Physical Aspects of the Theory of Nonlinear Networks

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Abstract: - In this paper implications of the theory of irreversible thermodynamics on the theory of nonlinear electrical networks are considered. It is shown that some well known theorems of the deterministic network theory have to be rediscussed.

Key-Words: - network theory, thermal noise, nonlinear networks, Nyquist's theorem

1 Introduction

The theory of electrical networks was founded by Ohm, Kirchhoff, Helmholtz and Maxwell in the middle of the nineteenth century (see Wunsch [1], Belevitch [2]). It became one of the most sucessful theoretical concepts with many applications in electrical engineering and physics. In contrast to the theory of mechanical systems – Newton's mechanics – only very few researchers were interested in developing systematic foundations of the theory of electrical networks. Based on Kirchoff's ideas the ingredients of network theory are some kind of discrete topology (e.g. graph theory or algebraic topology) and certain real- or complex-valued functions defined on it. A main aspect of network theory is that only a restricted number of these functions, which are called network elements, are sufficient for modelling a large class of electrical and electronic circuits.

Most of the flexibility of network theory is related to the fact that each network element can be described independently. From a physical point of view the functionality of circuit devices is represented by a certain collection of these "lumped" network elements. The interaction of network elements is represented in the case of Kirchhoff networks by an ideal transformer n-port (see e.g. Mathis [3]) where a description by means of network graphs is a special case. It can be shown that this kind of interaction is characterized by energy conservation and reciprocity (see e.g. Mathis, Pauli [4]). Although this concept of electrical networks can be used in a very flexible manner to describe models of electrical and electronic circuits it leads to unphysical situations. A simple idealized model of an amplifier that includes a controlled source can already violate the first law of thermodynamics.

On the other hand many useful theorems of linear as well as nonlinear network theorems are known (e.g. Mathis, Pauli [4], Hasler [8], Chua et al. [7]) and therefore the theory of electrical networks is a basic tool of electrical and electronic engineers. Considering the rare number of abstract formalizations of this theory (e.g. Ghenzi [6], Slepian [5], Reibiger [9]) we find a verification of the above mentioned statements that unphysical situations in network theory in the sense of thermodynamics cannot be avoided because corresponding thermodynamic restrictions are missing. In the following we will discuss further defects in network theory with respect to thermodynamics where this physical theory can be interpreted as a very general framework to classify physical and unphysical situations. For our discussion we consider some recent results in the theory of nonlinear noisy electrical networks (see Weiss, Mathis [13]) and its consequenses for the theoretical foundations of electrical networks. A main subject of this paper is a generalization of Nyquist's formula for thermal noise which has been derived for nonlinear reciprocal networks.

2 Fluctuation-Dissipation Theorems

2.1 Nyquist's Formula

The theory of statistical irreversible thermodynamics (see, e.g., [10], [11]) predicts an inseparable relation between dissipation and fluctuations. This relation manifests itself in so-called fluctuation-dissipation relations (theorems) FDRs: Any dissipative system exhibits fluctuations, which are caused by the thermal coupling between the system and its environment (a heat bath at absolute temperature T). In circuit theory, the linear FDR is known as Nyquist's formula [12]. The (thermal) fluctuations of a linear conductor of resistance R at temperature T are independent of device specific details like shape, size or current transport mechanism. Instead, the fluctuations only depend on resistance R and temperature T, and can be represented by a Gaussian white noise current source in parallel to the noise free resistor, with spectral density

$$S_I(f) = \frac{2kT}{R} \qquad (1)$$

and vanishing mean value, see Fig. 1. A dual formulation in terms of a voltage source with spectral density

$$S_U(f) \quad = \quad 2kTR \qquad (2)$$

in series to the noise free resistor exists. The equations (1), (2) hold provided the current through R is "sufficiently small". It is important to understand

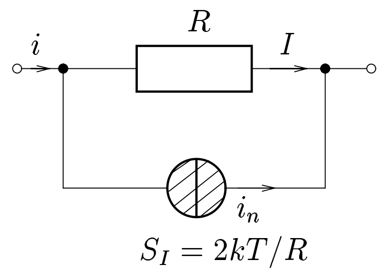

Figure 1: Norton noise equivalent circuit for linear resistor with thermal noise

that the circuit in Fig. 1 is merely a *representation* of the fluctuating current i through the resistor. The current's formal decomposition into a noise free current I and a noise current i_n, $i = I + i_n$, where $I =< I >$ is the statistical average of i, allows to represent internal fluctuations by the noise source (1), i.e. as if there were external noise. In contrast to external noise sources the source (1) cannot be switched off.

For nonlinear dissipative systems a similar "coupling" between dissipation and fluctuations exists, although it is much more difficult (if possible at all) to find explicit nonlinear FDRs. Stratonovich has derived nonlinear Markov FDRs for nonlinear dissipative systems up to dissipative nonlinearities of third order polynomials [11]. Weiss and Mathis have applied the concept to electrical networks (see, e.g., [13], [14]).

In the next section, we will sketch the derivation of linear and nonlinear Markov FDRs for electrical circuits, and we will discuss the implications of this (partial) embedding of circuit theory into nonequilibrium thermodynamics.

