Scattering on Fractal Potentials and Wavelet Dimensions

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Abstract

We consider the Schrödinger equation and the wave equation on the line when the interaction term is a fractal measure. We relate the long-time localization properties of the wave-packets to the fractal wavelet dimensions of this latter.

1 Introduction

An important application of scattering is the recognition and caracterization of fractal objects. This inverse problem is very complex in general and with limited results, but turns out to be more tractable in the simple case of the potential scattering on the line. Suppose we are given some formal hamiltonian $H = -\Delta + V$ on $L^2(\mathbb{R})$, where the potential term is fractal, in a sense to be precised, and consider two associated time evolution, namely the Schrödinger equation,

$$i\partial_t \Psi_t - H\Psi_t = 0, \tag{1.1}$$

and the wave equation,

$$\partial_{tt} \Psi_t + H\Psi_t = 0. \tag{1.2}$$

A natural question might be how the large time behaviour of the wave-packets Ψ_t mirrors the fractality of the potential, and what kind of fractal characterization of this latter can be obtained. The wavelet analysis and the wavelet dimensions will turn out to be an efficient tool to tackle this problem.

2 Scattering formalism for measures potentials

Dealing with truely fractal potentials requires to resort to classes of singular potentials, namely those given by singular measures. So let V be a finite real Borel measure on $\mathbb{R}$. For sake of simplicity, we will assume V has compact support, say within $[-1, 1]$, but this condition can be considerably relaxed. The operator sum $H = H_0 + V$ does in general not exist, but can be defined in the sense of quadratic forms. The domain $D(H)$ of the corresponsding operator is then a dense subset of $W^1(\mathbb{R})$ (Sobolev space of L^2 functions whose distributional derivative is also in L^2). It has been shown in [2] that an explicit scattering theory can be constructed for such operators. Basically all the results which are known for short-range regular potentials also hold for measures-potentials. In particular, the positive spectrum is purely absolutely continuous and the negative spectrum consists in a finite number of eigenvalues. The wave operators exist and are complete. The spectral measure and hence the time evolution can be computed in the usual way by means of the scattering wave functions, which are the solutions of the stationary Schrödinger equation satisfying the radiation condition. For an exhaustive presentation of these results, we refer to [2]. The solution Ψ_t of the Schrödinger equation (1.1) is given by the evolution operator e^{-iHt} and uniquely determined by the initial state $\Psi_{t=0} = f$:

$$\Psi_t = e^{-iHt} f, \ t \geq 0. \tag{2.1}$$

Analogously, the solution of the wave equation (1.2) is also given by an evolution operator, provided $H \geq 0$ (hence $V \geq 0$, an hypothesis which will always be implicitely assumed for the wave equation), and the initial data $\Psi_{t=0} = f \in D(H)$ and $\partial_t \Psi_{t=0} = g \in D(H)$:

$$\Psi_t = \cos(H^{1/2}t) f + H^{-1/2} \sin(H^{1/2}t) g, \ t \geq 0, \tag{2.2}$$

where the functions of H are defined by the usual functional calculus. In the following, we will assume that the initial wave-packet $\Psi_{t=0}$ is prepared by sending a incident wave train from the remote left on the scatterer. Thus, for $t < 0$, the wave function is simply a (free) solution of the wave equation propagating to the right, that is of the form $\Psi_t(x) = s(x - t)$. Therefore, it is natural to impose the initial conditions

$$g = \partial_t \Psi_{t=0} = -\partial_x \Psi_{t=0} = -f'. \tag{2.3}$$

Now let $\Omega^- =]-\infty, -1]$ and $\Omega^+ = [1, +\infty[$ be the regions to the left and to the right of the

support of V, respectively, and $\mathbb{I}_{\Omega^\pm}$ the corresponding characteristic functions, $\mathbb{I}_{\Omega^\pm}(x) = 1$ if $x \in \Omega^\pm$, $\mathbb{I}_{\Omega^\pm}(x) = 0$ else. Suppose the measurement of the wave packets Ψ_t is accessible outside the region of interaction, i.e in the regions $\Omega^\pm$. Our goal is to infer from this knowledge some fractal properties of the potential. A natural quantity to look at is the amount of $q-$energy present in each region:

$$p_q^\pm = \lim_{t\to+\infty} \left\| \mathbb{I}_{\Omega^\pm} \Psi_t \right\|_q^q, \tag{2.4}$$

assuming for the moment that the limits exist. Note that when f is normalized (i.e $\|f\| = 1$) and $q = 2$, these quantities can be interpreted as the final probability of finding a particle described by the wave function Ψ_t in the left, respectively right, region. The quantities $p_q^\pm$ turn out to be related in a simple way to the fractal wavelet dimensions.

