Quasiperiodic self-similar structures

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Abstract

In this paper we focus on point sets arising from a general cut and project scheme ('model sets'). We illustrate that only certain schemes may produce model sets with self-similarities. In this case the model set is based on a Pisot number and the structure of self-similarities is very rich. We recall the notion of λ-convexity (Pinch) and show that the model set is λ-convex for an infinity of self-similarity factors λ. However, not all λ-convex sets can be identified with model sets.

1 Introduction

A point set $\Lambda \subset R^n$ modelling recently discovered physical quasicrystals, should have several attributes. First of all, it should satisfy the Delone property (i.e. (i) the distances between any pair of points are greater than a fixed $\varepsilon > 0$, (ii) there exists $r > 0$, such that the entire space is covered by balls of radius r centered at points of Λ). Secondly, Λ should be aperiodic (without any periodic subsets), and repetitive (any finite configuration of points of Λ is repeated infinitely many times in Λ). In our contribution we provide an algebraic definition of such point sets, calling them 'model sets'. The elements of a model set Λ may be coordinatized by integers in an algebraic field. More precisely, Λ is a subset of a $Z[\beta]$-module, where $Z[\beta]$ is the ring of integers in the extension $Q[\beta]$ of rationals Q by an algebraic number β. The acceptance condition, which choses suitable points of the $Z[\beta]$-module to Λ is formulated using the Galois automorphisms on $Q[\beta]$. For an introduction to model sets and their basic properties we refer to the contribution of J. Patera [7].

The algebraic definition of model sets corresponds to a special case of the so-called cut and project scheme. Consider a Z-lattice $L = \sum_{i=1}^{n} Zw_i \subset R^{m+n}$, where $w_1, \ldots, w_{n+m}$ is a basis of R^{n+m}. Choose two subspaces V_1 and V_2, with $\dim V_1 = n$, $\dim V_2 = m$, and $V_1 \oplus V_2 = R^{m+n}$. The projections on V_1, V_2 are denoted by π_1 and π_2. The choice of the subspaces should be done in such a way that π_1 restricted to L is a 1-1 mapping and $\pi_2(L)$ is dense in V_2. One considers a bounded region $\Omega \subset V_2$, with non empty interior and such that $\overline{\Omega^\circ} = \overline{\Omega}$. Then a model set Λ is defined as

$$\Lambda = \{\pi_1(x) \mid x \in L\, , \pi_2(x) \in \Omega\}\ , \qquad (1)$$

and Ω is called the acceptance window of Λ. It was shown in [6] that any model set is a Meyer set, i.e. is Delone and satisfies the condition $\Lambda - \Lambda \subset \Lambda + F$, where F is a finite set. In [5] one finds there the following theorem concerning Meyer sets.

Theorem 1.1 (Y. Meyer). *If Λ is a Delone set satisfying $\Lambda - \Lambda \subset \Lambda + F$, for a finite set F, and if there exists a factor $\lambda > 1$, such that $\lambda\Lambda \subset \Lambda$, then λ is a Pisot or Salem number, i.e. λ is an algebraic integer and all the Galois conjugates of λ are in modulus less than 1 (strictly less for Pisot numbers, less or equal to for Salem numbers).*

Due to the above theorem, model sets with a self-similarity arise only in certain cut and project schemes. In Section 2 we illustrate on the simpliest example, that if a model set $\Lambda \in R^n$ has the self-similarity factor λ, then the coordinates of Λ belong to the algebraic field $Q[\lambda]$. For such Λ we determine in Section 3 the semigroup of affine symmetries. It turns out that any element of a model set Λ is a self-similarity center, i.e a fixed point of some affine transformation. To any such self-similarity center there correspond infinitely many affine mappings, having different scaling factors.

If for a fixed factor λ a point set Λ satisfies $\lambda(\Lambda - x) \subset \Lambda - x$ for all elements $x \in \Lambda$, then Λ is called λ-convex. The notion of λ-convexity was introduced by R. G. E. Pinch in [9]. Model sets with their abundant affine symmetries are λ-convex for a large family of factors λ. We show that only for certain of these factors any λ-convex set can be identified with a model set. For other factors one may expect existence of new interesting self-similar structures.

2 Algebraic definition of model sets

In this Section we illustrate the Meyer theorem mentioned above. We explain how a cut and project scheme, which produces a self-similar structure, is related to algebraic numbers. We provide an algebraic definition of model sets.

First of all, let us explain what we mean by a self-similarity of a point set. A self-similarity of an n-dimensional point set Λ is an affine transformation $T : R^n \to R^n$, defined for $\lambda \in R$, $t \in R^n$, by

$$Tx := \lambda x + t\,, \qquad x \in R^n\,, \tag{2}$$

which maps Λ into itself. In general, we may consider T to include also an orthogonal transformation $R \in O(n, R)$, i.e. $Tx := \lambda Rx + t$, but this is outside of the aims of this contribution. Thus the action of a self-similarity reduces to a rescaling by a factor λ and a translation by a vector t.