2.2 Derivation of Nyquist's Formula

In order to explain the basic principles behind the derivation of Nyquist's formula, it is sufficient to consider the linear RC circuit in Fig. 2

$$\frac{dV}{dt} \quad = \quad \Leftrightarrow \frac{1}{RC} V, \quad V(t_0) = V_0 \qquad (3)$$

which shall be close to thermodynamic equilibrium at temperature T. Again, we interpret V as the sta-

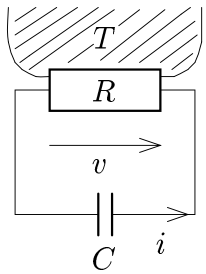

Figure 2: Linear RC circuit with thermal noise

tistical average $< v >$ of a stochastic process v. To describe the dynamics of the noisy circuit, a Fokker-Planck equation for the conditional probability density $P(V,t) \equiv P(V,t|V_0,t_0)$

$$\frac{\partial P(V,t)}{\partial t} = \Leftrightarrow \frac{\partial}{\partial V}\big(a(V)P(V,t)\big) + \frac{b}{2}\frac{\partial^2 P(V,t)}{\partial V^2} \qquad (4)$$

with initial condition

$$P(V,t_0|V_0,t_0) \quad = \quad \delta(V \Leftrightarrow V_0) \qquad (5)$$

can be used (see p. 115 in [10]). Using $V =< v >_{V_0}$, the drift coefficient $a(v)$ is identified with the right-hand side of the non-fluctuational equation (3)

$$a(V) \quad = \quad \Leftrightarrow \frac{1}{RC} V \qquad (6)$$

The conditional statistical average $< . >_{V_0}$ denotes the average over those ensemble members having the same initial voltage $V(t_0) = V_0$ (see, e.g., p. 61 in [15]). The diffusion coefficient b is found by inserting the stationary equilibrium distribution

$$P_{eq}(V) \quad = \quad \mathcal{N} \exp\left(\Leftrightarrow \frac{F(V)}{kT}\right) \qquad (7)$$

with the free energy F

$$F(V) \quad = \quad \frac{1}{2} C V^2 \qquad (8)$$

into (4)

$$b \quad = \quad \frac{2kT}{RC^2} \qquad (9)$$

(7) is known from statistical equilibrium thermodynamics. (One of the few basic principles of classical irreversible thermodynamics, which deals with system states in a close neighborhood of thermodynamic equilibrium, is to embed the equilibrium theory as a limiting case in the nonequilibrium theory (see "principle of dynamic equilibrium", p. 3 in [11]).) (9) is the linear Markov FDR for the circuit in Fig. 2. Its equivalence to Nyquist's formula (1) is easily shown via the stochastic differential equation (SDE)

$$dv \;=\; \Leftrightarrow \frac{1}{RC} v\, dt \,\pm\, \sqrt{\frac{2kT}{RC^2}}\, dw, \quad v(t_0) = v_0 \qquad (10)$$

which is stochastically equivalent to the FPE (4). dw denotes the differential of a Wiener process. A generalization to multidimensional linear RLC circuits is possible, *provided the networks under study are reciprocal* (see [16], [13]). Non-reciprocal networks are active systems and exhibit stationary non-equilibrium states. For such systems the stationary distribution P_{stat}, which replaces P_{eq}, cannot be determined from first principles (from equilibrium statistical thermodynamics).

2.3 Nonlinear Irreversible Thermodynamics

When the noisy circuit is nonlinear

$$\frac{dV}{dt} \quad = \quad \Leftrightarrow \frac{1}{C} g(V) \qquad (11)$$

a description in terms of an FPE, even with nonlinear coefficient functions $a(V)$, $b(V)$, is insufficient, because such a description does not account for non-Gaussian noise source contributions (see, e.g., [11], [14]). The same holds for the corresponding nonlinear SDE. Non-Gaussian noise source contributions are needed to ensure the thermodynamic correctness of the noise model, namely to ensure the validity of the second law of thermodynamics in the presence of dissipative nonlinearities. They can be expressed by "generalized FPEs" with third and higher order derivatives, so-called Kramers-Moyal equations

(KMEs)

$$\frac{\partial P(V,t)}{\partial t} = \sum_{n=1}^{\infty} \frac{(\Leftrightarrow 1)^n}{n!} \frac{\partial^n}{\partial V^n} \{\alpha_n(V)\, P(V,t)\} \quad (12)$$

Clearly, the KME (12) comprises the FPE (4). KMEs arise from formal power series expansions of the master equation for Markov processes (see, e.g., [17]). To establish nonlinear FDRs means to determine the unknown coefficients in (12). This will be done by a generalization of the method used to establish the linear FDR (9).

