Externally and Internally Positive 2-D Linear Systems

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Abstract: - A new class of externally positive 2D linear systems described by the general model and the Roesser model is introduced. It is shown that the 2D linear systems are externally positive if and only if their impulse responses are not negative. The relationship between external and internal positiveness of 2D linear systems is also discussed.

Key-Words: - Externally positive, 2-D linear system, impulse response, necessary, sufficients condition
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1 Introduction

The most popular models of two-dimensional (2D) linear systems are those introduced by Roesser [9], Fornasini and Marchesini [1,2] and Kurek [7]. A positive (non-negative) 2D Roesser type model has been introduced in [4]. Recently Valcher and Fornasini in [10] have investigated some interesting properties of homogeneous 2D positive systems described by the second Fornasini-Marchesini type models. The external positive 1D continuous-time and discrete-time linear systems have been studied in [8].

In this paper a new class of externally positive 2D linear systems described by the general model [7] and the Roesser model [9] is introduced. Necessary and sufficient conditions for the external positiveness of 2D linear systems are established and the relationship between external and internal positiveness of these systems is discussed.

2 Externally positive general model

Let Z_+ be the set of nonnegative integers and $R^{n\times m}$ be the set of $n\times m$ real matrices $\left(R^n := R^{n\times 1}\right)$. The set of $n\times m$ real matrices with nonnegative entries will be denoted by $R_+^{n\times m}$ $\left(R_+^n := R_+^{n\times 1}\right)$.

Consider the 2D general model

$$x_{i+1,j+1} = A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + \\ + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \tag{1a}$$

$$y_{ij} = Cx_{ij} + Du_{ij} \quad , i, j \in Z_+ \tag{1b}$$

where $x_{ij} \in R^n$ is the local state vector at the point (i, j), $u_{ij} \in R^m$ is the input vector, $y_{ij} \in R^p$ is the output vector and $A_k \in R^{n\times n}, B_k \in R^{n\times m}, k = 0,1,2, C \in R^{p\times n}, D \in R^{p\times m}$
Boundary conditions for (1a) are given by

$$x_{i0} \text{ for } i \in Z_+ \text{ and } x_{0j} \text{ for } j \in Z_+ \tag{2}$$

Definition 1. The model (system) (1) is called externally (input-output) positive if for all $u_{ij} \in R_+^m$, $i, j \in Z_+$ and zero boundary conditions (2) i.e. $x_{i0} = 0$, $i \in Z_+$ and $x_{0j} = 0$, $j \in Z_+$ its output vector $y_{ij} \in R_+^p$ for $i, j \in Z_+$

Theorem 1. The model (1) is externally positive if and only if its impulse response $g_{ij} \in R_+^{p\times m}$

Proof. It is well-known [5] that for any $u_{ij} \in R^m$, $i, j \in Z_+$ the output vector y_{ij} of (1) is given by

$$y_{ij} = \sum_{k=0}^{i}\sum_{l=0}^{j} g_{i-k,j-l}u_{kl} \qquad (3)$$

From (3) it follows that if $u_{ij} \in R_+^m$ and $g_{ij} \in R_+^{p\times m}$ for $i, j \in Z_+$ then $y_{ij} \in R_+^p$ for $i, j \in Z_+$.

To show the necessity suppose that for some $h,r,p,q \in Z_+$

$$g_{ij} < 0 \text{ for } h \le i \le r, p \le j \le q$$

and

$$u_{ij} \begin{cases} > 0 & for \quad h \le i \le r, p \le j \le q \\ = 0 & otherwise \end{cases}$$

Then from (3) we obtain

$$y_{ij} = \sum_{k=h}^{r}\sum_{l=p}^{q} g_{i-k,j-l}u_{kl} < 0$$

Therefore, the model (1) externally positive if and only if $g_{ij} \in R_+^{p\times m}$. □

Theorem 2. A 2D linear system described by the equation

$$y_{ij} = \sum_{\substack{k=0\\k+l>0}}^{n}\sum_{l=0}^{m} a_{kl}y_{i-k,j-l} + \sum_{\substack{k=0\\k+l>0}}^{n}\sum_{l=0}^{m} b_{k,l}u_{i-k,j-l} \qquad (4)$$

is externally positive if

$$a_{kl} \ge 0 \text{ and } b_{kl} \ge 0$$
$$\text{for } k = 0,1,\ldots,n; l = 0,1,\ldots,m \ (k+l > 0) \qquad (5)$$