3 The wavelet dimensions

The wavelet dimensions, introduced under this form by M. Holschneider ([4]), are a generalization of the usual box dimensions (see [3] for the original definition, [5] for more rigorous versions) that can be applied to any signed or complex signals, whereas the latter caracterize essentially probability measures. Following the standard notations, we shall write

$$g_{b,a}(x) = a^{-1} g(a^{-1}(x-b))$$

for the translated and dilated version of a function g. The wavelet transform of a function or more generally a distribution s with respect to a wavelet g is formally given by

$$\mathcal{W}_g s(b,a) = \int dt\ \overline{g_{b,a}}(t) s(t), \tag{3.1}$$

where the minimal requirement for a function g to be a wavelet is some localization both in position and frequency (typically $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$) and the vanishing moment condition, $\int g = 0$. Intuitively, the wavelet transform acts as a filter selecting the details present in s at scale a and position b. It is therefore a tool to analyse the local regularity of functions. The accuracy of the wavelet analysis depends on the choice of the wavelet. A convenient class of wavelets is the following. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of smooth functions which, together with their derivatives, decrease faster than any polynomial and let $\mathcal{S}_+(\mathbb{R})$ be the subspace of Schwartz functions with no negative frequencies (that is $\widehat{g}(\xi) = 0$ if $\xi \leq 0$). Functions in $\mathcal{S}_+(\mathbb{R})$ have all moments vanishing, that is

$$\int dx\ x^n g(x) = 0,\ \ n = 0, 1, 2, ..,$$

or, what amounts to the same, have a flat Fourier transform near zero. The wavelet dimensions are now constructed in the following way. Take some wavelet g in $\mathcal{S}_+(\mathbb{R})$ and set

$$G_g(a,q) = \|\mathcal{W}_g \mu(\cdot, a)\|_q^q .$$

The essential idea of the wavelet dimensions is to look at the scaling of this function near zero. However, for a good definition to be achieved, one more integration over the scales is needed, and therefore one defines

$$,\ _g(a,q) = \min\left\{ \int_a^1 \frac{d\alpha}{\alpha} G_g(\alpha, q), \int_0^a \frac{d\alpha}{\alpha} G_g(\alpha, q) \right\},$$

where the minimum is taken to excludes the case which is trivially convergent. The (upper and lower) wavelet dimensions $\kappa_q^\pm(\mu)$ are then defined by

$$\kappa_q^\pm(\mu) = \lim_{a\to 0} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\, ,\ _g(a,q)}{\log a}$$

and do not depend on the choosen wavelet g in $\mathcal{S}_+(\mathbb{R})$ provided $g \neq 0$. Note that a wavelet in $\mathcal{S}_+(\mathbb{R})$ cannot have compact support (otherwise its Fourier transform would be analytic and

hence could not vanish on a semi-axis without vanishing everywhere). However, admitting an arbitrarily small error on the dimensions $\kappa_q^{\pm}$, on always may suppose that the analysing wavelet is compactly supported. This is useful in the applications. An important particular case is the so-called wavelet correlation dimension $\kappa_2^{\pm}$, which is known to govern the L^2 asymptotic of its Fourier transform.

Theorem 3.1 ([4]) *If μ is a finite complex measure, then either $\mu \notin L^2(\mathbb{R})$ and*

$$\lim_{p\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \int_0^p d\xi \, |\widehat{\mu}(\xi)|^2}{\log p} = -\kappa_2^{\mp}[\mu] \qquad (3.2)$$

or $\mu \in L^2(\mathbb{R})$ and

$$\lim_{p\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \int_p^{+\infty} d\xi \, |\widehat{\mu}(\xi)|^2}{\log p} = -\kappa_2^{\mp}[\mu]. \qquad (3.3)$$

For positive measures, the wavelet correlation was shown ([1]) to coincide with the usual correlation dimension (hence the name).