Since nonlinear resistors cause a coupling between dissipative and resistive network elements [14], it is necessary to switch over to the multidimensional KME

$$\frac{\partial P(X,t)}{\partial t} = \quad (13)$$
$$\sum_{m=1}^{\infty} \frac{(\Leftrightarrow 1)^m}{m!} \sum_{\alpha_1 \ldots \alpha_m = 1}^{r} \frac{\partial^m [K_{\alpha_1 \ldots \alpha_m}(X) P(X,t)]}{\partial X_{\alpha_1} \ldots \partial X_{\alpha_m}}$$

with initial condition

$$P(X, t_0 | X_0, t_0) \quad = \quad \delta(X \Leftrightarrow X_0) \quad (14)$$

Here x denotes an r-dimensional vector of fluctuating state variables (inductor currents and capacitor voltages). The corresponding deterministic variable $X =< x >_{X_0}$ is assumed to obey an explicit nonlinear system of autonomous ordinary differential equations (ODEs)

$$\dot{X} \quad = \quad f(X), \quad X(t_0) = X_0 \quad (15)$$

In practise, only a small number of powers can be considered in the infinite series (13). For a systematic determination of the unknown coefficients $K_{\alpha_1 \ldots \alpha_m}$ it is necessary to transform the KME (13) into a power series in a small parameter (compare van Kampen's Ω-expansion [18]). In thermodynamic equilibrium the m-fold moments (statistical mean values) $\langle X_{\alpha_1} \, \ldots \, X_{\alpha_m} \rangle$ are proportional to $(kT)^{m-1}$

$$\langle X_{\alpha_1} \, \ldots \, X_{\alpha_m} \rangle \quad \sim \quad (kT)^{m-1} \quad (16)$$

Regarding m-fold moments in nonequilibrium situations close to thermodynamic equilibrium the same relation should be valid. Using this assumption and the relation between unknown coefficients and m-fold conditional nonequilibrium moments

$$K_{\alpha_1 \, \ldots \, \alpha_m}(X) = \lim_{\tau \to 0} \frac{\langle \Delta X_{\alpha_1} \, \ldots \, \Delta X_{\alpha_m} \rangle_X}{\tau} \quad (17)$$

one obtains

$$K_{\alpha_1 \ldots \alpha_m}(X) \sim (kT)^{m-1} \quad (18)$$

Hence it is natural to choose kT as the small expansion parameter.

As in the linear case, we again demand the equilibrium distribution known from equilibrium statistical thermodynamics

$$P_{eq}(X) \quad = \quad \mathcal{N} \exp \left(\Leftrightarrow \frac{F(X)}{kT} \right) \quad (19)$$

to be the unique stationary solution of (13). Existence and uniqueness of P_{eq} is obtained by a suitable restriction of dissipative nonlinearities and/or of the range of allowed X-values: *Only passive current- or voltage-controlled nonlinear resistors $I = g(V)$, $V = r(I)$, can be considered*

$$\begin{aligned} V\, g(V) \quad &\geq \quad 0 \quad \forall V, \quad V\, g(V) = 0 \Leftrightarrow V = 0 \\ I\, r(I) \quad &\geq \quad 0 \quad \forall I, \quad I\, r(I) = 0 \Leftrightarrow I = 0 \end{aligned} \quad (20)$$

Due to (20), none of the resistors supplies power to the circuit connected with its terminals. This prevents an isothermal conversion of heat to work (see p. 185 in [19]).

To determine the unknown coefficients $K_{\alpha_1 \ldots \alpha_m}$ in the multidimensional, nonlinear case, additional constraints are required. These contraints must not be too restrictive in order to leave the class of treatable systems large enough. Furthermore, the constraints must be in agreement with basic thermodynamic requirements, especially with the second law of thermodynamics.

One such general principle is the *principle of microscopic time-reversal symmetry.* To understand this principle, we assume the macroscopic equilibrium $P_{eq}(X)$ to be caused by an underlying non-dissipative microscopic Hamiltonian system with generalized coordinates q and momenta p. *We now demand that the microscopic dynamics is invariant under time-reversal transformation*

$$t \to \Leftrightarrow t, \quad q_i(t) \to q_i(\Leftrightarrow t) \;=\; q_i(t), \qquad (21)$$
$$p_i(t) \to p_i(\Leftrightarrow t) \;=\; \Leftrightarrow p_i(t)$$

i.e., the system Hamiltonian $\mathcal{H}(q,p)$ must be invariant under the transformation (21). Since the Hamiltonian determines the Gibbs equilibrium distribution $P_{eq}(q,p)$ of microscopic states via $P_{eq} \sim \exp(\Leftrightarrow \mathcal{H}/kT)$, the equilibrium distribution of microscopic states is also invariant under time reversal

$$P_{eq}(q(\Leftrightarrow t), p(\Leftrightarrow t)) \;=\; P_{eq}(q(t), p(t)) \qquad (22)$$

Using the functional relations $X(q,p)$ transformation rules for the macroscopic variables can be inferred

$$X_\alpha(t) \to X_\alpha(\Leftrightarrow t) = \epsilon_\alpha X_\alpha(t), \quad \epsilon_\alpha = \pm 1 \qquad (23)$$

Variables X_i with $\epsilon_\alpha = 1$ are called time-even variables, variables with $\epsilon_\alpha = \Leftrightarrow 1$ are called time-odd. Using microscopic time-reversal symmetry (22) and (23) we get the "principle of detailed balance" as a requirement for the macroscopic (conditional) probability densities (see, e.g., [20])

$$P_{stat}(\epsilon X) \;=\; P_{stat}(X) \;\Leftrightarrow \qquad (24)$$
$$P(X'|X)P_{stat}(X) \;=\; P(\epsilon X|\epsilon X')P_{stat}(X')$$

Detailed balance (24) is one of the corner-stones of linear and nonlinear irreversible thermodynamics.