Let us assume that we have a model set Λ constructed from a cut and project scheme with a lattice L in R^{n+m}, the dimensions of V_1, V_2 being $\dim V_1 = n$, $\dim V_2 = m$. According to (1), Λ is a subset of $\pi_1(L)$, which is a Z-module in V_1. If Λ is to have a self-similarity with a factor $\lambda > 1$, the same should be valid for the Z-module $\pi_1(L)$, i.e. we require $\lambda\pi_1(L) \subset \pi_1(L)$.

A sufficient condition for a model set Λ of (1) with the acceptance window Ω to have a self-similarity factor λ is given in [2].

Proposition 2.1. *Let $w_1, \dots, w_{m+n}$ be a basis of R^{m+n} such that $L = \sum Zw_i$. For $\lambda \in R$, $\lambda' \in (-1, 1)$ we denote by C the mapping $C := \lambda\pi_1 + \lambda'\pi_2$. Assume that following conditions are satisfied.*

(i) The matrix of C in the basis w_i, $i = 1, \dots, m+n$, has integer entries.

(ii) There exists $v \in L$ such that $T'(\Omega) := \lambda'\Omega + \pi_2(v)$ maps Ω into itself, $T'(\Omega) \subset \Omega$.

Then the mapping $T : R^n \to R^n$, acting by $Tx := \lambda x + \pi_1(v)$, is a self-similarity of the model set $\Lambda = \Lambda(\Omega)$.

Practically, from all the pairs λ, λ' satisfying (i), one chooses those which verify (ii) for given acceptance window Ω. Let us see what implications the assumption (i) has on the considered cut and project scheme.

Note that if the matrix of the mapping $C = \lambda\pi_1 + \lambda'\pi_2$ in the basis w_i has integer entries, then C preserves the lattice L. Simple algebraic manipulations allow us to find that if there exist such λ and λ', then λ is an algebraic integer of degree 2 (solution of a quadratic equation with integer coefficients), and the number λ' belongs to the quadratic field $Q[\lambda]$. Moreover, it turns out that λ, λ' are related by the Galois automorphism of the field, $\,'\, : x \mapsto x'$. Futher we find that the matrices of projections π_1, π_2 in the basis w_i have entries in $Q[\lambda]$, and that $\pi_2' = \pi_1$ componentwisely. This in turn implies, that the matrices of the two projections have the same rank and therefore the dimensions of the subspaces V_1, V_2, are equal.

We may conclude that the cut and project scheme should have the following form: We project the points of a lattice $L = \sum_{i=1}^{2n} Zw_i \subset R^{2n}$ to the subspaces V_1, V_2 with $\dim V_1 = \dim V_2 = n$. Their orientation is given by the conditions on matrices of the projections π_1, π_2. In order that λ is a self-similarity of a model set Λ, we need to assume moreover that $\lambda'\Omega + \pi_2(v) \subset \Omega$, for some $v \in L$, (see assumption (ii) of Proposition 2.1). It follows that $|\lambda'| < 1$, and hence λ is a quadratic Pisot number.

We have explained how the presence of a self-similarity forces the cut and project scheme to have certain properties, which allow us to rewrite the definition of a model set in an algebraic way, based on an algebraic number β. We shall focus on the case if β is a quadratic unitary Pisot number.

Let β and β' be the roots of the quadratic equation $x^2 = m\beta + 1$, $m \in N$, or $x^2 = m\beta - 1$, $m \in N$, $m \geq 3$. We choose for β the larger of the roots, therefore $\beta > 1$ and $|\beta'| < 1$. In particular $\beta\beta' = \mp 1$, which is why β is called a quadratic *unitary* Pisot number. Denote $Z[\beta] := Z + Z\beta$. In general, $Z[\beta]$ is a subring of the ring of integers of the quadratic field $Q[\beta]$. The correspondence $\beta \to \beta'$ gives the Galois automorphism on $Q[\beta]$. For an $x \in Q[\beta]$, i.e. $x = a + b\beta$, with $a, b \in Q$, we define $x' = a + b\beta'$.

Let α_i, α_i^*, $i = 1, \ldots, n$, be two bases of R^n. Then a model set $\Sigma(\Omega)$ is defined by

$$\Sigma(\Omega) := \left\{ X = \sum_{i=1}^{n} x_i \alpha_i \;\middle|\; x_i \in Z[\beta]\,,\, X^* = \sum_{i=1}^{n} x_i' \alpha_i^* \in \Omega \right\}, \tag{3}$$

for a bounded non empty region Ω with $\overline{\Omega^\circ} = \overline{\Omega}$, which is called the acceptance window of the model set $\Sigma(\Omega)$. It can be shown that this algebraic definition corresponds to the geometric one (1). The set $\Sigma(\Omega)$ is Delone [6], it contains no periodic subsets, and under some assumptions on Ω it is repetitive.