Proof. Using 2D z transform it is easy to show that the transfer function of the system described by (4) has the form

$$T(z_1,z_2) = \frac{\sum_{\substack{k=0\\k+l>0}}^{n}\sum_{l=0}^{m} b_{kl}z_1^{-k}z_2^{-l}}{1 - \sum_{\substack{k=0\\k+l>0}}^{n}\sum_{l=0}^{m} a_{kl}z_1^{-k}z_2^{-l}} \qquad (6)$$

The impulse response g_{kl} with $T(z_1,z_2)$ is related by

$$T(z_1,z_2) = \sum_{\substack{k=0\\k+l>0}}^{\infty}\sum_{l=0}^{\infty} g_{kl}z_1^{-k}z_2^{-l} \qquad (7)$$

From (6) and (7) we have

$$\sum_{\substack{k=0\\k+l>0}}^{n}\sum_{l=0}^{m} b_{kl}z_1^{-k}z_2^{-l} = \left(1 - \sum_{\substack{k=0\\k+l>0}}^{n}\sum_{l=0}^{m} a_{kl}z_1^{-k}z_2^{-l}\right)\left(\sum_{\substack{k=0\\k+l>0}}^{\infty}\sum_{l=0}^{\infty} g_{kl}z_1^{-k}z_2^{-l}\right) \qquad (8)$$

Comparison of the coefficients at the like powers of z_1 and z_2 of (8) we obtain

$$g_{k0} = b_{k0} \text{ for } k = 1,\ldots,n ,$$

$$g_{0l} = b_{0l} \text{ for } l = 1,\ldots,m$$

$$g_{kl} = \begin{cases} b_{kl} + \sum_{p=0}^{k}\sum_{q=0}^{l} a_{pq}g_{k-p,l-q} & for \quad 0 \le k \le n; 0 \le l \le m \\ \sum_{p=0}^{n}\sum_{q=0}^{m} a_{pq}g_{k-p,l-q} & otherwise \end{cases} \qquad (9)$$

From (9) it follows that $g_{kl} \ge 0$ for $k,l \in Z_+$. By theorem 1 the system described by (4) is externally positive. □

From theorem 2 and its proof we have.

Remark. The model (1) with transfer function (6) is externally positive if (5) hold.

3 Relationship between external positive and positive systems

Definition 2. The model (system) (1) is called internally positive (shortly positive) if for all boundary conditions $x_{i0} \in R_+^n, x_{0j} \in R_+^n, i, j \in Z_+$ and all input sequences $u_{ij} \in R_+^m, i, j \in Z_+$ we have $x_{ij} \in R_+^n$ and $y_{ij} \in R_+^p$ for all $i, j \in Z_+$.

From Definition 2 it follows that the model (1) is positive only if its impulse response $g_{kl} \in R_+^{p\times m}$ for $k,l \in Z_+$.

Therefore if the model (1) is positive it is also externally positive.

Theorem 3. The model (1) is positive if and only if

$$A_k \in R_+^{n\times n}, B_k \in R_+^{n\times m} \quad \text{for } k = 0,1,2$$
$$\text{and} \qquad C \in R_+^{p\times n}, D \in R_+^{p\times m} \tag{10}$$

Proof. If (10) holds and $x_{i0} \in R_+^n, x_{0j} \in R_+^n$, $u_{ij} \in R_+^m$ for $i, j \in Z_+$ then from (1a) and (1b) we have recurrently $x_{11} \in R_+^n, x_{12} \in R_+^n, x_{21} \in R_+^n, \ldots$ and $y_{11} \in R_+^p, y_{12} \in R_+^p, y_{21} \in R_+^p, \ldots$
To show the necessity we assume that $x_{00} = e_i$, $i = 1,\ldots,n$ (the ith column of the $n\times n$ identity matrix), $x_{10} = x_{01} = 0$ and $u_{ij} = 0$, $i, j \in Z_+$. Then for $i = j = 0$ from (1a) we obtain $x_{11} = A_{0i} \in R_+^n$ where A_{ki} is the ith column of A_k $(k = 0,1,2)$. If we assume $x_{10} = e_i, x_{00} = x_{01} = 0$ and $\in {}_+$ $= =$ $A_{1i} \in R_+^n$. Similarily if we $x_{01} = e_i, x_{00} = x_{10} = 0$ and $u_{ij} = 0, i, j \in Z_+$ then from (1a) for $i = j = 0$ we have $x_{11} = A_{2i} \in R_+^n$. In a similar way we may shown that $x_{11} = B_{ki} \in R_+^n$, $k = 0,1,2$ where B_{ki} is the ith column of B_k. By similar arguments using (1b) we may show the necessity of $C \in R_+^{p\times n}$ and $D \in R_+^{p\times m}$. □
The transition matrix T_{ij} for the model (1) is defined by [7,3]