As mentioned before, only a few powers of the series (13) can be considered. Due to Equation (17), the nth power of the series (13) corresponds to the $(n \Leftrightarrow 1)th$ power of nonlinearity in the deterministic ODE (15). Therefore, Equation (15) is expanded in a Taylor series at $X = 0$. This expansion is made in the *forces Y_α conjugate to the X_α*

$$Y_\alpha \;:=\; \frac{dF(X)}{dX_\alpha} \qquad (25)$$

$$\dot{X}_\alpha \;=\; f(X(Y)) =: \varphi(Y) \qquad (26)$$
$$= \sum_\beta \varphi_{\alpha,\beta}(0)\, Y_\beta \;+\; \frac{1}{2}\sum_{\beta,\gamma} \varphi_{\alpha,\beta\gamma}(0)\, Y_\beta\, Y_\gamma \;+\; \ldots$$

where the abbreviation

$$\varphi_{\alpha,\beta_1\ldots\beta_m} \equiv \frac{\partial^m \varphi_\alpha}{\partial Y_{\beta_1}\ldots\partial Y_{\beta_m}} \qquad (27)$$

has been used. As a consequence of time reversal symmetry, the Jacobian of $\varphi(Y = 0)$ exhibits a generalized symmetry, which is known as *Onsager-Casimir reciprocity*

$$\varphi_{\beta,\alpha}(0) \;=\; \epsilon_\alpha \epsilon_\beta \varphi_{\alpha,\beta}(0) \qquad (28)$$

with $\epsilon_\alpha = 1$ for time-even Y_α and $\epsilon_\alpha = \Leftrightarrow 1$ for time-odd Y_α. Equation (28) only holds for reciprocal electrical networks. *The standard theory of irreversible thermodynamics, which describes fluctuations in a neighborhood of thermodynamic equilibrium, is not applicable to non-reciprocal networks!*

Finally, to ensure consistency, the coefficients $K_{\alpha_1\ldots\alpha_m}(X)$ of the KME (13), which may be general nonlinear functions of X, also have to be expanded in a Taylor series at $X = 0$. In these series, the highest relevant order is determined by the highest order considered in the ODE (26) and by the kT-dependence (18) of the $K_{\alpha_1\ldots\alpha_m}(X)$.

For classes of networks which fit into the frame of the thermodynamic theory discussed here, it is possible to determine the KME (13) from the knowledge of the deterministic ODE and the networks free energy F (see [13], [14]). The knowledge of the KME amounts to a description of deterministic and fluctuational behavior, where of course only the "thermodynamic fluctuation types" thermal noise and shot noise can be included. Before we proceed we have to discuss the implications of the "thermodynamic frame" for electrical networks, in order to choose a class of networks to which the concept is applicable.

2.4 Implications for Electrical Networks

1.) It can be shown that voltages V and charges Q are time-even whereas currents I and fluxes Φ are time-odd variables. Thus, from a thermodynamic point of view, *voltages and currents are not dual.* As a consequence, FDRs in current and voltage formulation are not dual in the general nonlinear case. The duality of the linear

FDR (1), (2) appears to be an exception which is caused by the spectral densities' independence of state variables.

2.) In general, external driving forces, which yield a non-equilibrium stationary state instead of thermodynamic equilibrium, destroy time-reversal symmetry (see, e.g., [21]). Hence the concept based on detailed balance is no longer applicable. As a consequence, *the linear theory of networks with thermal coupling to their environment cannot be generalized to the nonlinear case by linearization in a bias point.* The only exception from this rule is the case of zero bias (see, e.g., [22], [23]). For Nyquist's formulas, this means that they can be generalized to

$$S_I(f) = 2kT\, \frac{dI}{dV}\bigg|_{V=0},\; S_U(f) = 2kT\, \frac{dV}{dI}\bigg|_{I=0} \qquad (29)$$

but not to other bias points $V_0 \neq 0$ and $I_0 \neq 0$. Thus, we restrict our discussions to sourceless networks. (In the linear case, time constant independent sources can be considered, see, e.g., [14]).

3.) In networks with controlled sources it is not possible to determine the stationary distribution P_{stat} from first principles (see [24]). Since detailed balance is (in general) not valid, the resulting linearized differential equations are not Onsager-Casimir reciprocal. Hence *non-reciprocal networks cannot be treated in the framework of standard thermodynamic fluctuation theory of electrical networks.* (Of course, there are thermodynamic concepts for fluctuations in stationary nonequilibrium states, see, e.g., [25], [21]. These concepts are fundamentally different from the theory of fluctuations in the neighborhood of thermodynamic equilibrium, they are much less powerful, and they are not used to derive Nyquist's formula (1)).

4.) The free energy F of a source-free passive nonlinear RLC network equals the total energy of the dynamic elements. As a consequence, the equilibrium distribution (19) of such networks is completely determined by the energies of the resistive elements. To ensure detailed balance (24) in case of nonlinear current controlled two-terminal inductors $L(I) := d\Phi/dI$ and voltage controlled two-terminal capacitors $C(V) := dQ/dV$, the inductor energies $W(\Phi) = \int I(\Phi)d\Phi$ have to be even functions of the fluxes Φ, so that $W(\Phi(\Leftrightarrow t)) = W(\Phi(t))$ is valid. No such requirement for capacitors is necessary. Due to $Q(\Leftrightarrow t) = Q(t)$ we always have $W(Q(\Leftrightarrow t)) = W(Q(t))$. Since the inverse functions of $\Phi(I)$ and $Q(V)$ are needed (see [13]), both characteristics must be strictly monotonely increasing.

