Substitution systems for quasiperiodic sequences

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Abstract

We consider cut-and-project sequences Λ for which there exists a self-similarity factor θ such that $\theta\Lambda \subset \Lambda$. We show that a large class of them is closely related to substitution systems. In particular, we show that there exist a substitution system and suitable lengths of letters such that Λ may be identified with a fixed point of the substitution.

1 Introduction

We consider real strictly increasing sequences $\Lambda = \{x_n \mid n \in Z\}$ where the set of distances $\{x_{n+1} - x_n \mid n \in Z\}$ between adjacent points is finite. There are two well-known way how to obtain such sequences. The first construction is so called cut-and-project scheme, where $(m+1)$-dimensional lattice $Z^{m+1} \subset R^{m+1}$ is projected to a real line V_1.

Let $R^{m+1} = V_1 \oplus V_2$. Denote by π_1 and π_2 the projections on V_1 and V_2, resp. If $\Omega \subset V_2$ is a convex set with nonempty interior then

$$\Lambda = \Lambda(\Omega) = \{\pi_1(x) \mid x \in Z^{m+1}\,,\, \pi_2(x) \in \Omega\}\,, \tag{1}$$

is a set with finite number of distances between neighbouring points. Moreover, such $\Lambda(\Omega)$ has so called Meyer property, i.e. there exists finite set F such that $\Lambda(\Omega) - \Lambda(\Omega) \subset \Lambda(\Omega) + F$.

The second way to obtain strictly increasing sequences $\{x_n \mid n \in Z\}$, where $\{x_{n+1} - x_n \mid n \in Z\}$ is finite, is based on substitution rules which produce bidirectional infinite word in a finite alphabet. One may associate each letter of the alphabet with some length and put bricks of these lengths on a real line according to the order of letters in the word.

Generaly, these two types of sequences have different properties. If we focus our attention only to sequences which are selfsimilar (i.e. there exists a nontrivial factor θ such that $\theta\Lambda \subset \Lambda$), the situation is changed.

Bombieri and Taylor showed in [1] that if a sequence Λ arose from a substitution rule for which the Perron-Frobenius eigenvalue λ of the substitution matrix is a Pisot number then it is possible to chose lengths corresponding to letters in such a way that Λ is contained in a cut-and-project sequence and λ is a selfsimilarity factor of Λ.

On the other hand, it is known [3] that if θ is a selfisimilarity factor of the cut-and-project sequence Λ then θ is a Pisot or Salem number. In this contribution, we would like to discuss the question opposite to those of Bombieri and Taylor: Which selfsimilar cut-and-project sequences can be generated by a substitution rule?

We will demonstrate relationship between cut-and-project sequences and substitution rules on the case of a projection of two dimensional lattice Z^2 into a real line. Let Λ be a sequence given by (1) where $m = 1$. The presence of a nontrivial selfsimilarity θ then forces θ to be a quadratic Pisot number and the slope β of the line V_1 and the slope γ of V_2 are forced to be mutually conjugate elements of the quadratic field $Q[\theta]$. Let us restrict to β which is a quadratic unitary Pisot number. All such numbers β are given by quadratic equations $x^2 = mx + 1$, $m \in N$, or $x^2 = mx - 1$, $m \in N$, $m \geq 3$. Such equations have two mutually conjugate roots, say

$$\beta = \frac{m + \sqrt{m^2 \pm 4}}{2} \qquad \text{and} \qquad \beta' = \frac{m - \sqrt{m^2 \pm 4}}{2}$$

which means that $\beta > 1$ and $|\beta'| < 1$. For such a number β, the definition (1) of Λ can be rewritten as

$$\Lambda = \Lambda(\Omega) = \{a + b\beta \mid a, b \in Z\,,\, a + b\beta' \in \Omega\} \tag{2}$$

where Ω is a bounded non empty interval in R. We shall assume (without loss of generality) that Ω contains the origin. The elements of the sequence may be enumerated in such a way that $\Lambda = (x_n)_{n\in Z}$ is a strictly increasing sequence and $x_0 = 0$. It can be derived from the 3-gap theorem [4] that such a Λ has at most three (2 or 3) possible distances $x_{n+1} - x_n$ between adjacent points for any choice of the interval Ω.

