A Matrix-Measure Condition for the Robust Stabilization of a Class of Bilinear Systems

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Abstract: - The robust asymptotic stabilization of a class of uncertain bilinear systems with delays in the state variables is considered. Sufficient conditions are derived to guarantee the stability of the time-delay system under state-feedback control in the presence of time-varying, nonlinear uncertainties. Use is made of the matrix measure to yield a general robust-stability condition and a characterization of the associated domain of attraction for the nonlinear system. IMACS/IEEE CSCC'99 Proceedings, Pages:5431-5433

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1 Introduction

A great number of processes in science and engineering can be modeled as a bilinear system. Earlier publications concern the analysis and structural properties of bilinear systems [1]. Later, optimal control theory and quadratic indices were used for the design of controllers for bilinear systems (*cf.* [2] and included references). The stability of perfectly known bilinear models is studied in [1], [3], and [4].

Various engineering systems involve time delays in the state or the control variables. These delays, which are often ignored to make the theoretical analysis simpler, can be a source of instability. Much work has concentrated on the analysis of linear systems with delays ([5], [6] and references). In contrast, the problem of analyzing bilinear time-lag models has not been given comparable attention. In a recent publication Lu and Wey [7] examine the stability of a bilinear system with delay in the state, and derive a sufficient condition using the Lyapunov direct method. The Lu-Wey approach assumes that the system model is exactly known; however, the mathematical modeling of physical systems always involves uncertainties associated with the nominal model. The treatment of uncertain bilinear systems with time delays and their stability properties appears to be absent from the literature.

In this paper a state-feedback controller approach is employed to stabilize a continuous-time uncertain bilinear system with delay in the state variables.

2 Notation

The matrix measure is a function μ: ${}^{nxn} \rightarrow$

$$\mu(\boldsymbol{A}) = \lim_{\varepsilon \rightarrow 0^+} \frac{\|\boldsymbol{I} + \varepsilon \boldsymbol{A}\| - 1}{\varepsilon}$$

where $\|\ \|$ is an induced matrix-norm on ^{nxn}. For the usual 1, 2, and infinity induced norms the matrix measure is given by the following simple formulas:

$$\mu_1(\boldsymbol{A}) = \max_j (a_{jj} + \sum_{i \neq j} |a_{ij}|)$$

$$\mu_2(\boldsymbol{A}) = \lambda_{\max}(\frac{\boldsymbol{A}^* + \boldsymbol{A}}{2})$$

$$\mu_\infty(\boldsymbol{A}) = \max_i (a_{ii} + \sum_{j \neq i} |a_{ij}|)$$

where $\boldsymbol{A}^*$ is the Hermitian of matrix $\boldsymbol{A}$, and $\lambda_{\max}$ represents the maximum eigenvalue.

2 Problem Formulation

Consider the uncertain MIMO bilinear system with time-delay in the state represented by the equations

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_1\boldsymbol{x}(t) + \boldsymbol{A}_2\boldsymbol{x}(t-\tau) + \boldsymbol{B}\boldsymbol{u}(t) + \sum_{i=1}^{m} [\boldsymbol{N}_i \boldsymbol{x}(t) u_i(t) + \boldsymbol{g}_1(\boldsymbol{x}(t),t) + \boldsymbol{g}_2(\boldsymbol{x}(t-\tau),t)] \quad (1)$$

$$\boldsymbol{x}(\theta) = \phi(\theta) \quad , \quad \theta \quad [-\tau, 0] \quad (2)$$

where $\boldsymbol{x}(t)$ ^{n} is the state vector with initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$; $\boldsymbol{u}(t)$ ^{m} is the input vector; $\boldsymbol{A}_i$, $\boldsymbol{B}$, and $\boldsymbol{N}_i$, are constant matrices of appropriate dimensions; $\phi(t)$ is a continuous vector-valued initial function; and $\tau > 0$ is the time delay. The vector functions $\boldsymbol{g}_1(\boldsymbol{x}(t),t)$ ^{n} and $\boldsymbol{g}_2(\boldsymbol{x}(t-\tau),t)$ ^{n} represent nonlinear modeling perturbations that depend on the current state $\boldsymbol{x}(t)$ and the delayed state $\boldsymbol{x}(t-\tau)$ of the system, respectively. It is assumed that the modeling uncertainties satisfy the bounds

$$\|\boldsymbol{g}_1(\boldsymbol{x}(t),t)\| \leq \gamma_1 \|\boldsymbol{x}(t)\| \quad (3)$$

