Bifurcation Control and Stability Monitoring

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Abstract: This paper provides a brief summary of bifurcation control and of a recent related development, namely closed-loop monitoring systems for detecting incipient bifurcation. These monitoring systems use noisy precursors as a robust means for detecting an incipient loss of stability. Both Hopf bifurcation and stationary bifurcation are considered.

Key-Words: Bifurcation, Stability, Nonlinear Systems, Control Systems, Monitoring, Modes

1 Introduction

Bifurcations are transitions in the steady state behavior of a dynamical system that are triggered when a slight parameter variation renders unstable a nominally stable operating condition. Although prediction of bifurcations is tantamount to prediction of stability loss, a standard problem in linear system analysis, bifurcations themselves entail changes that cannot be predicted by linear system analysis alone. In this paper, only local bifurcations are considered, i.e., bifurcations from an equilibrium point of a nonlinear system. Some recent achievements in bifurcation control and in the allied subject of non-model based stability monitoring are summarized.

Since a detailed overview of bifurcation control is already given in the paper [4], here we give only a brief discussion of bifurcation control. We discuss in somewhat more detail the subject of closed-loop monitoring systems for detecting incipient instability, following the work of Kim and Abed [5, 6]. The monitoring systems employ results on so-called "noisy precursors," which are features observed in the output power spectral density of a system that is only marginally stable and excited by a noise input. As these features become more pronounced, they give a reliable indication of an incipient loss of stability.

The paper proceeds as follows. In Section 2, necessary concepts on bifurcations and bifurcation control are reviewed. In Section 3, noisy precursors of local bifurcations are explained. In Section 4, a closed-loop stability monitoring technique that uses noisy precursors is summarized.

2 Local Bifurcations and Local Bifurcation Control

A *bifurcation* is a change in the number of steady state solutions of a nonlinear system that occurs as a parameter is quasistatically varied. The parameter being varied is referred to as the bifurcation parameter. A value of the bifurcation parameter at which a bifurcation occurs is called a critical value of the bifurcation parameter. Bifurcations from a nominal operating condition can only occur at parameter values for which the condition (say, an equilibrium point or limit cycle) either loses stability or ceases to exist.

Consider a general one-parameter family of ordinary differential equation systems

$$\dot{x} = F^{\mu}(x) \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $\mu \in \mathbb{R}$ denotes the bifurcation parameter, and F is smooth in x

and μ.

Local bifurcations are those that occur in the vicinity of an equilibrium point. For example, a small-amplitude limit cycle can emerge (bifurcate) from an equilibrium as the bifurcation parameter is varied. Local bifurcations can occur only when the linearized system loses stability. Suppose, for example, that the origin is the nominal operating condition for some range of parameter values. That is, let $F^\mu(0) = 0$ for all values of μ for which the nominal equilibrium exists. Denote the Jacobian matrix of (1) evaluated at the origin by

$$A(\mu) := \frac{\partial F^\mu}{\partial x}(0).$$

Local bifurcations from the origin can only occur at parameter values μ where $A(\mu)$ loses stability.

In a very real sense, the fact that bifurcations occur when stability is lost is helpful from the perspective of control system design. To explain this, suppose that a system operating condition (the "nominal" operating condition) is not stabilizable beyond a critical parameter value. Suppose a bifurcation occurs at the critical parameter value. That is, suppose a new steady state solution emerges from the nominal one at the critical parameter value. Then it may be that the new operating condition is stable and occurs beyond the critical parameter value, providing an alternative operating condition near the nominal one. This is referred to as a *supercritical bifurcation.* For example, a supercritical Hopf bifurcation is depicted in Fig. 1(a). In contrast, it may happen that the new operating condition is unstable and occurs prior to the critical parameter value. In this situation (called a *subcritical bifurcation*), the system state must leave the vicinity of the nominal operating condition for parameter values beyond the critical value. A subcritical Hopf bifurcation is depicted in Fig. 1(b). Feedback offers the possibility of rendering such a bifurcation supercritical. This is true even if the nominal operating condition is not stabilizable. If such a feedback control can be found, then the system behavior beyond the stability boundary can remain close to its behavior at the nominal operating condition.