5.) Finally the concept of independent lumped network elements becomes obscure because a nonlinear noise causes a coupling of network elements which are independent from a deterministic point of view (see also Müller [26]). This is illustrated in the example in section IV; see especially equ. (42) where there is a correction term $\Leftrightarrow \gamma kT/4C$ which depends on the capacitor <u>and</u> the resistor at the same time.

III. NONEQUILIBRIUM THERMODYNAMICS OF COMPLETE NETWORKS

As an example we consider the application of Stratonovich's Nonequilibrium Thermodynamics to the class of complete networks. (For a more detailed description of complete networks, see ([28], [36] or [37].) Complete networks are reciprocal nonlinear RLC networks with either voltage- or current controlled resistors. Their network equations can be written in terms of a single scalar function, the *mixed potential function* $\mathcal{P}(i,v)$

$$\begin{aligned} L_\rho(i_\rho)\,\frac{di_\rho}{dt} &= \frac{\partial \mathcal{P}(i,v)}{\partial i_\rho} \\ C_\sigma(v_\sigma)\,\frac{dv_\sigma}{dt} &= \Leftrightarrow \frac{\partial \mathcal{P}(i,v)}{\partial v_\sigma} \end{aligned} \qquad (30)$$

The equations (30) are called the *Brayton-Moser equations.* $\mathcal{P}$ is a function of the dynamical variables $(i_L, v_C) \equiv (i,v)$ and is given in terms of the *current potential* $F(i)$, the *voltage potential* $G(v)$ and a constant matrix γ

$$\mathcal{P}(i,v) \quad = \quad F(i) \Leftrightarrow G(v) + (i, \gamma v) \qquad (31)$$

$\mathcal{P}$ describes the influence of independent sources and (nonlinear) resistors. For complete networks, a Hamiltonian description is known ([36], [37]). The Hamiltonian function $\mathcal{H}$ equals the total energy of the dynamical network elements

$$\begin{aligned} \mathcal{H}(q,\phi,p_q,p_\phi) &= W_L + W_C \\ = \Big(\sum_\rho \int \dot{q}_\rho(\phi_\rho)\, d\phi_\rho \qquad & \qquad (32) \\ + \sum_\sigma \int \dot{\phi}_\sigma(q_\sigma)\, dq_\sigma \Big) \Big|_{(\dot{q},\dot{\phi})=h(q,\phi,p_q,p_\phi)} & \end{aligned}$$

q and ϕ are the charges and fluxes associated with i and v. To ensure the existence of the Hamiltonian (32) and time reversal symmetry, the inductor fluxes $\phi_\rho(\dot{q}_\rho)$ and the capacitor charges $q_\sigma(\dot{\phi}_\sigma)$ must be odd functions of their arguments.

When the Brayton-Moser equations (31) are expanded into the Taylor series (26) and the free energy $\mathcal{F}$ of the electrical networks is associated with the energy of the capacitors and the inductors, the unknown coefficients $K_{\alpha_1 \ldots \alpha_m}$ of the KME (13) can be derived from Stratonovich's results (pp. 128, [11]). For different powers of nonlinearity in the expansion of (30) different coefficients are obtained. For simplicity, we only consider source-free networks with linear dynamical elements here. (For the more general case see [13]).

Linear Approximation

When the network equations (30) are linear or linearized, the KME (13) becomes a "linear" FPE

$$\begin{aligned} \frac{\partial P(i,v)}{\partial t} \quad = \quad \mathcal{L}_1\, P(i,v) \; = \qquad & \qquad (33) \\ \Big\{ \Leftrightarrow \sum_{\rho_1\rho_2} \frac{1}{L_{\rho_1}} F_{,\rho_1\rho_2}(0) \frac{\partial}{\partial i_{\rho_1}} i_{\rho_2} \Leftrightarrow \sum_{\rho_1\sigma_2} \gamma_{\rho_1\sigma_2} \frac{v_{r+\sigma_2}}{L_{\rho_1}} \frac{\partial}{\partial i_{\rho_1}} & \\ + \sum_{\sigma_1\rho_2} \frac{\gamma_{\rho_2\,\sigma_1 - r}}{C_{\sigma_1}} i_{\rho_2} \frac{\partial}{\partial v_{\sigma_1}} \Leftrightarrow \sum_{\sigma_1\sigma_2} \frac{1}{C_{\sigma_1}} G_{,\sigma_1\sigma_2}(0) \frac{\partial}{\partial v_{\sigma_1}} v_{\sigma_2} & \\ \Leftrightarrow \sum_{\rho_1\rho_2} \frac{kT}{L_{\rho_1}L_{\rho_2}} F_{,\rho_1\rho_2}(0) \frac{\partial^2}{\partial i_{\rho_1} \partial i_{\rho_2}} & \\ \Leftrightarrow \sum_{\sigma_1\sigma_2} \frac{kT}{C_{\sigma_1}C_{\sigma_2}} G_{,\sigma_1\sigma_2}(0) \frac{\partial^2}{\partial v_{\sigma_1} \partial v_{\sigma_2}} \Big\} P(i,v) & \end{aligned}$$