and

$$\|\boldsymbol{g}_2(\boldsymbol{x}(t-\tau),t)\| \leq \gamma_2 \|\boldsymbol{x}(t-\tau)\| \quad (4)$$

where γ_1 and γ_2 are known positive real constants, and the operator $\|\ \|$ may be any vector norm. Note that in the case where the nonlinear uncertainty $\boldsymbol{g}_1(\boldsymbol{x}(t),t) = 0$, it is accepted that $\gamma_1 = 0$ and likewise for the case in which $\boldsymbol{g}_2(\boldsymbol{x}(t-\tau),t) = 0$ where $\gamma_2 = 0$. It is also assumed that the following inequality is satisfied for all $\theta \in [-\tau, 0]$ and for all real $q > 0$:

$$\|x(t+\theta)\| \le q\|x(t)\| \tag{5}$$

Similar assumptions have been used in the context of Razumikhin-type theorems where Lyapunov functionals are employed for stability analysis. For an extensive discussion see [8, pg. 127].

Using a state feedback control law

$$u(t) = Fx(t) \tag{6}$$

where F is a constant matrix, the objective is to find sufficient conditions that F must satisfy in order to asymptotically stabilize the bilinear system (1)-(2) for any modeling uncertainties that satisfy the norm bounds (3)-(4).

Theorem. *Suppose that the bilinear system (1)-(2) satisfies the uncertainty bounds (3)-(4) and inequality (5). Then (6) is a robustly stabilizing state feedback control if*

$$\mu(A_1 + BF) + q\|A_2\| + \gamma_1 + q\gamma_2 < 0 \tag{7}$$

and the initial state lies in the domain of attraction defined by the inequality

$$\|x_0\| < -\frac{\mu(A_1 + BF) + q\|A_2\| + \gamma_1 + q\gamma_2}{\sum_{i=1}^{m}\|N_i\|\|F\|} \tag{8}$$

Proof. Let a component of the input vector (6) be $u_i(t) = f_i^{\mathrm{T}} x(t)$ where f_i^{T} is the i-th row of matrix F. From (1) the closed-loop system is written as

$$\dot{x}(t) = \bar{A}_1 x(t) + A_2 x(t-\tau) + \sum_{i=1}^{m} N_i x(t) f_i^{\mathrm{T}} x(t) + g_1(x(t),t) + g_2(x(t-\tau),t) \tag{9}$$

where $\bar{A} = A_1 + BF$. The solution to (9) for $t \ge 0$, is readily expressed as the integral equation

$$x(t) = e^{\bar{A}t}x_0 + \int_0^t e^{\bar{A}(t-s)}[A_2 x(s-\tau) + \sum_{i=1}^{m} N_i x(s) f_i^{\mathrm{T}} x(s) + g_1(x(s), s) + g_2(x(s-\tau), s)]\, ds \tag{10}$$

Taking the norm of both sides in (10), using the inequality [9]

$$\|e^{At}\| \le e^{\mu(A)t}, \quad t \ge 0 \tag{11}$$

and invoking the bounding inequalities (3)-(5) yields

$$\|x(t)\| \le e^{\mu(\bar{A})t}\|x_0\| + \int_0^t e^{\mu(\bar{A})(t-s)}(q\|A_2\| + \gamma_1 + q\gamma_2)\|x(s)\| ds + \int_0^t e^{\mu(\bar{A})(t-s)} \sum_{i=1}^{m}\|N_i\|\|F\|\|x(s)\|^2 ds \tag{12}$$

Now consider the scalar differential equation

$$\dot{z}(t) = [\mu(\bar{A}) + q\|A_2\| + \gamma_1 + q\gamma_2]z(t) + \sum_{i=1}^{m}\|N_i\|\|F\| z(t)^2 \tag{13}$$

with initial condition $z(0) = \|x_0\|$. The solution to (13) is unique and is given by the integral expression

$$z(t) = e^{\mu(\bar{A})t}z(0) + \int_0^t e^{\mu(\bar{A})(t-s)}(q\|A_2\| + \gamma_1 + q\gamma_2)z(s)ds + \int_0^t e^{\mu(\bar{A})(t-s)} \sum_{i=1}^{m}\|N_i\|\|F\| z(s)^2 ds \tag{14}$$