The most important local bifurcations are stationary bifurcation, Hopf bifurcation and saddle node bifurcation. These are discussed at length in books on nonlinear dynamics; for references see [4]. The stationary bifurcation and the Hopf bifurcation can be either subcritical or supercritical. The saddle node bifurcation entails annihilation of the nominal equilibrium, and is always a so-called "dangerous" event. However, feedback control can often be used to render supercritical a stationary bifurcation or a Hopf bifurcation. This is discussed in [1, 2, 4].

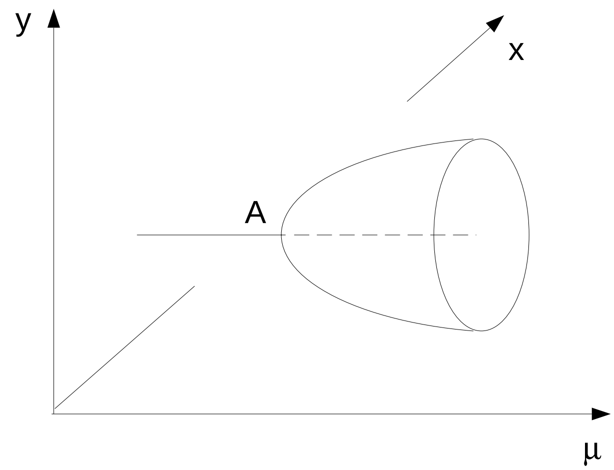


Fig. 1(a). Supercritical Andronov-Hopf Bifurcation

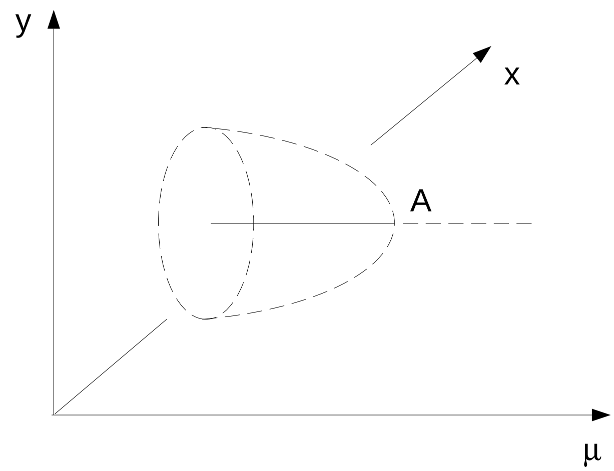


Fig. 1(b). Subcritical Andronov-Hopf Bifurcation

3 Noisy Precursors of Local Bifurcations

In this section, results are recalled from [5, 6] that extend the noisy precursor analysis of Wiesenfeld [8] to systems operating at an equilibrium point.

Wiesenfeld considered systems driven by white noise and operating near a periodic steady state. In [5, 6], it is shown that the power spectrum of a measured output for a system undergoing bifurcation from an equilibrium exhibits sharply growing peaks. This is used as a basis for the design of closed-loop monitoring systems for detecting incipient instability. The advantage of this method is that it does not require availability of an accurate system model—the noisy precursor characteristics occur if a bifurcation is imminent, regardless of system details.

Consider a nonlinear dynamic system ("the plant")

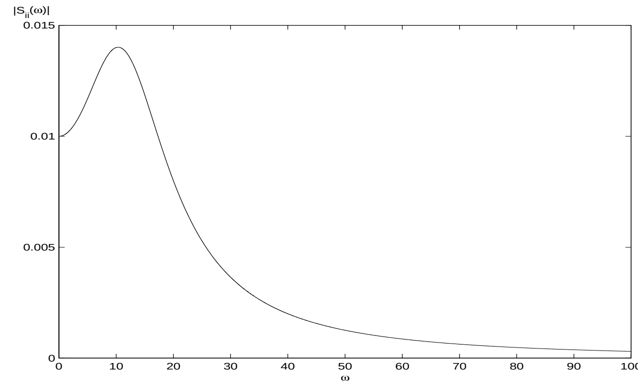
$$\dot{\tilde{x}} = f(\tilde{x}, \mu) + N(t) \tag{2}$$

