Generalized Algebraic Multiports and the Axiom of Choice

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Abstract: -Multiports which possess characteristics play a key role in electrical circuit theory. Resistors, capacitors, inductors and memristors as well as an infinite variety of higher order basic circuit elements belong to the class of algebraic multiports. In this paper the general coordinate free definition of an n-port having the characteristic is introduced and investigated in details. A concept of generalized algebraic multiport is also introduced. A deep relationship between the existence of time-varying characteristics of n-ports and the Axiom of Choice is emphasized..

Key-Words: -Characteristic of a Multiport, Generalized Algebraic Multiports, The Axiom of Choice

1 Preliminaries

Consider a collection of *abstract objects* each with two access points, called *terminals.* At this stage the objects are not restricted; for example they may be electrical, mechanical, thermodynamical, depending on what physical attributes we associate with them. In order to obtain a mathematical model, with each pair of terminals we associate two types of oriented scalar variables: "through" variables and "across" variables. Such pairs of variables are called *conjugate variables* since their scalar product always has the dimension of power. Typical conjugate variables are current and voltage (in case of electrical abstract objects), force and velocity (in case of mechanical objects), temperature and entropy change (in case of thermodynamical objects) etc. It is generally impossible to specify in advance the actual orientations of these variables. We therefore set up the so-called *frame of reference* in terms of which the actual orientation of "through" and "across" variables, can be specified.

A form of interdependencies between two-terminal objects, called *coupling,* can occur. A collection of two-terminal objects is said to be *closed* with respect to couplings if there is no object external to the collection influencing objects in it through couplings. A closed collection is *minimal* if no proper subcollection has the same closure property. Roughly speaking, an *multiport* is a minimal collection of two-terminal object, closed by means of couplings. A two-terminal object that belong to a multiport is commonly referred to as a *port.* Suppose that each port of a network has an associated pair of well defined oriented scalars: a *current* passing through the port from one terminal to another and a *voltage* between the pair of ordered terminals. Then, the multiport is called an *electrical multiport.* From this point on, without loss of generality, we shall restrict our consideration to electrical multiports.

A map whose domain is time is called a *signal.* Normally, the time domain is simply the set of real numbers which will denote by R. A signal whose co-domain is also the set of real numbers we call a real signal. Formally, an electrical n-port is characterized by a set $\mathcal{B}$ of all possible $2n$-tuples of real signals $(u_1(t), i_1(t); \ldots; u_n(t), i_n(t))$ that are simultaneously allowed on the ports, providing that $u_k(t)$ is the voltage signal and $i_k(t)$ is the current signal associated with an oriented port k, for $k = 1, \ldots n$ (see e.g. [1] and [5]). We assume that all these signals belong to the same universal set of real signals which we shall denote by S. The set S will be denoted by U when its members have the dimension of voltage and by I if its members have the dimension of current. Clearly, $\mathcal{B} \subseteq (U \times I)^n = S^{2n}$ and therefore $\mathcal{B}$ defines a relation in the signal set S which constitutes a multiport. This explains why the set $\mathcal{B}$ of all allowed $2n$-tuples of an n-port is called a *constitutive relation*. Sometimes $\mathcal{B}$ can be identified with the set of solutions of a system of equations. If this is the case such system of equations itself is called the system of constitutive relations.

Consider the structure $(S, + \mid \cdot)$ where $+$ is internal binary operation (sum operation) induced point by point from the corresponding binary operations in the field of real numbers $(\mathsf{R}, +, \cdot)$ and $\cdot$ is external binary operation (multiplication with real numbers), also induced point by point from the corresponding binary operations in the field of real numbers $(\mathsf{R}, +, \cdot)$. It is easy to show that this structure is a linear (vector) space. It is also clear that the structure $(S^{2n}, + \mid \cdot)$ where $+$ is a internal sum binary operation in S^{2n} induced from S in a natural way component by component and $\cdot$ is an external binary operation (multiplication with real numbers), also defined component by components, is a linear (vector) space.