One may describe (see [5]) all selfsimilarities of such point sets. In particular, we shall be interested in all the factors s for which $s\Lambda \subset \Lambda$ (i.e. selfsimilarities fixing the origin). It may be easily shown that all such factors should be of the form $s = a + b\beta$, $a, b \in Z$ and $s' := a + b\beta'$ being $0 < s' < 1$. For our construction of substitution rules, we will make use of the factor $s = \beta^2$.

2 Substitution systems

The relation between the sequences (2) and substitution rules has been recognized by many authors. An easy example can be seen on the well known Fibonacci chain. Consider a sequence Λ, based on the golden mean irrationality $\beta = \tau = \frac{1}{2}(1+\sqrt{5})$ with the interval Ω of the unit length. In this case, the distances between adjacent points of Λ have only two possible values $\{\tau, \tau^2\}$. The sequence Λ is created by the substitution

$$\begin{array}{ccc} A & \to & AB \\ B & \to & A \end{array}$$

If we start with a pair of letters $B|A$ and carry out the substitution, we produce step by step

$$B|A \to A|AB \to AB|ABA \to ABA|ABAAB \to ABAAB|ABAABABA \to \dots$$

After an infinite number of steps, one obtains an infinite word in the alphabet $\{A, B\}$ which is invariant with respect to the substitution. One may imagine letters as tiles of given lengths layed down onto the real line starting from the origin in the order prescribed by the infinite word. Associating the lengths $l(A) = \tau^2$ and $l(B) = 1$ to the letters and the origin to the delimiter $|$, we obtain the division of the real line as by the point set Λ.

In a similar way, the substitutions $A \to A^m B$, $B \to A$ correspond to sequences based on $\beta^2 = m\beta + 1$, with Ω of the unit length.

Generally, a substitution system is given by a finite alphabet $A = \{\alpha_1, \dots, \alpha_k\}$ and a substitution rule $\phi : A \to A^*$, which assigns to each letter α_i a word $\phi(\alpha_i)$ in the alphabet A. We say that a bidirectional infinite word

$$w = \dots \alpha_{i_{-3}}\alpha_{i_{-2}}\alpha_{i_{-1}}|\alpha_{i_0}\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\dots$$

is a fixed point of our substitution iff

$$w = \dots \phi(\alpha_{i_{-3}})\phi(\alpha_{i_{-2}})\phi(\alpha_{i_{-1}})|\phi(\alpha_{i_0})\phi(\alpha_{i_1})\phi(\alpha_{i_2})\phi(\alpha_{i_3})\dots$$

An infinite word w may be represented on the real axis in the following way. We associate to each letter α_i a length $l(\alpha_i)$, and starting from 0 forwards and backwards we put side-by-side tiles of length $l(\alpha_i)$ according to the order of letters in the infinite word w.

Naturally, one may associate a matrix P with non negative entries P_{ij}, $i, j = 1, \dots, k$ to a given substitution ϕ. The element P_{ij} is the number of letters α_j in the word $\phi(\alpha_i)$. The substitution is said to be primitive, if there exists an exponent $k \in N$ such that $P^k_{ij} > 0$ for each i, j. If the substitution is primitive then the largest eigenvalue λ of P is called the Perron-Frobenius eigenvalue and the eigenvector v corresponding to λ has positive entries $\mathbf{v} = (v_1, \dots, v_k)$. If we set $l(\alpha_i) = v_i$, then the point set associated to a fixed point w of the substitution ϕ has a self-similarity with respect to the origin with the scaling factor λ.

It can be also shown, that the entries $\tilde{v}_i$ of the eigenvector $\tilde{\mathbf{v}} = (\tilde{v}_1, \dots, \tilde{v}_k)$ of the transposed matrix P^T corresponding to the same eigenvalue λ, determine the relative frequences of letters α_i in the infinite word w.

Apparently, all the examples of substitutions related to cut-and-project sets found in literature, are given only for special intervals Ω, which reduces the amount of such examples to one per a given irrationality. In this article, we explain that for a fixed quadratic unitary Pisot number there is an infinite family of substitution rules producing non equivalent three-distance sequences. In particular, we provide a necessary and sufficient condition for an interval Ω in order that there exist a substitution system and suitable lengths of letters such that Λ may be identified with a fixed point of the substitution. This would mean that Λ may be built starting from two tiles - left and right neighbours of zero - using repeated iterations of the substitution ϕ.

