otherwise continue to Step 3.

Step 3: Shift the system eigenvalues to the left using $\lambda_i = \lambda_{ii} - \Delta\lambda_i,\ \ i=1,2,\ldots,n$ where $\Delta\lambda_i > 0$; then go back to Step 2.

From the inverse point of view, one can estimate the sector where the input nonlinearities must lie so that the system remains asymptoticaly stable. In such case the first step of the algorithm remains the same. In Step 2, check condition (12) as if the nonlinearities were not present. If inequality (12) is satisfied go to Step 4. Otherwise, continue with Step 3.

Step 4: From inequality (12) and the equality $(p+q)/2=1$, find variables p and q such that the input nonlinearities lie in the sector $[p,q]$. The uncertain time-delay feedback system is then guaranteed to be stable.

As pointed out earlier, the present work can accomodate uncertain time delays. If this is the case, inequality (13) should be used instead of (12) in the above algorithm.

5 Example

This section demonstrates the applicability of the robustness conditions developed earlier to the stabilization of the Van de Vusse reaction kinetic scheme (Van de Vusse, 1964; [13]), characterized by the irreversible reactions

$$\mathrm{A} \rightarrow \mathrm{B} \rightarrow \mathrm{C},$$
$$2\mathrm{A} \rightarrow \mathrm{D},$$

taking place in an isothermal CSTR with recycle stream. The mass balances for components A and B are as follows:

$$\dot{C}_A(t') = -k_1 C_A(t') - k_3 C_A^2(t') + \lambda \frac{F}{V} C_{Af} + (1-\lambda)\frac{F}{V} C_A(t'-h') - \frac{F}{V} C_A(t') \qquad (17)$$

$$\dot{C}_B(t') = k_1 C_A(t') - k_2 C_B(t') + (1-\lambda)\frac{F}{V} C_B(t'-h') - \frac{F}{V} C_B(t') \qquad (18)$$

The control design problem in this case focuses on regulating the concentration of component B (output C_A) by manipulating the inlet flow rate (control F). Despite its simplicity, the Van de Vusse reaction scheme displays some interesting behavior: the reactor exhibits a change in gain at peak conversion level, and has nonminimum-phase characteristics for operation to the left of this peak and minimum phase for operation to the right.

Define $t = t' F/V$ and $h = h' F/V$ as the dimensionless time and time-delay respectively. Linearizing about the steady state operating point (C_{As}, C_{Bs}, F_s), and writing all variables in deviation form yields the following time-delayed model for the reactor:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{A}_d \boldsymbol{x}(t-h) + \boldsymbol{B}u(t) + \boldsymbol{g}(\boldsymbol{x}(t),t) + \boldsymbol{g}_d(\boldsymbol{x}(t-h),t) \qquad (19)$$

where $\boldsymbol{x} = (\tilde{C}_A\ \tilde{C}_B)^{\mathrm{T}}$, $\boldsymbol{u} = \tilde{F}$, $\tilde{C}_A = C_A - C_{As}$, $\tilde{C}_B = C_B - C_{Bs}$, $\tilde{F} = F - F_s$, and

$$\boldsymbol{A} = \begin{bmatrix} -k_1 \dfrac{V}{F_s} - 2k_3 \dfrac{V}{F_s} C_{As} - 1 & 0 \\ k_1 \dfrac{V}{F_s} & -k_2 \dfrac{V}{F_s} - 1 \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} \lambda \dfrac{(C_{Af} - C_{As})}{F_s} \\ -\lambda \dfrac{C_{Bs}}{F_s} \end{bmatrix},$$

and

$$\boldsymbol{A}_d = \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}$$

Nominal values for the physical constants of the model and the operating conditions are given in Table 1. The recycle ratio is taken to be $\lambda = 0.75$, and the uncertainty bounds are $k = k_d = 0.25$. The input variable is saturated as described by (15)-(16) with $\underline{u} = 0$ and $\bar{u} = 1$. The time delay h is not exactly known and an upper bound for it is sought.

Using a standard pole-placement technique, fix the eigenvalues of $\bar{\boldsymbol{A}}$ to be {-3.94, -3.58}, and find the feedback matrix $F = [1.3 \quad 0.2]$. Condition (13) then gives the upper bound on h: (i) for the 1-norm, $\bar{h} = 2.09$, (ii) for the 2-norm, $\bar{h} = 3.12$, and (iii) for the ∞-norm, $\bar{h} = 1.94$. Therefore, for this specific example, $\bar{h} < 3.12$ guarantees the asymptotic stability of the system. For the case where no uncertainty is present the best upper bound on h is $\bar{h} = 4.1$.