Let $\mathcal{B} \subseteq (U \times I)^n = S^{2n}$ be the constitutive relation of an n-port. If $(\mathcal{B}, + \mid \cdot)$ is a linear (vector) subspace of S^{2n} (clearly, there are subsets of S^{2n} which are not linear), than the n-port characterized by $\mathcal{B}$ is a *linear n-port*. Denote $\mathcal{B}_o = \mathcal{B} \setminus \{(u_{1o}(t), i_{1o}(t); \ldots; u_{no}(t), i_{no}(t)\}$. Suppose that for every $2n$-tuple $(u_{1o}(t), i_{1o}(t); \ldots; u_{no}(t), i_{no}(t))$ in $\mathcal{B}$, the associated structure $(\mathcal{B}_o, +, \circ)$ is a linear space of S^{2n}. Then $\mathcal{B}$ is an affine subspace of S^{2n} and the n-port characterized by $\mathcal{B}$ is an *affine n-port*. Notice that for each affine n-port there is unique associated linear n-port. An n-port which is neither linear or affine is *nonlinear*. We say that an n-port is *time-invariant* if for each real number T and for each $2n$-tuple $(u_1(t), i_1(t); \ldots; u_n(t), i_n(t))$ from $\mathcal{B}$, the $2n$-tuple $(u_1(t-T), i_1(t-T); \ldots; u_n(t-T), i_n(t-T))$ also belongs to $\mathcal{B}$.

2 The concept of characteristic of an abstract subset of S^{2n}

Each $2n$-tuple of signals $(\hat{s}_1(t), \check{s}_1(t); \ldots; \hat{s}_n(t), \check{s}_n(t)) \in S^{2n}$ can be viewed as a maps from R to R^{2n}. Therefore, any subset $\mathcal{B}$ of S^{2n} can be treated as a collection of maps from R to R^{2n}. For some $t_o \in \mathsf{R}$ denote by B_{t_o} the set obtained from $\mathcal{B}$ by restricting the domain of maps from R to $\{t\}$. In other words, B_{t_o} is the set of all $2n$-tuples of numbers $(\hat{s}_1(t_o), \check{s}_1(t_o); \ldots; \hat{s}_n(t_o), \check{s}_n(t_o))$ where $(\hat{s}_1(t), \check{s}_1(t); \ldots; \hat{s}_n(t), \check{s}_n(t)) \in \mathcal{B}$ and represents a subset of R^{2n}. We shall call it the *chart* of $\mathcal{B}$ at the moment $t_o \in \mathsf{R}$. The collection $A(\mathcal{B}) = \{B_t \mid B_t \subseteq \mathsf{R}^{2n},\ t \in \mathsf{R}\}$ of all charts of $\mathcal{B}$ we shall call the *atlas* of $\mathcal{B}$. Notice that with every set $\mathcal{B}$ there is a uniquely associated an atlas. We say that a map from R to R^{2n} is in accordance with the atlas of $\mathcal{B}$ if for every $t \in \mathsf{R}$ the image of t belongs to $B_t \subseteq \mathsf{R}^{2n}$. Denote by $\mathcal{A}$ be the set of *all* maps from R to R^{2n} that are in accordance with the atlas of $\mathcal{B}$. If $\mathcal{A}$ exists then clearly $\mathcal{B} \subseteq \mathcal{A} \subseteq S^{2n}$.

Notice that $\mathcal{A}$ exists iff there exists a *choice function* on $A(\mathcal{B})$ which simultaneously chooses an element from each member of $A(\mathcal{B})$, that is, which selects a 2n-tuple of real numbers from *each* chart B_t of the atlas $A(\mathcal{B})$. An easy inductive argument shows that such a function exists if atlas $A(\mathcal{B})$ is finite. Otherwise, if $A(\mathcal{B})$ is infinite, the existence of choice function and hence the existence of the set $\mathcal{A}$ cannot be guaranteed unless we postulate that The Axiom of Choice holds (see for example [7]). This Axiom and number of its equivalent formulations have far-reaching consequences in all fields of mathematics.

The most important family of subsets of S^{2n} which members have finite atlases, is the family of time-invariant subsets.

Theorem 1: Let $\mathcal{B}$ be a subset of S^{2n} and let $A(\mathcal{B})$ be its atlas. If $\mathcal{B}$ is time-invariant then atlas $A(\mathcal{B})$ contains exactly one chart.