3 Substitutions for the three-distance sequences

For convenience, we shall state all the results for the most simple example of a quadratic unitary Pisot number, the golden ratio τ. Here $\tau = \frac{1}{2}(1+\sqrt{5})$ and $\tau' = \frac{1}{2}(1-\sqrt{5})$ are two roots of the quadratic equation $x^2 = x+1$. We denote the set $Z[\tau] := Z + Z\tau$ and, for any element $x = a + b\tau \in Z[\tau]$, we define $x' = a + b\tau'$. $Z[\tau]$ denotes the ring of integers in $Q[\tau]$, the mapping $'$ is the Galois automorphism on this field. It is everywhere discontinuous. With such a notation, let us recall the definition of a sequence $\Lambda(\Omega)$,

$$\Lambda(\Omega) := \{x \in Z[\tau] \mid x' \in \Omega\},$$

where Ω is a bounded non empty interval in R. In order that 0 be a member of Λ, we need $0 \in \Omega$.

On an easy example of a sequence $\Lambda(\Omega)$ with $\Omega = [0, 1+1/\tau^3)$, we shall describe the algorithm for obtaining the substitution rule.

As it was mentioned, there are three different distances between adjacent members of the sequence Λ: small one $S = 1$, middle one $M = \tau$ and large one $L = \tau^2$. Directly from the definition of Λ, one may generate several right and left neighbours of the origin.

$\ldots$	$x_{-3} = -\tau^3 - \tau$	$x_{-2} = -\tau^3$
$x_{-1} = -\tau$	$x_0 = 0$	$x_1 = 1$
$x_2 = \tau^2$	$x_3 = 2\tau^2$	$x_4 = \tau^4$
$x_5 = \tau^4 + 1$	$x_6 = \tau^4 + \tau^2 + 1$	$\ldots$

The important property for generating the sequence Λ is that, for any point $x_n \in \Lambda$, we are able to determine its first right neighbour. In particular, we have in our case

(i) x_n is followed by the distance S, i.e. $x_{n+1} = x_n + 1$, iff $x'_n = [0, 1/\tau^3) =: \Omega_S$.

(ii) x_n is followed by the distance M, i.e. $x_{n+1} = x_n + \tau$, iff $x'_n = [1/\tau, 1 + 1/\tau^3) =: \Omega_M$.

(iii) x_n is followed by the distance L, i.e. $x_{n+1} = x_n + \tau^2$, iff $x'_n = [1/\tau^3, 1/\tau) =: \Omega_L$.

Assume that we have generated the whole sequence $\Lambda = \Lambda[0, 1 + 1/\tau^3)$. Multiplying this set by its self-similarity factor τ^2, we obtain the set $\tau^2\Lambda$ whose adjacent points have distances $\tau^2 S$, $\tau^2 M$, and $\tau^2 L$. Since generally we have $\tau^2\Lambda(\Omega) = \Lambda(\Omega/\tau^2)$, we obtain

$$\tau^2\Lambda\left[0, 1 + \frac{1}{\tau^3}\right) = \Lambda\left[0, \frac{1}{\tau^3} + \frac{1}{\tau^5}\right) \subset \Lambda\left[0, 1 + \frac{1}{\tau^3}\right).$$

Let us fill in the gap following the point $\tau^2 x_n$ in $\tau^2\Lambda$ where $x_{n+1} = x_n + 1$ i.e., the point x_n was followed by S in Λ. For such an x_n, the case (i) was necessarily true which means that $x'_n \in [0, 1/\tau^3)$. The rescaled point $\tau^2 x_n$ has the Galois image $(\tau^2 x_n)' = x'_n/\tau^2 \in [0, 1/\tau^5) \subset \Omega_S$. Therefore, the first right neighbour of the point $\tau^2 x_n$ in Λ is $y_1 = \tau^2 x_n + S$. Since $y'_1 = x'_n/\tau^2 + 1 \in [1, 1 + 1/\tau^5) \subset \Omega_M$, the next member of the sequence Λ will be the point $y_2 = y_1 + M = \tau^2 x_n + S + M$. For the point y_2, it holds that $y'_2 = x'_n/\tau^2 + 1 - 1/\tau \in [1/\tau^2, 1/\tau^2 + 1/\tau^5)$. Therefore $y_2 \in \Lambda[0, 1/\tau^2 + 1/\tau^5) = \tau^2\Lambda[0, 1 + 1/\tau^3)$ and hence we have already found all members of the sequence Λ in the gap $\tau^2 S$ between points $\tau^2 x_n$ and $\tau^2 x_{n+1}$.