Here, the abbreviation $\partial^p F / \partial i_{\rho_1} \ldots \partial i_{\rho_p} \equiv, F_{,\rho_1 \ldots \rho_p}$ has been used. The matrix γ has elements $\gamma_{\rho\,\sigma}$. Equation (33) equals the conventional description of thermal noise in a linear, source-free complete network and reproduces the multidimensional linear FDR. Thus, (33) is a time domain formulation of Nyquist's theorem. An equivalent result can be obtained with the well known linear Langevin theory of thermal noise. The thermal noise is fully determined by the deterministic network equations, i.e., by the deterministic network parameters, by the functions F, G and by the matrix γ. Consequently, arbitrary moments of functions of (i,v) can be calculated, after (33) has been solved for P. (Instead of solving (33) for P, the so-called "method of moments" can be used to derive ordinary differential equations for the moments, which are usually much easier to solve (see, .e.g, [17]).

Quadratic Approximation

When the right-hand side of (30) contains linear and quadratic terms, additional derivatives of first and second order occur. Besides, derivatives of order three occur. The KME (13) becomes

$$\begin{aligned} \frac{\partial P(i,v)}{\partial t} = \mathcal{L}_1\, P(i,v) + \mathcal{L}_2\, P(i,v) = \mathcal{L}_1\, P(i,v) \quad & (34) \\ + \frac{1}{2} \Big\{ \Leftrightarrow \sum_{\rho_1\rho_2\rho_3} \frac{1}{L_{\rho_1}} F_{,\rho_1\rho_2\rho_3}(0) \frac{\partial}{\partial i_{\rho_1}} i_{\rho_2} i_{\rho_3} & \\ + kT \sum_{\rho_1\rho_2} \frac{1}{L_{\rho_1}L_{\rho_2}} F_{,\rho_1\rho_2\rho_2}(0) \frac{\partial}{\partial i_{\rho_1}} & \\ \Leftrightarrow \sum_{\sigma_1\sigma_2\sigma_3} \frac{1}{C_{\sigma_1}} G_{,\sigma_1\sigma_2\sigma_3}(0) \frac{\partial}{\partial v_{\sigma_1}} v_{\sigma_2} v_{\sigma_3} & \\ + kT \sum_{\sigma_1\sigma_2} \frac{1}{C_{\sigma_1}C_{\sigma_2}} G_{,\sigma_1\sigma_2\sigma_2}(0) \frac{\partial}{\partial v_{\sigma_1}} & \\ \Leftrightarrow 3 \sum_{\rho_1\rho_2\rho_3} \frac{kT}{L_{\rho_1}L_{\rho_2}} F_{,\rho_1\rho_2\rho_3}(0) \frac{\partial^2}{\partial i_{\rho_1} \partial i_{\rho_2}} i_{\rho_3} & \end{aligned}$$

$$\Leftrightarrow \sum_{\sigma_1\sigma_2\sigma_3} \frac{kT}{C_{\sigma_1}C_{\sigma_2}} G_{,\sigma_1\sigma_2\sigma_3}(0) \frac{\partial^2}{\partial v_{\sigma_1} v_{\sigma_2}} v_{\sigma_3}$$

$$\Leftrightarrow \sum_{\rho_1\rho_2\rho_3} \frac{2(kT)^2}{L_{\rho_1}L_{\rho_2}L_{\rho_3}} F_{,\rho_1\rho_2\rho_3}(0) \frac{\partial^3}{\partial i_{\rho_1}\partial i_{\rho_2}\partial i_{\rho_3}} \Bigg\} P(i,v)$$

Again, the "thermodynamic noise" is fully determined by the deterministic system. Equation (34) contains derivatives of third order and is more general than a FPE. Therefore, *when (quadratic) nonlinearities are present, neither a FPE nor a (generalized) Langevin equation can serve as a correct (complete) description of thermal noise.* In other words, in general, no correct noise source approach exists. Nevertheless, in certain special cases it is possible to derive a correct FPE from (34): (i.) When no inductors are present, the current potential F vanishes and (34) becomes a nonlinear FPE. (This different behavior of inductors and capacitors seems to be strange from the viewpoint of network theory but appears to be natural in the framework of nonequilibrium thermodynamics. It is due to the different behavior of currents and voltages under time reversal symmetry.) (ii.) For the computation of moments up to second order (spectral densities), the coefficients $O((kT)^2)$ of second order in kT can be omitted from (34). This holds, because second moments are $O((kT)^1)$. (Generally speaking, m-th moments are $O((kT)^{m-1})$, see equation (16). Therefore, terms of $O((kT)^m)$ have no physically relevant meaning for the calculation of m-th moments.) *When quadratic nonlinearities are present, a "nonlinear" FPE (a generalized Langevin equation) is sufficient to determine all first and second moments of* (i,v).