3 Self-similarities and λ-convexity

Let us now describe the self-similarities of model sets given by the definition (3). Observe that any transformation T of the form (2) has a fixed point $u = (1-\lambda)^{-1}t$, for which $Tu = u$. We may rewrite (2) in a form which tells more about the action of T,

$$Tx = \lambda x + (1-\lambda)u = \lambda(x-u) + u =: T_{(u,\lambda)}(x)\,. \tag{4}$$

If such a transformation leaves a point set Λ invariant, ($T\Lambda \subset \Lambda$), the point u is said to be the center of the scaling symmetry, or 'inflation center'. Our aim is to find, for a given center u, all factors λ such that $T_{(u,\lambda)}(\Lambda) \subset \Lambda$. Of course, we shall be interested in $\lambda \neq 0, 1$.

The following theorem describes a large family of self-similarity centers, namely those, which are elements of the model set (internal inflation centers).

Theorem 3.1. *Let β be a quadratic unitary Pisot number, Ω a bounded convex region in R^n. Then*

$$\lambda(\Sigma(\Omega) - x) \subset \Sigma(\Omega) - x\,,$$

for any $x \in \Sigma(\Omega)$ and any $\lambda \in Z[\beta]$ such that $\lambda' \in [0, 1]$. In other words, any element x of a model set with convex acceptance window is its self-similarity center for an infinite family of scaling factors.

In [3] it is shown that in addition to inflation centers among the points of $\Sigma(\Omega)$, the model set has infinitely many self-similarity centers which are not its elements (external inflation centers). The family of scaling factors corresponding to each such non trivial inflation center $u \notin \Sigma(\Omega)$ is infinite, but not as large as for internal inflation centers.

Consider all the transformations $T_{(u,\lambda)}$ of (4) preserving a given model set Λ, i.e. with u being any inflation center external or internal, and λ a corresponding scaling factor. Together with standard composition of mappings they form a semi-group of affine symmetries of Λ.

In [9] R. G. E. Pinch indtroduces the notion of λ-convexity, which is closely connected to the self-similarities. A point set $\Lambda \subset R^n$ is called λ-convex, if

$$\lambda x + (1-\lambda)y \in \Lambda\,, \qquad \text{for any } x, y \in \Lambda\,.$$

According to Theorem 3.1, model sets with convex acceptance windows are λ-convex for any $\lambda \in Z[\beta]$, such that $0 < \lambda' < 1$. One may ask more about the relation of λ-convex sets and model sets. For example, if $\lambda = -\tau$, (i.e. $\lambda \in Z[\tau]$, $\lambda' = 1/\tau \in (0,1)$), then any λ-convex set Λ is either dense in R^n or an n-dimensional model set. The assertion is formulated precisely in the following two theorems. Their proofs may be found in [4].

Theorem 3.2. *Let $\Lambda \subset R^n$ be a Delone $(-\tau)$-convex set. Then there exist bases α_i, α_i^* of R^n, such that Λ is a model set in the sense of equation (3) for some bounded region Ω with non empty convex interior and such that $\overline{\Omega^\circ} = \overline{\Omega}$.*

Theorem 3.3. *Let $F \subset R^n$ be such that $0 \in F$ and F spans R^n over real numbers. Let Λ be the smallest $(-\tau)$-convex set containing F. Then $\Lambda \subset R^n$ is Delone if and only if there exists $w_1, \ldots, w_n \in F$ such that $F \subset \sum_{i=1}^{n} Q[\tau]w_i$. If this is the case, then according to Theorem 3.2, Λ is a model set.*

Both theorems above concern λ-convex sets for $\lambda = -\tau$. Similar properties do not hold for every considered λ, i.e. every $\lambda \in Z[\beta]$, such that $0 < \lambda' < 1$. In particular, not every λ-convex set may be identified with a model set based on the corresponding irrationality. A counter example for this is found by the following proposition.

Proposition 3.4. *Let β be a quadratic unitary Pisot number, and let $\lambda \in Z[\beta]$, such that $0 < \lambda' < 1$. The smallest λ-convex set, containing $\{0,1\}$ is a model set $\Sigma[0,1] \subset Z[\beta]$ if λ or $1-\lambda$ takes one of the three values $-\frac{1}{2}(1+\sqrt{5})$, $-1-\sqrt{2}$, or $2+\sqrt{3}$.*

4 Conclusion

In the contribution we have explained how the requirement of the presence of a self-similarity factor for a model set Λ limits the cut and project scheme in which Λ is constructed. For such scheme, the definition may be formulated in a nice algebraic way, see (3). Model sets given by (3) have a rich structure of self-similarities, described in the terms of a semi-group. The presence of a self-similarity factor of a model set Λ may allow one to generate Λ using substitution rules [8].

We have established the relation between model sets and λ-convex sets. Any model set with convex acceptance window is λ-convex for infinitely many factors λ, but up to three exceptional cases, given such λ, there are λ-convex sets, which cannot be identified with any model set. This conclusion mativates one for further study of the structure and properties of λ-convex sets.

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