Since for any member of Λ, which is followed by the distance S, we may carry out the same considerations we may conclude that all small gaps S after rescaling by the factor τ^2 are filled in the same way, namely by distances S and M in this order. Symbolically, we write $S \to SM$. Using the same procedure, we may obtain that $M \to LM$ and $L \to SMLM$. Note that the left neighbour of the origin in the sequence Λ is the point $x_{-1} = -\tau$, corresponding to the gap M, and the first right neighbour of 0 is $x_1 = 1$, corresponding to the gap S. Thus from the starting pair $M|S$ by repeated multiplication by the factor τ^2 and by filling the stretched gaps according to the derived rules, we may generate the entire sequence $\Lambda[0, 1 + 1/\tau^3)$,

$$M|S \to LM|SM \to SMLMLM|SMLM \to$$
$$\to SMLMSMLMLMSMLMLM|SMLMSMLMLM \to \dots$$

Not always the situation is so favourable as in the previous example. ¿For different interval Ω, it may happen that not all the gaps labeled by the same letter, say M, are after multiplication by the self-similarity factor filled in the same way. Nevertheless, we may be able to divide these gaps M into a finite number of groups $M_1, \dots, M_k$, in order that all gaps within the same group M_i are filled equally. For example, the sequence $\Lambda = \Lambda[0, 1 + 1/\tau^2)$ is generated by the substitution rule

$$\begin{array}{lcl} S & \to & SM_2 \\ M_2 & \to & SM_2M_1 \\ M_2 & \to & LM_1 \\ L & \to & SM_2LM_1 \end{array}$$

The lengths of the gaps denoted by M_1 and M_2 are of course both equal to τ^2. Such a substitution rule uses instead of a three letter alphabet an alphabet consisting of four letters $\{S, M_1, M_2, L\}$.

The necessary and sufficient condition for an interval Ω in order that the corresponding Λ could be obtained using substitution rules is stated in the following theorem.

Theorem 3.1. [2] *There exists an alphabet $A = \{a_1, \dots, a_k\}$, $k \geq 2$, and a substitution $\phi : A \to A^*$, reproducing the sequence $\Lambda[c, d)$ starting from 0 if and only if $c, d \in Q[\tau]$.*

If $c, d \in Q[\tau]$ and $y \in \Lambda[c, d) \subset Z[\tau]$, then $c - y', d - y' \in Q[\tau]$ and $\Lambda[c, d) = y + \Lambda[c - y', d - y')$. This simple property which may be easily proven directly from the definition (2), has an interesting consequence: If it is possible to find a substitution which generates the given sequence Λ starting form the origin then for any element $x \in \Lambda$, there exists another substitution generating Λ from the point x.

4 Comments

Let us make several comments to the above results.

- If Ω is an interval of the type $[c, d]$, (c, d) or $(c, d]$, we may stipulate similar assertion as in Theorem 3.1. In the case that $c, d \in Z[\tau]$ and $\Omega = [c, d]$, or $\Omega = (c, d)$, the substitution rule is not primitive, it means that there does not exist a positive power P^k of the matrix corresponding to the substitution rule, having all elements strictly positive.

- If the definition of Λ involves a quadratic unitary Pisot number distinct from the golden ratio, the substitution rules may be derived in the same way.

- For cubic irrationalities, the construction of substitution rules is remarkable more complicated. Consider $\beta > 1$, the root of the equation $x^3 = 2x^2 - x + 1$ and denote by β' and β'' the other two roots of the equation. Then a sequence Λ may be defined, in analogue to (2), by

  $$\Lambda = \Lambda(\Omega_1, \Omega_2) = \{a + b\beta + c\beta^2 \mid a, b, c \in Z\ ,\ a + b\beta' + c\beta'^2 \in$$

  Such a sequence has 9 to 11 types of distances between adjacent points [6] dependingly on the intervals Ω_1 and Ω_2. While for the quadratic case, the different letters correspond to subintervals Ω_S, Ω_M, and Ω_L of the original Ω, for the cubic irrationalities, a given type of gap is related to a connected region of the rectangle $\Omega_1 \times \Omega_2$.

- In [1], Bombieri and Taylor present a 3-distance sequence generated by substitution

$$a \to aac\ , \qquad b \to ac\ , \qquad c \to b$$

with characteristic equation of the substitution matrix $x^3 = 2x^2 - x + 1$. Because of the small number of distances, such a sequence is only a proper subset of some $\Lambda(\Omega_1, \Omega_2)$.

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