where $\tilde{x} \in R^n$, μ is a bifurcation parameter, and $N(t) \in R^n$ is a zero-mean vector white Gaussian noise process. Let the system possess an equilibrium point $\tilde{x}_0$. For small perturbations and noise, the dynamical behavior of the system can be described by the linearized system in the vicinity of the equilibrium point $\tilde{x}_0$. The linearized system corresponding to (2) with a small noise forcing $N(t)$ is given by

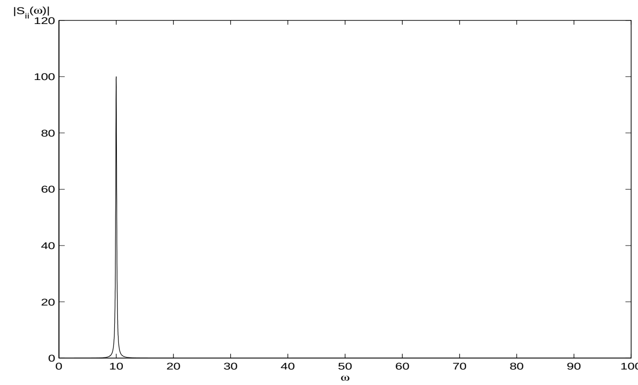
$$\dot{x} = Df(\tilde{x}_0, \mu)x + N(t) \tag{3}$$

where $x := \tilde{x} - \tilde{x}_0$ and $N(t) \in R^n$ is a vector white Gaussian noise having zero mean. For the results of the linearized analysis to have any bearing on the original nonlinear model, we must assume that the noise is of small amplitude. This assumption of small noise will be explicated below, in terms of smallness of correlation and cross-correlation coefficients. The distinct notation for the system state $\tilde{x}$ and the linearized system state x was used here for clarity. In the sequel, we will simply use the notation x and the meaning will be clear from the context.

In [5, 6], the power spectrum is calculated by taking the Fourier transform of the autocovariance function for $x_i(t)$. The details will not be repeated here in the interest of brevity. Suffice it to say that analytical formulas are found for the power spectrum both in the case of Hopf bifurcation and in the case of stationary bifurcation. The formulas are asymptotic and assume that either a complex conjugate pair of eigenvalues is nearing the imaginary axis or that a real eigenvalue is nearing the origin. Fig. 2 depicts the nature of the result for Hopf bifurcation, and Fig. 3 does the same for stationary bifurcation.



a. $\epsilon = 10$

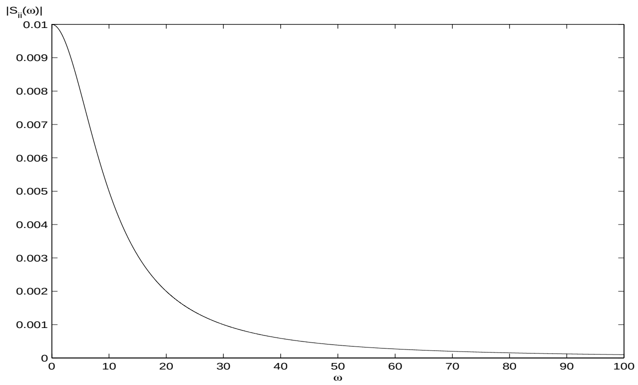


b. $\epsilon = 0.1$

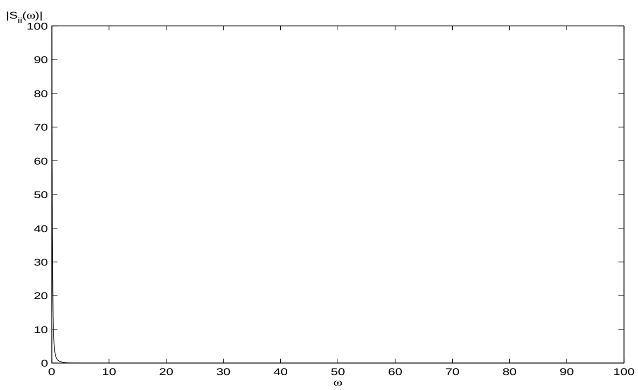
Fig. 2. Power spectrum magnitude for Hopf bifurcation when $\omega = 10$ for two values of ϵ