4 Recovering the wavelet dimensions

We shall now propose a method to retrieve the wavelet dimensions of the potential from the quantities $p_q^{\pm}$ by choosing an appropriate family of initial states. In the following, we will adopt the notation $p_q^{\pm}[f]$ and $\Psi_t[f]$ for the quantities referring to the initial state f [respectively $(f, -f')$] in the case of the Schrödinger [resp. wave] equation.

4.1 Schrödinger equation

In that case, the results are limited to the wavelet correlation dimension $\kappa_2^{\pm}$. Suppose we can prepare some initial state f in $\mathcal{S}_+(\mathbb{R})$. Then

Proposition 4.1 *For all $b \notin supp(V)$, we have*

$$p^-[f_{b,a}] = a^2 \left\| \mathcal{W}_g V(\cdot, a) \right\|^2 + O(a^2), \ a \to 0, \qquad (4.1)$$

where the wavelet $g \in \mathcal{S}_+(\mathbb{R})$ is essentially a primitive of f: $\widehat{g}(k) = \sqrt{\pi/8}\widehat{f}(k/2)/k$.

Remark: the condition $b \notin supp(V)$ is a technical condition to avoid the contribution of the bound states.
It follows directly from the last result that the wavelet correlation dimension can be calculted by varying the family of initial states f_{ba}:

Corollary 4.2 *For all $b \notin supp(V)$, we have*

$$\lim_{a\to 0} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \rho(b,a)}{\log a} = \min\{\kappa_2^{\pm}[V], 0\},$$

where

$$\rho(b,a) = \int_a^1 \frac{d\alpha}{\alpha^3} \, p^-[f_{b,\alpha}].$$

Remark: this result is interesting only if $\kappa_2^{\pm}[V] \leq 0$, that is $V \notin L^2(\mathbb{R})$. But this is always the case for singular measures.

4.2 Wave equation

Stronger results can be shown in the case of the wave equation. The wave-packets Ψ_t turn out to be an experimental wavelet transform of the potential, where the time plays the role of the position parameter and the central frequency of the initial state the scale parameter.

Theorem 4.3 *Let the initial state $\Psi_{t=0} = f \in W^1(\mathbb{R})$ have support on Ω^- (i.e left of the potential). Choose b_{in}, b and b_{out} and set $t = 2b - b_{in} - b_{out}$. We then have for b_{out} large enough*

$$\Psi_t[f_{b_{in},a}](b_{out}) = a\mathcal{W}_h V(b, a/2) + O(a), \qquad (4.1)$$

where h is essentially a primitive of f: $\widehat{h}(k) = i\sqrt{\pi/8}\overline{\widehat{f}(-k)}/k$.

Hence, given some source point b_{in} and some other observation point b_{out} *outside* of the region of interaction, we can analyse the potential at any position *inside* by choosing the time accordingly. Another consequence is that the whole set of wavelet dimensions can be recovered.

Proposition 4.4 *Suppose V is sufficiently singular in the sense that*

$$\lim_{a\to 0} \|\mathcal{W}_h V(\cdot, a)\|_q^q = \infty \qquad (4.2)$$

for some (and hence all) $h \in \mathcal{S}_+(\mathbb{R})$. Then for the integrated q-energy of the wave-packets

$$\rho_q(a) = \int_a^1 \frac{d\alpha}{\alpha^{q+1}} p_q^-[f_{b_{in},\alpha}]$$

we have

$$\lim_{a\to 0} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \rho_q(a)}{\log a} = \kappa_q^{\pm}[V] + \epsilon_n, \qquad (4.3)$$

where the error term ϵ_n goes to zero as the number of vanishing moments n of the initial state f increases: $\epsilon_n \to 0,\ n \to \infty$.

Remark: the condition (4.2) is always satisfied for singular measures.

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