Proof: Consider two charts $B_{t'}$ and $B_{t''}$ corresponding to two time instances t' and t''. Let $(\hat{b}'_1, \check{b}'_1; \ldots; \hat{b}'_n, \check{b}'_n)$ be an arbitrary member of

$B_{t'}$. Then there exists a $2n$-tuple of real signals $(\hat{s}_1(t), \check{s}_1(t); \ldots; \hat{s}_n(t), \check{s}_n(t))$ in $\mathcal{B}$ such that $\hat{b}'_k = \check{s}_k(t')$ and $\check{b}'_k = \check{s}_k(t')$ for $k = 1, ..., n$. Since $\mathcal{B}$ is time-invariant it follows that $(\hat{s}_1(t + t' - t''),$ $\check{s}_1(t + t' - t''); \ldots; \hat{s}_n(t + t' - t''),$ $\check{s}_n(t + t' - t''))$ also belongs to $\mathcal{B}$. For $t = t''$ this $2n$-tuple of real signals produces a $2n$-tuple of real numbers $(\hat{b}''_1, \check{b}''_1; \ldots; \hat{b}''_n, \check{b}''_n)$ which is clearly a member of $B_{t''}$. But $\check{s}_k(t'' + t' - t'') = \check{s}_k(t')$ and $\check{s}_k(t' + t' - t'') = \check{s}_k(t')$ and hence $\hat{b}'_k = \hat{b}''_k$ and $\check{b}'_k = \check{b}''_k$. Consequently, $(\hat{b}'_1, \check{b}'_1; \ldots; \hat{b}'_n, \check{b}'_n) \in B_{t''}$ and hence $B_{t'} \subseteq B_{t''}$. Using dual arguments we can prove that $B_{t''} \subseteq B_{t'}$. Accordingly $B_{t'} = B_{t''}$ for any pair of time instances t' and t''. ■

Let us now consider a general time-variant case of a subset $\mathcal{B}$ of S^{2n} and suppose that the associated set $\mathcal{A}$ exists. The existence of $\mathcal{A}$ can either be proved or forced by postulating The Axiom of Choice. Clearly any member of $\mathcal{B}$ is a maps from R to R^{2n}. Since $\mathcal{A}$ is defined to be the set of *all* maps from R to R^{2n} in accordance with the atlas of $\mathcal{B}$ and since S^{2n} contains *all* maps from R to R^{2n} it follows immediately that the following relations hold:

$$\mathcal{B} \subseteq \mathcal{A} \subseteq S^{2n}. \qquad (1)$$

Relating to the question whether the equality in the relation $\mathcal{B} \subseteq \mathcal{A}$ occurs or not all subsets of S^{2n} can be divided into two classes: those for which this equality does occur, and the remainder (in which it does not). This motivates the following definition of characteristics of a subset $\mathcal{B}$ of S^{2n}.

Definition 1: Let $\mathcal{B}$ be a subset of S^{2n} and let $\mathcal{A}$.be the set of *all* maps from R to R^{2n} that are in accordance with the atlas $A(\mathcal{B})$. If $\mathcal{B} = \mathcal{A}$ then atlas $A(\mathcal{B})$ is called the *characteristic of* $\mathcal{B}$.

If subset $\mathcal{B}$ of S^{2n} is time-variant then generally speaking the corresponding atlas $A(\mathcal{B})$ consists of infinite number of charts and consequently the concept of characteristic could be ill-defined, unless the Axiom of Choice is postulated. This explains why this Axiom is so important in theory of time-variant subsets of S^{2n}.

Notice that the concept of characteristic is normally addressed to the subsets $\mathcal{B}$ of S^{2n} which are time-invariant. Why is that? Simply because in this case the substitution of $\mathcal{B}$ with its characteristic provides the most efficient compression of information since all charts coincide (atlas consists of only one chart). On the other words, for time-invariant subset $\mathcal{B}$ of S^{2n} always there is $B \subseteq \mathsf{R}^{2n}$ such that for all $t \in \mathsf{R}$, $B_t = B$. Consequently, any subset $\mathcal{B}$ which is time-invariant is uniquely determined and completely characterized by the associated set $B \subseteq \mathsf{R}^{2n}$ [8]. For time-variant subsets $\mathcal{B}$ of S^{2n} which possess characteristics the compression obtained by substitution of $\mathcal{B}$ with its characteristic is not so evident.