Fig. 2 is from an example in [5, 6]. It shows the magnitude of the power spectral density of the i-th state $S_{ii}(\eta)$ for $\omega = 10$, for two values of ϵ. Note the sharp peak around $\omega = 10$ that appears as $\epsilon \to 0$. The power spectrum peak near the bifurcation is located at ω, and the magnitude of this peak grows without bound as ϵ approaches to zero. This property will be used as a precursor signaling the closeness to Hopf bifurcation.

In [5, 6], for the case of a stationary bifurcation it is found that the magnitude of the power spectrum peak grows as ϵ approaches to zero and the location of this peak is $\eta = 0$. Fig. 3 is from an example in [5, 6]. It shows the magnitude of $S_{ii}(\eta)$. Note the sharp growing peak around $\omega = 0$ as $\epsilon \to 0$.



a. $\epsilon = 10$



b. ϵ= 0.1

Fig. 3. Power spectrum magnitude for stationary bifurcation for two values of ϵ

4 Closed-Loop Monitoring System

Suppose the plant of interest is susceptible to loss of stability through a stationary bifurcation. Since Hopf bifurcation is easier to detect than stationary bifurcation through noisy precursors, we introduce a monitoring system that replaces the stationary bifurcation with a Hopf bifurcation of tunable frequency.

Let the plant obey the dynamics

$$\dot{x} = f(x, \mu) \tag{4}$$

and suppose the following assumptions hold:

(S1) The origin is an equilibrium point of system (4) for all values of μ.

(S2) System (4) undergoes stationary bifurcation at $\mu = \mu_c$. (i.e., there is a simple eigenvalue $\lambda(\mu)$ of $Df(x^0(\mu), \mu)$ such that for some value $\mu = \mu_c$, $\lambda(\mu_c) = 0$ and $\frac{d\lambda(\mu_c)}{d\mu} \neq 0$)

(S3) All other eigenvalues of $Df(0, \mu_c)$ are in the open left half complex plane.

We introduce the following *augmented system* corresponding to (4):

$$\begin{aligned} \dot{x}_i &= f(x, \mu) - cy_i \\ \dot{y}_i &= cx_i \end{aligned} \tag{5}$$

Here, $y \in R^n$, $c \in R$ and $i = 1, 2, \ldots, n$. Eq. (5) will later be viewed as a basic monitoring system whose use facilitates detection of either stationary or Hopf bifurcation. Note that the state vector consists of the original physical system states x augmented with the states y of the monitoring system.

Proposition 1 *Under assumptions (S1)-(S3), the augmented system (5) undergoes a Hopf bifurcation from the origin at $\mu = \mu_c$. Moreover, if for any value of μ the origin of the original system (4) is asymptotically stable (resp. unstable), then the origin is asymptotically stable (resp. unstable) for the augmented system (5).*

Note that since the value c in equation (5) is adjustable, we can control the crossing frequency of the complex conjugate pair of eigenvalues of the augmented system. Thus, for detecting stationary bifurcation, we only need to monitor a frequency band around the chosen value of c. It is also possible to slowly vary c in a controlled fashion, giving added confidence in our assessment that an instability is imminent.

There are some other advantages of our monitoring system. The augmented system (5) has the same critical parameter value (μ_c) as the original system. This is actually not a luxury but a necessity for the system to be practically useful. In addition, the proof [6] shows that augmenting the states y_i and applying the feedbacks cy_i to the original system does not change the local stability of the system. Moreover, to apply the monitoring system, we do not need knowledge of the original system.

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