Cubic Approximation

When nonlinearities of third order are considered, additional derivatives up to fourth order occur

$$\begin{aligned}
\frac{\partial P(i,v)}{\partial t} \;=\; & \mathcal{L}_1\,P(i,v) + \mathcal{L}_2\,P(i,v) \qquad (35)\\
+\Bigg(& \Leftrightarrow\frac{1}{6}\sum_{\rho_1\rho_2\rho_3\rho_4} \frac{1}{L_{\rho_1}} F_{,\rho_1\rho_2\rho_3\rho_4}(0)\frac{\partial}{\partial i_{\rho_1}} i_{\rho_2} i_{\rho_3} i_{\rho_4}\\
& +\frac{1}{2}\sum_{\rho_1\rho_2\rho_3} \frac{kT}{L_{\rho_1}L_{\rho_3}} F_{,\rho_1\rho_2\rho_3\rho_3}(0)\frac{\partial}{\partial i_{\rho_1}} i_{\rho_2}\\
& \Leftrightarrow\frac{1}{6}\sum_{\sigma_1\sigma_2\sigma_3\sigma_4} \frac{1}{C_{\sigma_1}} G_{,\sigma_1\sigma_2\sigma_3\sigma_4}(0)\frac{\partial}{\partial v_{\sigma_1}} v_{\sigma_2} v_{\sigma_3} v_{\sigma_4}\\
& +\frac{1}{2}\sum_{\sigma_1,\,\sigma_2,\,\sigma_3} \frac{kT}{C_{\sigma_1}C_{\sigma_3}} G_{,\sigma_1\sigma_2\sigma_3\sigma_3}(0)\frac{\partial}{\partial v_{\sigma_1}} v_{\sigma_2}\\
& +\frac{1}{4}\sum_{\rho_1\rho_2\rho_3\rho_4} \frac{kT}{L_{\rho_1}L_{\rho_2}} \Big(c_{\rho_1\,\rho_2,\,\rho_3\,\rho_4} \Leftrightarrow 2F_{,\rho_1\rho_2\rho_3\rho_4}(0)\Big)\\
& \qquad\times\frac{\partial^2}{\partial i_{\rho_1}\partial i_{\rho_2}}\Big(i_{\rho_3} i_{\rho_4} \Leftrightarrow \frac{kT}{L_{\rho_4}}\delta_{\rho_3\rho_4}\Big)\\
& +\frac{1}{4}\sum_{\sigma_1\sigma_2\sigma_3\sigma_4} \frac{kT}{C_{\sigma_1}C_{\sigma_2}} \Big(c_{\sigma_1\,\sigma_2,\,\sigma_3\,\sigma_4} \Leftrightarrow 2G_{,\sigma_1\sigma_2\sigma_3\sigma_4}(0)\Big)\\
& \qquad\times\frac{\partial^2}{\partial v_{\sigma_1}\partial v_{\sigma_2}}\Big(v_{\sigma_3} v_{\sigma_4} \Leftrightarrow \frac{kT}{C_{\sigma_4}}\delta_{\sigma_3\sigma_4}\Big)\\
& \Leftrightarrow\frac{1}{6}\sum_{\rho_1\rho_2\rho_3\rho_4} \frac{(kT)^2}{L_{\rho_1}L_{\rho_2}L_{\rho_3}} \Big(4F_{,\rho_1\rho_2\rho_3\rho_4}(0)\\
& \qquad \Leftrightarrow c_{\rho_1\,\rho_2,\,\rho_3\,\rho_4} \Leftrightarrow c_{\rho_1\,\rho_3,\,\rho_2\,\rho_4} \Leftrightarrow c_{\rho_2\,\rho_3,\,\rho_1\,\rho_4}\Big)\\
& \qquad\times\frac{\partial^3}{\partial i_{\rho_1}\partial i_{\rho_2}\partial i_{\rho_3}}\Big(i_{\rho_4} + \frac{kT}{2L_{\rho_4}}\frac{\partial}{\partial i_{\rho_4}}\Big)\\
& \Leftrightarrow\frac{1}{6}\sum_{\sigma_1\sigma_2\sigma_3\sigma_4} \frac{(kT)^2}{C_{\sigma_1}C_{\sigma_2}C_{\sigma_3}} \Big(4G_{,\sigma_1\sigma_2\sigma_3\sigma_4}(0)\\
& \qquad \Leftrightarrow c_{\sigma_1\sigma_2,\,\sigma_3\sigma_4} \Leftrightarrow c_{\sigma_1\sigma_3,\,\sigma_2\sigma_4} \Leftrightarrow c_{\sigma_2\sigma_3,\,\sigma_1\sigma_4}\Big)\\
& \qquad\times\frac{\partial^3}{\partial v_{\sigma_1}\partial v_{\sigma_2}\partial v_{\sigma_3}}\Big(v_{\sigma_4} + \frac{kT}{2C_{\sigma_4}}\frac{\partial}{\partial v_{\sigma_4}}\Big)\Bigg)\;P(i,v)
\end{aligned}$$