Now suppose that the input nonlinearities lie in the sector [0.3 0.8] and the time delay is known to be $h = 2.8$. Choose the feedback matrix to be $F = [-2.81 \quad -0.52]$ which places the closed-loop poles at {-4, -3.84}. Then, using the 1-norm and matrix measure when no uncertainty is present, inequality (12) yields 0.17 which is greater than zero; therefore the system is stable.

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Notation:

B Dimensionless adiabatic temperature rise, $[(-\Delta H)C_{Af}\zeta \,/\, \rho C_p T_f]$

C_A Reactant concentration in the CSTR

C_{Af} Reactant feed concentration

C_p Specific heat

D_a Damköhler number, $(k^o e^{-\zeta} V / F)$

D Damköhler number, (D_a / λ)

E Activation energy

F Total flow rate into the CSTR

ΔH Heat of reaction

k^o Arrhenius factor

t Time

t_n Dimensionless time, $(t_n F / V)$

T Reactor tank temperature

T_c Temperature of coolant

T_f Feed temperature

V Reactor volume

x_1 Dimentionless concentration, $[(C_{Af} - C_A)/C_{Af}]$

$x_1(t-h)$ Dimentionless concentration x_1 at time *t - h*

x_2 Dimensionless temperature, $[(T - T_f)\zeta / T_f]$

x_{2c} Dimensionless coolant temperature, $[(T_c - T)\zeta / T_f]$

$x_2(t-h)$ Dimensionless concentration x_2 at time *t - h*

h Delay time for recycle stream

Greek Letters:

β Dimensionless heat transfer coefficient, $(vA / F\rho C_p)$

β_n Dimensionless heat transfer coefficient, (β / λ)

ζ Dimensionless activation energy, (E / RT_f)

λ Recycle ratio

λ_i i-th eigenvalue

ρ Density

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Appendix

A.1 Proof of Theorem 1.

Lemma [7]. *Let a scalar function* $f(t)$ *satisfy the inequality* $\dot{f}(t) \le -\alpha f(t) + \beta \sup_{t-h \le s \le t} f(s)$, $t \ge t_0$, *where* α, β *are real constants such that* $\alpha > \beta \ge 0$. *Then there exists scalars* $\gamma > 0$ *and* $K > 0$ *such that*

$$f(t) \le K \exp(-\gamma (t - t_0))$$

for $t \ge t_0$.

Consider the initial time to be zero and let $\boldsymbol{x}(t)$ be the solution of (7) for $t \ge 0$. Since $\boldsymbol{x}(t)$ is continuously differentiable for $t \ge 0$ write

$$\boldsymbol{x}(t) - \boldsymbol{x}(t-h) = \int_{t-h}^{t} \dot{\boldsymbol{x}}(s)\,ds = \int_{t-h}^{t} [\boldsymbol{A}\boldsymbol{x}(s) + \boldsymbol{A}_d \boldsymbol{x}(s-h) + \boldsymbol{B}\boldsymbol{N}(\boldsymbol{u}(s)) + \boldsymbol{g}(\boldsymbol{x}(s), s) + \boldsymbol{g}_d(\boldsymbol{x}(s-h), s)]\,ds \quad \text{(A.1)}$$

Substitute for $x(t-h)$ in (5) using (11) to obtain

$$\dot{\boldsymbol{x}}(t) = (\bar{\boldsymbol{A}} + \boldsymbol{A}_d)\boldsymbol{x}(t) - \boldsymbol{A}_d \int_{t-h}^{t} \{\bar{\boldsymbol{A}}\boldsymbol{x}(s) + \boldsymbol{A}_d \boldsymbol{x}(s-h) + \boldsymbol{B}[\boldsymbol{N}(\boldsymbol{u}(s)) - \tfrac{p+q}{2}\boldsymbol{u}(s)] + \boldsymbol{g}(\boldsymbol{x}(s), s) + \boldsymbol{g}_d(\boldsymbol{x}(s-h), s)\}\,ds + \boldsymbol{g}(\boldsymbol{x}(t), t) + \boldsymbol{g}_d(\boldsymbol{x}(t-h), t) \quad \text{(A.2)}$$

where the term $\frac{p+q}{2}\boldsymbol{Bu}(t)$ has been added and subtracted, and matrix $\bar{\boldsymbol{A}}$ is defined as $\bar{\boldsymbol{A}} = \boldsymbol{A}+\frac{p+q}{2}\boldsymbol{BF}$. The solution to (A.2) for $t \ge 0$ is expressed as the integral equation

