# Integer Wavelet Transforms using the Lifting Scheme

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*Abstract:* Due to its good decorrelating properties, the wavelet transform is a powerful tool for signal analysis. The lifting scheme is an efficient algorithm to calculate wavelet transforms and it allows for the construction of second-generation wavelets. Furthermore it can easily be converted to an integer transform, which is of great importance for multimedia applications. We show how the lifting scheme can be used for one- and two-dimensional signals. In the 2D case, we consider a rectangular grid on which we construct second-generation wavelets based on a red-black blocking scheme. Compared to classical tensor product wavelets on the same grid, these wavelets show less anisotropy.

*Key-Words:* Wavelet transform, lifting, integer, quincunx lattice CSCC'99 Proceedings, Pages:6251-6253

## 1 Introduction

The lifting scheme is an algorithm to calculate wavelet transforms in an efficient way. It is also a generic method to create so-called second-generation wavelets. They are much more flexible and can be used to define a wavelet basis on an interval or on an irregular grid, or even on a sphere. Furthermore all classical wavelets can be generated by the lifting scheme. Several introductions to the lifting scheme are available, e.g. [4].

## 2 Predict and Update

The wavelet transform of a one-dimensional signal is a multi-resolution representation of that signal where the wavelets are the basis functions which at each resolution level give a highly decorrelated representation. Thus at each level, the (low-pass part of the) signal is split into a high-pass and a low-pass part. These high-pass and low-pass parts are obtained by applying corresponding wavelet filters.

The lifting scheme is an efficient implementation of these filtering operations. So suppose that the low-resolution part of a signal at level $j+1$ is given as a data set $\lambda_{j+1}$. This set is transformed into two other sets at level $j$: the low-resolution part $\lambda_j$ and the high-resolution part $\gamma_j$. This is obtained first by just splitting the data set $\lambda_{j+1}$ into two separate data subsets (usually called the *lazy wavelet transform*). The next step is to recombine these two sets in several subsequent lifting steps which decorrelate the two signals.

A *dual lifting* step can be seen as a prediction: the data $\gamma_j$ are "predicted" from the data $\lambda_j$. When the signals are still highly correlated, then such a prediction will usually be very good, and we can store only the part of $\gamma_j$ that differs from its prediction (the prediction error). Thus $\gamma_j$ is replaced by $\gamma_j - P(\lambda_j)$, where $P$ represents the prediction operator.

However, the new representation has lost certain basic properties, like for example the mean value of the signal. To restore this property, one needs a *primal lifting* step, whereby the set $\lambda_j$ is updated with data computed from the (new) subset $\gamma_j$. Thus $\lambda_j$ is replaced by $\lambda_j + U(\gamma_j)$, with $U$ some updating operator.

In general, several such lifting steps can be applied in sequence to go from level $j+1$ to level $j$. To recapitulate, let us consider a simple lifting scheme with only one pair of lifting steps to go from level $j+1$ to level $j$.

**Splitting** (*lazy wavelet transform*) Partition the data set $\lambda_{j+1}$ into two distinct data sets $\lambda_j$ and $\gamma_j$.

**Prediction** (*dual lifting*) Predict the data in the set $\gamma_j$ by the data set $\lambda_j$: $\gamma_j \leftarrow \gamma_j - P(\lambda_j)$.

**Update** (*primal lifting*) Update the data in the set $\lambda_j$ by the data in set $\gamma_j$: $\lambda_j \leftarrow \lambda_j + U(\gamma_j)$.

These steps can be repeated by iteration on the $\lambda_j$, creating a multi-level transform or multi-resolution decomposition. The inversion rules are obvious: revert the order of the operations, invert the signs in the lifting steps, and replace the splitting step by a merging step:

**Inverse update** $\lambda_j \leftarrow \lambda_j - U(\gamma_j)$,

**Inverse prediction** $\gamma_j \leftarrow \gamma_j + P(\lambda_j)$,

**Merge** $\lambda_{j+1} \leftarrow \lambda_j \cup \gamma_j$.

# 3 The Integer Wavelet Transform

In many applications (e.g. multimedia) the input data consist of integer samples. Fortunately the lifting scheme can be modified easily to a transform that maps integers to integers and that is still reversible [1, 6]. This is done by adding some rounding operations (indicated by curly braces), at the expense of introducing a nonlinearity in the transform:

$$\gamma_j \leftarrow \gamma_j - \left\{P(\lambda_j)\right\}, \qquad \lambda_j \leftarrow \lambda_j + \left\{U(\gamma_j)\right\}.$$

# 4 One-dimensional Lifting

For the one-dimensional case, we illustrate the lifting scheme with a simple example, using linear prediction. One transform step of a discrete one-dimensional signal $x = \{x_k\}$ looks like:

**Splitting** Split the signal $x$ into even samples and odd samples: $s_i \leftarrow x_{2i}$, $d_i \leftarrow x_{2i+1}$.