3 Algebraic multiports

Let $\mathcal{B} \subseteq S^{2n}$ be the set of all $2n$-tuples of *voltage and current signals* $(i_1(t), \ldots, i_n(t), u_1(t), \ldots, u_n(t))$ that are allowed by an n-port, that is, let $\mathcal{B} \subseteq S^{2n}$ be the constitutive relation of an n-port. If subset $\mathcal{B}$ possesses the characteristic then we shall call it a (v, i)-characteristic. Accordingly, all multiports can be divided in two classes: those which does possess (v, i)-characteristics, called *resistive n-ports,* and the remainder (in which it does not). In case when n-port is time-invariant then all its charts coincide, that is, atlas consists of only one chart $B \subseteq \mathsf{R}^{2n}$. Consequently, any resistive n-port which is time-invariant is uniquely determined by a set $B \subseteq \mathsf{R}^{2n}$ of $2n$-tuples of values of voltages and currents. Usually we deal with resistive n-ports which atlases can be interpreted as the set of solutions of a system of algebraic time-varying equations in terms of port voltages and currents. These equations are also called constitutive relations of the resistive n-port.

Denote by q and ϕ electric flux (or simply *charge*) and is magnetic flux (or simply *flux*), respectively. These two variables are related to another two variables i (current) and v (voltage) respectively through the following two equations:

$$q^{(1)}(t) = i(t) \qquad (2)$$
$$\phi^{(1)}(t) = u(t), \qquad (3)$$

where $q^{(1)}(t)$ and $\phi^{(1)}(t)$ denote first derivatives of q and ϕ respectively. These two equations can be treated as degenerate versions of Maxwell equations. We presume here that signals $q(t)$ and $\phi(t)$ are differentiable at least in the sense of generalized functions. We shall adopt the following notation from [5]:

$$i^{(-1)} = i^{(-1)}(0)+\int_0^t i(\tau)d\tau \qquad (4)$$

$$u^{(-1)} = u^{(-1)}(0)+\int_0^t v(\tau)d\tau. \qquad (5)$$

where $i^{(-1)}(0)$ and $u^{(-1)}(0)$ are two arbitrary constants. Then, the equations (2) and (3) can be written in the following equivalent form: $q(t) = i^{(-1)}$ and $\phi(t) = u^{(-1)}$, respectively. We shall also adopt the following recurrent notation from [5]:

$$i^{(-p)}(t) = i^{(-p)}(t)+\int_0^t i^{(-p+1)}(t)d\tau \qquad (6)$$

$$u^{(-q)}(t) = u^{(-q)}(t)+\int_0^t u^{(-q+1)}(t)d\tau. \qquad (7)$$

where p and q are positive integers.

Let $\mathcal{B} \subseteq S^{2n}$ be the set of all $2n$-tuples of *voltage and current signals* $(i_1(t), \dots, i_n(t), u_1(t), \dots, u_n(t))$ that are allowed by an n-port and suppose that there exist a set $\mathcal{F} \subseteq S^{2n}$ of $2n$-tuples of signals $(a_1(t), \dots, a_n(t), b_1(t), \dots, b_n(t))$ such that for each $k = 1, \dots, n$, $a_k(t)$ is either current or charge and $b_k(t)$ is either voltage or flux provided that the set of all $2n$-tuples of signals $\mathcal{A}$ obtained from $\mathcal{F}$ by differentiating charge and flux signals and leaving current and voltage signals, coincides with $\mathcal{B}$. Suppose now that $\mathcal{F}$ has the characteristic. Since this characteristic is simply the atlas of $\mathcal{F}$ we may construct the set $\mathcal{F}$ as the set of *all* maps from R to R^{2n} in accordance with this atlas. Furthermore, from $\mathcal{F}$ we can derive the set $\mathcal{A}$ which coincides with $\mathcal{B}$ and hence the characteristic of $\mathcal{F}$ uniquely determine the constitutive relation of the considered n-port. Such multiport is called an *algebraic multiport*[2, 3]. Special algebraic multiports are *resistors* (if for all $k = 1, ..., n$, the signals $a_k(t)$ are currents and $b_k(t)$ are voltages), *capacitors* (for all $k = 1, ..., n$, the signals $a_k(t)$ are charges and $b_k(t)$ are voltages), *inductors* (for all $k = 1, ..., n$, the signals $a_k(t)$ are currents and $b_k(t)$ are fluxes) and *memristors* (for all $k = 1, ..., n$, the signals $a_k(t)$ are charges and $b_k(t)$ are fluxes).