$$\begin{aligned}\boldsymbol{x}(t) = {} & e^{(\bar{\boldsymbol{A}}+\boldsymbol{A}_d)t}\boldsymbol{x}(0) \\ & + \int_0^t e^{\mu(\bar{\boldsymbol{A}}+\boldsymbol{A}_d)(s-h)}\int_{t-h}^{t}(-\boldsymbol{A}_d)[\bar{\boldsymbol{A}}\boldsymbol{x}(\vartheta)+\boldsymbol{A}_d\boldsymbol{x}(\vartheta-h) \\ & + \boldsymbol{B}[\boldsymbol{N}(\boldsymbol{u}(\vartheta)) - \tfrac{p+q}{2}\boldsymbol{u}(\vartheta)] + (\boldsymbol{g}(\boldsymbol{x}(\vartheta),\vartheta) \\ & + \boldsymbol{g}_d(\boldsymbol{x}(\vartheta-h),\vartheta))]d\vartheta + \boldsymbol{g}(\boldsymbol{x}(s),s) + \boldsymbol{g}_d(\boldsymbol{x}(s-h),s)ds \end{aligned} \quad (A.3)$$

An upper bound on the norm of the solution of (A.3) can be found after taking the norm of both sides, using known norm properties, using inequality (6), and using $\|e^{\boldsymbol{A}t}\| \le e^{\mu(\boldsymbol{A})t}, t \ge 0$ [12], to get

$$\begin{aligned}\|\boldsymbol{x}(t)\| \le {} & \sup_{-h\le t\le h}\|\boldsymbol{x}(t)\| e^{\mu(\bar{\boldsymbol{A}}+\boldsymbol{A}_d)(t-h)} \\ & + \int_0^t e^{\mu(\bar{\boldsymbol{A}}+\boldsymbol{A}_d)(s-h)}\int_{t-h}^{t}[(\|\boldsymbol{A}_d\bar{\boldsymbol{A}}\| + \tfrac{q-p}{2}\|\boldsymbol{A}_d\|\|\boldsymbol{BF}\|)\|\boldsymbol{x}(\vartheta)\| \\ & + \|\boldsymbol{A}_d\boldsymbol{A}_d\|\|\boldsymbol{x}(\vartheta-h)\| + \|\boldsymbol{A}_d\|\|\boldsymbol{g}(\boldsymbol{x}(\vartheta),\vartheta)\| \\ & + \|\boldsymbol{g}_d(\boldsymbol{x}(\vartheta-h),\vartheta)\|]d\vartheta + \|\boldsymbol{g}(\boldsymbol{x}(s),s)\| + \|\boldsymbol{g}_d(\boldsymbol{x}(s-h),s)\|ds \end{aligned} \quad (A.4)$$

Now use the plant uncertainty bounds (9) and (10) in (A.4) and define $\chi := \sup_{-h\le t\le h}\|\boldsymbol{x}(t)\|$ to obtain

$$\begin{aligned}\|\boldsymbol{x}(t)\| \le {} & \chi e^{\mu(\bar{\boldsymbol{A}}+\boldsymbol{A}_d)(t-h)} \\ & + \int_0^t e^{\mu(\bar{\boldsymbol{A}}+\boldsymbol{A}_d)(s-h)}\int_{t-h}^{t}[\|\boldsymbol{A}_d\bar{\boldsymbol{A}}\|\|\boldsymbol{x}(\vartheta)\| + \|\boldsymbol{A}_d\boldsymbol{A}_d\|\|\boldsymbol{x}(\vartheta-h)\| \\ & + \|\boldsymbol{A}_d\|((\tfrac{q-p}{2}\|\boldsymbol{A}_d\|\|\boldsymbol{BF}\| + k)\|\boldsymbol{x}(\vartheta)\| + k_d\|\boldsymbol{x}(\vartheta-h)\|)]d\vartheta \\ & + k\|\boldsymbol{x}(s)\| + k_d\|\boldsymbol{x}(s-h)\|ds \end{aligned} \quad (A.5)$$

After carrying out the inner integration, inequality (A.5) can be written as

$$\begin{aligned}\|\boldsymbol{x}(t)\| \le {} & \chi e^{\mu(\bar{\boldsymbol{A}}+\boldsymbol{A}_d)(t-h)} \\ & + \int_0^t e^{\mu(\bar{\boldsymbol{A}}+\boldsymbol{A}_d)(s-h)} hM \sup_{s-2h\le\vartheta\le s}\|\boldsymbol{x}(\vartheta)\| \\ & + k\|\boldsymbol{x}(s)\| + k_d\|\boldsymbol{x}(s-h)\|ds \end{aligned} \quad (A.6)$$

where $M = \|\boldsymbol{A}_d\bar{\boldsymbol{A}}\| + \|\boldsymbol{A}_d\boldsymbol{A}_d\| + \|\boldsymbol{A}_d\|(\frac{q-p}{2}\|\boldsymbol{BF}\| + k + k_d)$. Let $z(t)$ be a signal that attains the equality sign in (A.6), *i.e.*,

$$\begin{aligned} z(t) = {} & \chi e^{\mu(\bar{\boldsymbol{A}}+\boldsymbol{A}_d)(t-h)} \\ & + \int_h^t e^{\mu(\bar{\boldsymbol{A}}+\boldsymbol{A}_d)(s-h)} hM \sup_{s-2h\le\vartheta\le s}\|\boldsymbol{x}(\vartheta)\| \\ & + k\|\boldsymbol{x}(s)\| + k_d\|\boldsymbol{x}(s-h)\|ds \end{aligned} \quad (A.7)$$