**Prediction** Predict the odd samples using linear interpolation: $d_i \leftarrow d_i - \left\{\frac{1}{2}(s_i + s_{i+1})\right\}$.

**Update** Update the even samples to preserve the mean value of the samples: $s_i \leftarrow s_i + \left\{\frac{1}{4}(d_{i-1} + d_i)\right\}$.

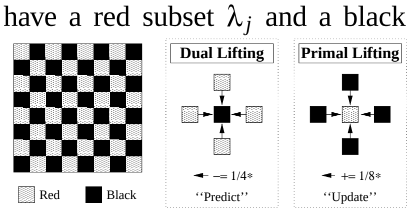
The resulting signal $s = \{s_k\}$ is a coarse representation of the original signal $x$, while $d = \{d_k\}$ contains the high-frequency information that is lost when going from resolution level $j+1$ to resolution level $j$.

This example is the lifting equivalent of the popular Cohen, Daubechies and Feauveau [2] classical biorthogonal wavelets with two vanishing moments for both the primal and dual wavelet (CDF $(2,2)$). Thanks to the lifting scheme, the accompanying wavelet transform can be implemented in an efficient way (see e.g. [6]).

# 5 The Red-Black Wavelet Transform

Classical one-dimensional wavelet transforms can be extended to more dimensions using tensor products, yielding a separable multi-dimensional transform. A disadvantage of this technique is the introduction of an anisotropy in the wavelet decomposition. The lifting scheme allows to design second-generation wavelets that are non-separable. We illustrate this on a rectangular grid $X = x_{i,j}$, using a two-dimensional analog of the one-dimensional CDF (2, 2) wavelet. This kind of lattice is known as a quincunx lattice, but we prefer the name Red-Black wavelet transform because it is more appropriate to describe the splitting step in the lifting scheme [5].

First we *split* the data in two subsets. The even/odd splitting from the one-dimensional case is replaced by a checkerboard splitting, with red and black squares. Thus we have a red subset $\lambda_j$ and a black subset $\gamma_j$.



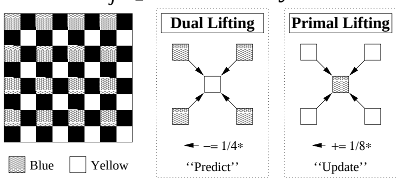

Then we *predict* the values of the red subset $\lambda_j$ by the average of its immediate horizontal and vertical neighbors in the black set and replace them by their prediction errors:

$$x_{i,j} \leftarrow x_{i,j} - \left\{\frac{1}{4}\left(x_{i-1,j} + x_{i,j-1} + x_{i,j+1} + x_{i+1,j}\right)\right\}$$

(for $i \bmod 2 \neq j \bmod 2$). Next, we *update* the data in the red subset $\gamma_j$ to preserve the mean value of the data:

$$x_{i,j} \leftarrow x_{i,j} + \left\{ \frac{1}{8} \left( x_{i-1,j} + x_{i,j-1} + x_{i,j+1} + x_{i+1,j} \right) \right\}$$

(for $i \bmod 2 = j \bmod 2$). For the subsequent resolution level $j-1$, we are left with the (new) red set $\lambda_j$ from the previous level. Because these data are arranged along diagonals on the grid, we repeat the same operation on the diagonals, i.e. we consider the checkerboard to be rotated over 45°. Again we start by splitting $\lambda_j$ into two subsets: the blue subset $\lambda_{j-1}$ and the yellow subset $\gamma_{j-1}$.




The same kind of averaging and updating steps are then computed for the blue-yellow squares:

$$x_{i,j} \leftarrow x_{i,j} - \left\{ \frac{1}{4} \left( x_{i-1,j-1} + x_{i-1,j+1} + x_{i+1,j-1} + x_{i+1,j+1} \right) \right\}$$

(for $i \bmod 2 = 1$ and $j \bmod 2 = 1$).

$$x_{i,j} \leftarrow x_{i,j} + \left\{ \frac{1}{8} \left( x_{i-1,j-1} + x_{i-1,j+1} + x_{i+1,j-1} + x_{i+1,j+1} \right) \right\}$$

(for $i \bmod 2 = 0$ and $j \bmod 2 = 0$). The values corresponding to the *red* and *blue* squares are low-resolution representations of the original image, while the *black* and *yellow* squares contain detail information. The inverse transform is straightforward.

For the next resolution level, we have to decompose the blue subset further, which is again arranged in a horizontal/vertical manner. Thus we can do again a red-black transform, yielding a multi-resolution decomposition of alternating red-black and a blue-yellow splitting.

A more general consideration of using lifting to construct wavelets in arbitrary dimensions can be found in [3].

## 6 Conclusion

The lifting scheme allows for the efficient implementation of integer wavelet transforms. It is not restricted to one-dimensional signals. Red-Black wavelets are less anisotropic than tensor product wavelets. The principle of the Red-Black wavelet transform is not restricted to a quincunx lattice. It can also be extended to triangular or hexagonal regular grids.

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