$$T_{ij} = \begin{cases} I_n \ (\textit{the } n\times n \textit{ identity matrix}) & \textit{for } i = j = 0 \\ A_0 T_{i-1,j-1} + A_1 T_{i,j-1} + A_2 T_{i-1,j} & \textit{for } i, j \ge 0 \ (j + j > 0) \\ 0 \ (\textit{the zero matrix}) & \textit{for } i < 0 \textit{ or/and } j < 0 \end{cases} \tag{11}$$

Theorem 4. If the model (1) is positive then

$$T_{ij} \in R_+^{n\times n} \quad \text{for} \quad i, j \in Z_+ \tag{12}$$

Proof. The proof will be accomplished by induction on i and j. From (11) we have $T_{10} = A_2 \in R_+^{n\times n}$, $T_{01} = A_1 \in R_+^{n\times n}$, $T_{11} = A_0 + A_1 T_{10} + A_2 T_{01} \in R_+^{n\times n}$. Assuming that the hypothesis is true for the pairs $(p,q),(p+1,q)$ and $(p,q+1)$ we shall shown that it is also valid for the pair $(p+1,q+1)$. From (11) for $i = p+1,\ j = q+1$ we have

$$T_{p+1,q+1} = A_0 T_{pq} + A_1 T_{p+1,q} + A_2 T_{p,q+1} \in R_+^{n\times n} \qquad \square$$

4 Externally positive Roesser model

Consider the 2D Roesser model [9,5,6]

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij}, \ i, j \in Z_+ \tag{12a}$$

$$y_{ij} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + D u_{ij} \tag{12b}$$

where $x_{ij}^h \in R^{n_1}$ is the horizontal state vector at the point (i, j), $x_{ij}^v \in R^{n_2}$ is the vertical state vector, $u_{ij} \in R^m$ is the input vector, $y_{ij} \in R^p$ is the output vector and $A_{11} \in R^{n_1\times n_1}$, $A_{22} \in R^{n_2\times n_2}$, $B_1 \in R^{n_1\times m}, B_2 \in R^{n_2\times m}, C_1 \in R^{p\times n_1}, C_2 \in R^{p\times n_2}$, $D \in R^{p\times m}$
Boundary conditions for (12a) are given by

$$x_{0j}^h,\ j \in Z_+ \quad \text{and} \quad x_{i0}^v,\ i \in Z_+ \tag{13}$$

Definition 3. The Roesser model (12) is called externally (input-output) positive if for all $u_{ij} \in R_+^n$, $i, j \in Z_+$ and zero boundary conditions (13), i.e. $x_{0j}^h = 0, x_{i0}^v = 0,\ i, j \in Z_+$ its output vector $y_{ij} \in R_+^p$ for $i, j \in Z_+$.

Theorem 5. The Roesser model (12) is externally positive if and only if its impulse response $g_{ij} \in R_+^{p\times m}$.
The proof is similar to the one of theorem 1.
In [4] the positive (non-negative) Roesser model has been defined and it has been shown that the model (12) is positive if and only if

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in R_+^{(n_1+n_2)\times(n_1+n_2)},$$
$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in R_+^{(n_1+n_2)\times m}, C = [C_1 \quad C_2] \in R_+^{p\times(n_1+n_2)}, \quad (14)$$
$$D \in R_+^{p\times m}$$

The 2D Roesser model is positive only if its impulse response $g_{kl} \in R_+^{p\times m}$ for $k,l \in Z_+$

Therefore, if the 2D Roesser model is positive it is also externally positive.

Theorem 6. A single-input single-output 2D Roesser model with transfer function (6) is externally positive if the conditions (5) are satisfied.