δ_{ij} is the Kronecker-δ. The coefficients $c_{\rho_1\,\rho_2,\,\rho_3\,\rho_4}$, $c_{\sigma_1\sigma_2,\,\sigma_3\sigma_4}$ cannot be determined in the framework of the theory of nonequilibrium thermodynamics. When nonlinearities of third or higher order are given, it is impossible to determine the unknown coefficients of the KME (13) completely, i.e., *the thermal noise is not fully determined by the deterministic network equations.* Unfortunately, it is not known of which order in kT the unknown coefficients $c_{\rho_1\,\rho_2,\,\rho_3\,\rho_4}$, $c_{\sigma_1\sigma_2,\,\sigma_3\sigma_4}$ are. They might be $O((kT)^1)$. Thus, neither an omission of all derivatives of orders higher than two nor an omission of all unknown coefficients leads to a fully determined systematic approximation. *When arbitrary nonlinearities are present, no FPE (Langevin equation) can be derived as a systematic approximation for the calculations of second or higher moments.* Of course, one can omit all unknown terms and *hope* that their influence on second and higher moments will be small. On the other hand, it should be possible to determine the unknown coefficients experimentally by comparison with measurements. For every unknown coefficient one 2nd or higher order moment has to be measured. This procedure leads to a system of m algebraic equations for m unknowns.

Approximations of Fourth and Higher Orders

As the number of undeterminable coefficients increases when higher orders of nonlinearity are considered, the number of measurements of higher order moments has to be increased. Therefore, it does not seem to be a good idea to consider Taylor polynomials of much higher order. Since the basic effects of the nonlinearities in the neighborhood of an equilibrium (operating point) should already occur in lower order approximations, it is not even necessary to proceed much further.

IV. EXAMPLE

As an example we consider the RC network in Figure 1 with a nonlinear resistance in cubic approximation

$$\frac{du}{dt} = \Leftrightarrow \frac{1}{C}\left[Gu + \frac{1}{2}\gamma u^2 + \frac{1}{6}\delta u^3\right] \qquad (36)$$

The corresponding mixed potential function is

$$P(u) = \Leftrightarrow G(u) = \int \left[Gu + \frac{1}{2}\gamma u^2 + \frac{1}{6}\delta u^3\right] du \qquad (37)$$

Using (33) - (35) we find

$$\frac{\partial P(u,t)}{\partial t} = (\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3) P(u,t) \qquad (38)$$

with the operator $\mathcal{L}_1$ of the linear approximation

$$\mathcal{L}_1 = \frac{\partial}{\partial u}\frac{G}{C}u + \frac{\partial^2}{\partial u^2}\frac{GkT}{C^2} \qquad (39)$$

the operator L_2 of the quadratic approximation

$$L_2 = \frac{\partial}{\partial u}\left(\frac{\gamma}{2C}u^2 \Leftrightarrow \frac{\gamma kT}{2C^2}\right) + \frac{\partial^2}{\partial u^2}\frac{\gamma kT}{2C^2}u \qquad (40)$$

and the operator $\mathcal{L}_3$ of the cubic approximation

$$\begin{aligned} \mathcal{L}_3 = & \frac{\partial}{\partial u}\left\{\frac{\delta}{6C}u^3 \Leftrightarrow \frac{kT\delta}{2C^2}u\right\} \qquad (41) \\ & + \frac{\partial^2}{\partial u^2}(2\delta+\epsilon)\left\{\frac{kT}{4C^2}u^2 \Leftrightarrow \frac{(kT)^2}{4C^3}\right\} \\ & + \frac{\partial^3}{\partial u^3}(4\delta+3\epsilon)u\frac{(kT)^2}{6C^3} + \frac{\partial^4}{\partial u^4}\frac{(kT)^3}{12C^4}(4\delta+3\epsilon) \end{aligned}$$

ϵ denotes a theoretically undeterminable parameter. In the quadratic approximation $\delta = 0 \Rightarrow \mathcal{L}_3 = 0$ of (39) the KME becomes a pure FPE with an equivalent Langevin equation

$$\frac{dU}{dt} = \Leftrightarrow \frac{1}{C}\left\{GU + \frac{1}{2}\gamma U^2 \Leftrightarrow \frac{\gamma kT}{4C}\right\} + \left\{1 + \frac{\gamma U}{2G}\right\}^{1/2}\xi(t)$$

$$< \xi(t)\xi(t') > = \frac{2GkT}{C^2}\,\delta(t \Leftrightarrow t') \qquad (42)$$

As the correction term $\frac{\gamma kT}{4C}$ is of order $O(kT)$ it has no influence on the mean value $< U > = u$, which is $O(1)$. Equation (42) can be interpreted in terms of a controlled noise source with $g(u) = Gu + 1/2\gamma u^2$ and $h(u) = \{1 + \frac{\gamma U}{2G}\}^{1/2}$

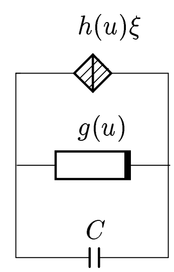

Figure 3: Equivalent noise source model

V. CONCLUSIONS

In this paper we consider the implications on the theory of electrical networks imposed by nonlinear irreversible thermodynamics which forms the basis of the theory of nonlinear noisy electrical networks. It is found that some of the well known theorems of deterministic network theory are no longer valid if the network description meets thermodynamic requirements. Examples are duality of currents and voltage and linearization in operating points. Finally the nonlinear noise causes a coupling between different network elements which obscures the network concept of independent lumped network elements.

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