4 Higher order and generalized algebraic multiports

Let $\mathcal{B} \subseteq S^{2n}$ be the set of all $2n$-tuples of *voltage and current signals* $(i_1(t), \dots, i_n(t), u_1(t), \dots, u_n(t))$ that are allowed by an n-port. Suppose there exists a $2n$-tuple of nonnegative integers $(p_1, \dots, p_n, q_1, \dots, q_n)$ and a set $\mathcal{F} \subseteq S^{2n}$ of $2n$-tuples of *signals* $(j_1(t), \dots, j_n(t), v_1(t), \dots, v_n(t))$ with appropriate dimensions such that for $k = 1, \dots, n$, the signals $j_k^{(p_k)}(t)$, and $v_k^{(q_k)}(t)$ belong to S and the set of all $2n$-tuples of signals $(j_1^{(p_1)}(t), ..., j_n^{(p_n)}(t), v_1^{(q_1)}(t), ..., v_n^{(q_n)}(t))$ coincides with $\mathcal{B}$. Then, if such set $\mathcal{F}$ exists it completely characterize $\mathcal{B}$ and can be used as an alternative description. We shall call it the $(p_1, \dots, p_n, q_1, \dots, q_n)$-description of the n-port. In this context the description of the n-port via voltage and current signals coincides with $(0, \dots, 0, 0, \dots, 0)$-description. It is easy to construct simple example of an n-port which has no characteristic in $(0, \dots, 0, 0, \dots, 0)$-description but has it in $(p_1, \dots, p_n, q_1, \dots, q_n)$-description. Therefore if $\mathcal{B}$ does not have characteristic it is natural to ask whether $\mathcal{F}$ does. Suppose that $\mathcal{F}$ possesses the characteristic, to which we shall also address as $(p_1, \dots, p_n, q_1, \dots, q_n)$-characteristic of the n-port. Since this characteristic is simply the atlas of $\mathcal{F}$ we may construct the set $\mathcal{F}$ as the set of *all* maps from R to R^{2n} in accordance with this atlas. Furthermore, from $\mathcal{F}$ we can derive $\mathcal{B}$ and hence we conclude that $(p_1, \dots, p_n, q_1, \dots, q_n)$-characteristic of an n-port uniquely determines its constitutive relation $\mathcal{B}$. In particular the multiports with $(0, \dots, 0, 0, \dots, 0)$-characteristic are called resistors, the multiports having $(-1, \dots, -1, 0, \dots, 0)$-characteristic are called capacitors, those having $(0, \dots, 0, -1, \dots, -1)$-characteristic are called inductors and multiports with $(-1, \dots, -1, -1, \dots, -1)$-characteristic are called memristors. Beside resistors, capacitors, inductors and memristors a variety of multiports with $(p_1, \dots, p_n, q_1, \dots, q_n)$-characteristic, where for each $k \in \{1, \dots, n\}$, $p_k^2 + q_k^2 \in \{0, 1, 2\}$ belong to algebraic multiports with *mixed order.* All other multiports with $(p_1, \dots, p_n, q_1, \dots, q_n)$-characteristic, where at least for one $k \in \{1, \dots, n\}$, $p_k^2 + q_k^2 > 2$ are *higher order algebraic multiports* [5, 6].