Proof. In a similar way as in proof of theorem 2 it can be show that (9) holds and $g_{kl} > 0$ for $k,l \in Z_+$. □

Theorem 7. Let (14) with $D = 0$ be matrices of a positive Roesser model (12). Then the Roesser model with $A,-B,-C,D=0$ is externally positive, but it is not positive.

Let (10) with $D = 0$ be matrices of a positive model (1). Then the model (1) with $A_k,-B_k,-C,D=0$ $(k=0,1,2)$ is externally positive but it is not positive.

Proof. If (14) (with $D = 0$) holds then it is easy to show [9,3,5,6] that the solution of (12a)

$$\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} = \sum_{p=0}^{i} T_{i-p,j} \begin{bmatrix} 0 \\ x_{p0}^v \end{bmatrix} + \sum_{q=0}^{j} T_{i,j-q} \begin{bmatrix} x_{0q}^h \\ 0 \end{bmatrix} +$$
$$+ \sum_{p=0}^{i}\sum_{q=0}^{j}\left(T_{i-p-1,j-q} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + T_{i-p,j-q-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \right) u_{pq} \in R_+^{n_1+n_2} \quad (15)$$

for any boundary conditions (13) $x_{oj}^h \in R_+^{n_1}$, $x_{io}^v \in R_+^{n_2}$ and all $u_{ij} \in R_+^m$, $i,j \in Z_+$, where the transition matrix of (12) is defined by [9,3]

$$T_{ij} = \begin{cases} I_n & for \ i=j=0 \ (n = n_1 + n_2) \\ T_{10}T_{i-1} + T_{01}T_{j-1} & for \ i \ge 0, j \ge 0 \ (i+j \ne 0) \\ 0 & for \ i<0 \ or/and \ j<0 \end{cases} \quad (16)$$

and

$$T_{10} := \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, T_{01} := \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

satisfies the condition $T_{ij} \in R_+^{n\times n}$.

Note that if in (15) the matrix B is substituted by -B then $\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} \notin R_+^{n_1+n_2}$ and the Roesser model is not positive, but its output $y_{ij} = -C \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} \in R_+^p$.

Therefore, the impulse response of the Roesser model with $A,-B,-C,D=0$ satisfies the condition $g_{kl} \in R_+^{p\times m}$ for $k,l \in Z_+$ and it is externally positive.

The proof for the model (1) is similar.

Example. Consider the Roesser model (12) with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \hdashline 1 & 0 & 0 \end{array}\right], B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \left[\begin{array}{c} -1 \\ 0 \\ \hdashline -1 \end{array}\right], \quad (17)$$
$$C = [0 \quad -1 \ \vdots \ -1], D = 0$$

Using (16) we obtain

$$T_{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in R_+^{3\times 3}, T_{01} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in R_+^{3\times 3},$$
$$T_{11} = T_{10}T_{01} + T_{01}T_{10} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in R_+^{3\times 3}$$

and recurrently $T_{ij} \in R_+^{3\times 3}$ for $i,j \ge 0$

The solution (15) of Roesser model (12) with (17) for zero boundary conditions (13) and $u_{ij} = 2^i e^{-j}$, $i,j \in Z_+$ given by

$$\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} = \sum_{p=0}^{i}\sum_{q=0}^{j}\left(T_{i-p-1,j-q}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + T_{i-p,j-q-1}\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\right) 2^p e^{-q}$$

has not positive components. Therefore, the Roesser model (12) with (17) is not positive (the same follows from (17) and (14)) but it is externally positive since its impulse response [3,5]

$$g_{kl} = C\left(T_{k-1,l}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} + T_{k,l-1}\begin{bmatrix} 0 \\ B_2 \end{bmatrix}\right) =$$

$$= \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}\left(T_{k-1,l}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + T_{k,l-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \in R_+$$

is not negative.

5 Concluding remarks

It has been show that the 2D linear systems are externally positive if and only if their impulse responses are non-negative (theorems 1 and 5). Sufficient conditions have been established under which the 2D linear systems with given transfer functions are externally positive (remark and theorem 6).

With slight modifications the presented results can be easily extended for 2D linear continuous systems.

The considerations can be also extended for the $n-D$ $(n>2)$ linear systems [3,6].

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