Then,

$$\begin{aligned} \dot{z}(t) = {} & \mu(\bar{\boldsymbol{A}}+\boldsymbol{A}_d)z(t) + hM \sup_{t-2h\le\vartheta\le t}\|\boldsymbol{x}(\vartheta)\| \\ & + k\|\boldsymbol{x}(t)\| + k_d\|\boldsymbol{x}(t-h)\| \end{aligned} \quad (A.8)$$

From equations (A.6) and (A.7) it is obvious that $\|\boldsymbol{x}(t)\| \le z(t)$ for $t \ge 0$. Hence,

$$\begin{aligned} & \sup_{t-2h\le\vartheta\le t}\|\boldsymbol{x}(\vartheta)\| \le \sup_{t-2h\le\vartheta\le t} z(\vartheta) \\ & \|\boldsymbol{x}(t-h)\| \le z(t-h) \le \sup_{t-2h\le\vartheta\le t} z(\vartheta) \end{aligned} \quad (A.9)$$

After substituting (A.9) in (A.8) the following differential inequality results:

$$\dot{z}(t) \le -(-\mu(\bar{\boldsymbol{A}}+\boldsymbol{A}_d) - k)z(t) + (hM + k_d)\sup_{t-2h\le\vartheta\le t} z(\vartheta) \quad (A.10)$$

After invoking the Lemma, $z(t) \le Ke^{-\gamma(t-h)}$, *i.e.*, $z(t)$ is asymptotically stable if

$$(-\mu(\bar{\boldsymbol{A}}+\boldsymbol{A}_d) - k) > (hM + k_d) > 0 \quad (A.11)$$

which concludes the proof. *Q.E.D.*

A.2 Matrix measure computation.

For the usual 1, 2, and infinity induced norms, the matrix measure is given by the simple formulas below where the induced norms are also included for completeness.

$$\|\boldsymbol{A}\|_1 = \max_j \sum_i |a_{ij}|, \quad \mu_1(\boldsymbol{A}) = \max_j (a_{jj} + \sum_{i\ne j}|a_{ij}|)$$

$$\|\boldsymbol{A}\|_2 = \lambda_{\max}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})^{1/2}, \quad \mu_2(\boldsymbol{A}) = \lambda_{\max}(\frac{\boldsymbol{A}^{\mathrm{T}}+\boldsymbol{A}}{2})$$

$$\|\boldsymbol{A}\|_\infty = \max_i \sum_j |a_{ij}|, \quad \mu_\infty(\boldsymbol{A}) = \max_i (a_{ii} + \sum_{j\ne i}|a_{ij}|)$$

where $\boldsymbol{A}^{\mathrm{T}}$ is the transpose of matrix $\boldsymbol{A}$, and $\lambda_{\max}$ denotes the maximum eigenvalue.

Table 1. Kinetic parameters and operating variables for the model used in the example.

k_1	50 h^{-1}
k_2	100 h^{-1}
k_3	10 L mol^{-1} h^{-1}
C_{as}	10 mol L^{-1}
V	1 L
C_{as}	3.0 mol L^{-1}
C_{Bs}	1.12 mol L^{-1}
F_s	34.3 L h^{-1}

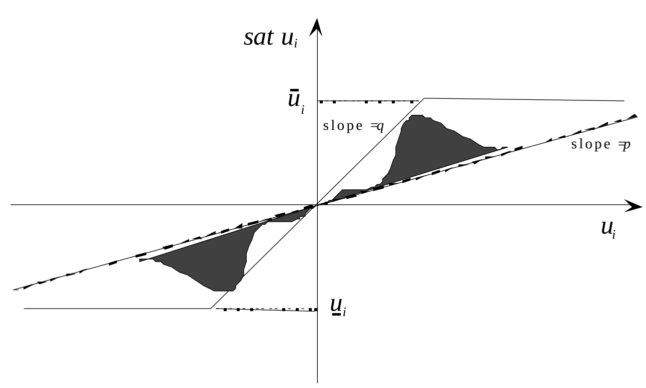

Figure 1. Actuator saturation with bounds $\underline{u}_i$ and $\bar{u}_i$, and sector nonlinearity $[p, q]$.