Let $\mathcal{B} \subseteq S^{2n}$ be the set of all $2n$-tuples of *voltage and current signals* $(i_1(t), \ldots, i_n(t), u_1(t), \ldots, u_n(t))$ that are allowed by an n-port. Suppose there exists a $2n$-tuple of *operators* $(P_1, \ldots, P_n, Q_1, \ldots, Q_n)$ and a set $\mathcal{F} \subseteq S^{2n}$ of $2n$-tuples of *signals* $(j_1(t), \ldots, j_n(t), v_1(t), \ldots, v_n(t))$ with appropriate dimensions such that for all $k = 1, \ldots, n$, the signals $P_k j_k(t)$ and $Q_k v_k(t)$ belong to S and the set of all $2n$-tuples of signals $(P_1 j_1(t), \ldots, P_n j_n(t), Q_1 v_1(t), \ldots, Q_n v_n(t))$ coincides with $\mathcal{B}$. Then, if such set $\mathcal{F}$ exists it completely characterize $\mathcal{B}$ and may be used as an alternative description. We shall call it the $(P_1, ..., P_n, Q_1, ..., Q_n)$-description of $\mathcal{B}$. Suppose now that $\mathcal{F}$ has characteristic. Since this characteristic is simply the atlas of $\mathcal{F}$ we can built the set $\mathcal{F}$ as the set of *all* maps from R to R^{2n} in accordance with this atlas. Furthermore we can derive the set $\mathcal{B}$, from $\mathcal{F}$ and hence we conclude that the characteristic of $\mathcal{F}$ uniquely determine the constitutive relation $\mathcal{B}$ of the considered n-port. Therefore we shall call it the $(P_1, ..., P_n, Q_1, ..., Q_n)$-characteristic of $\mathcal{B}$. Those n-ports with possess $(P_1, ..., P_n, Q_1, ..., Q_n)$-characteristics we call *generalized algebraic n-ports* [8]. Notice that higher order algebraic multiports are special cases of generalized algebraic multiports obtained by interpreting the operators P_k and Q_k as high order differential operators, that is, by assuming $P_k j_k(t) = j_k^{(p_k)}(t)$ and $Q_k v_k(t) = v_k^{(q_k)}(t)$ for all $k = 1, \ldots, n$.

5 Conclusion

It was recognized by many authors (L. O. Chua [2, 4, 5, 6], Y-F. Lam [2, 3] and E. W. Szeto [6]) that multiports which possess characteristics play a major role in electrical circuit theory. The concept of characteristic was implicitly introduced in the context of algebraic multiports. Resistors, capacitors, inductors and memristors as well as an infinite variety of higher order basic circuit elements belong to this class. In this paper we formalize the definition of characteristic and raise to a general abstract level which is essentially coordinate-free. A deep relationship between the existence of time-varying characteristics and the Axiom of Choice is also pointed out. In particular a new notion of generalized algebraic multiport as a broad generalization of the notion of algebraic multiport, is introduced.

References:

[1] R. W. Newcomb, *Linear Multiport Synthesis*, McGraw-Hill, New York (1966).

[2] L. O. Chua and Y.-F. Lam, A theory of algebraic n-ports, *IEEE Transactions of Circuit Theory*, CT-20, No.4, pp.370-382 (1973).

[3] Y.-F. Lam, Characterization of time-invariant network elements, *Journal of The Franklin Institute*, Vol.298, No.1, pp.1-7 (1974).

[4] L. O. Chua, Nonlinear Circuit Theory in R. K. Brayton, L. O. Chua, J. D. Rhodes and R. Spence, *Modern Network Theory-An Introduction*, Guest lectures of the 1978 European Conference on Circuit Theory and Design, Editors: G. S. Moschtz and J. Neiryrck, Georgi. St. Saphorin, Switzerland (1978).

[5] L. O. Chua, Device modeling via basic nonlinear circuit elements, *IEEE Transactions of Circuits and Systems*, CAS-27, No.11, pp.1014-1044 (1980).

[6] L. O. Chua and E. W. Szeto, Synthesis of higher order nonlinear circuit elements, *IEEE Transactions on Circuits and Systems*, CAS-31, No.2, pp.231-235 (1984).

[7] S. Warner, *Modern Algebra*, Dover Publications, Inc., New York (1990).

[8] L. A. Novak, Fundamental Aspects of Circuit Theory, *Internal Report* (in Serbo-Croatian), graduate Course on Circuit Theory at the Department of Electrical Engineering and Computing, University of Novi Sad (1992).