From inequality (12) and invoking the Comparison Theorem [8], it follows that

$$\|x(t)\| \le z(t), \quad \text{for } t \ge 0 \tag{15}$$

therefore, asymptotic stability for (13) (*i.e.*, $z(t) \to 0$ as $t \to \infty$) implies asymptotic stability for $x(t)$. From the Poincaré-Lyapunov theorem (*cf.* [9]) it follows that (13) is asymptotically stable if $\mu(\bar{A}) + q\|A_2\| + \gamma_1 + q\gamma_2 < 0$ and if $z(0) = z_0$ is sufficiently small. A characterization of the smallness of z_0 can be obtained by examining the asymptotic behavior of $z(t)$ as a function of z_0. First, the equilibrium points of (13) are found by setting the derivative of $z(t)$ equal to zero to obtain $z_1 = 0$ and

$$z_2 = -\frac{\mu(\bar{A}) + q\|A_2\| + \gamma_1 + q\gamma_2}{\sum_{i=1}^{m}\|N_i\|\|F\|} \tag{16}$$

It follows that when $\mu(\bar{A}) + q\|A_2\| + \gamma_1 + q\gamma_2 < 0$, then $z(t) \to 0$ for all $z_0 < z_2$, and that $z(t) \to \infty$ in finite time for all $z_0 > z_2$. Using the fact that $z_0 = \|x_0\|$ and recognizing (15), it immediately follows that when the inequality condition (7) is satisfied then $x(t) \to 0$ as $t \to \infty$ provided that the initial state in turn satisfies condition (8) of the Theorem as specified by (16). *Q.E.D.*

As can easily be verified, the bounds obtained using the sufficient conditions (7) and (8) vary with the chosen norm and the corresponding matrix measure [9]. It is then possible that for a given norm and matrix measure one can conclude stability, while with other matrix norms the stability condition may not hold. In this respect, the problem of choosing a suitable norm and matrix measure to tighten the stability condition is similar to the problem of finding an appropriate Lyapunov function candidate in the well-known and widely used Lyapunov techniques for determining the stability of control systems.

Besides the well-known 1, 2, and infinity norms, other induced norms and matrix measures involving weighting parameters may be utilized in the stability conditions. As an example consider the following weighted matrix norm and corresponding matrix measure:

$$\|A\|_w = \max_i \sum_j \frac{w_j}{w_i}|a_{ij}|,$$

$$\mu_w(A) = \max_i \{a_{ii} + \sum_{j \ne i} \frac{w_j}{w_i}|a_{ij}|\}$$

3 Example

Consider an uncertain bilinear system defined as in (1) with dynamics described by

$$A_1 = \begin{bmatrix} 0.1 & -0.2 \\ 0.8 & -2.1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1.1 \\ 0.2 \end{bmatrix} \quad N = \begin{bmatrix} 0 & -0.2 \\ 0.1 & 0 \end{bmatrix}$$

and the nonlinear uncertainty bounds $\gamma_1 = 0.3$. For simplicity, let $g_2(x(t-\tau), t) = 0$, which is equivalent to setting $\gamma_2 = 0$. Notice that the open-loop system is unstable since matrix A has one positive eigenvalue.

By a standard pole-placement technique, we take the eigenvalues of $\bar{A} = A_1 + BF$ to be -2.09 ± 0.4i, and then find the feedback matrix $F = [2.95 \quad 0.25]$. Using condition (7), we obtain results for the usual 1, 2 and infinity norms. For the 1-norm,

$$\mu_1(\bar{\boldsymbol{A}}) + q\|\boldsymbol{A}_2\| + \tfrac{1}{3}\|\boldsymbol{B}\|\|\boldsymbol{F}\| + \gamma_1 = 0.15$$

which is greater than zero; therefore nothing can be concluded for the stability of the system.

For the 2-norm,

$$\mu_2(\bar{\boldsymbol{A}}) + q\|\boldsymbol{A}_2\| + \tfrac{1}{3}\|\boldsymbol{B}\|\|\boldsymbol{F}\| + \gamma_1 = -0.54$$

with a corresponding region of attraction given by $\|\boldsymbol{x}_0\| \le 0.91$ while for the ∞-norm,

$$\mu_\infty(\bar{\boldsymbol{A}}) + q\|\boldsymbol{A}_2\| + \tfrac{1}{3}\|\boldsymbol{B}\|\|\boldsymbol{F}\| + \gamma_1 = -0.18$$

with $\|\boldsymbol{x}_0\| \le 0.30$. Hence, it follows from the Theorem that the delayed bilinear system is asymptotically stable when the initial state belongs to a region of attraction defined by $\|\boldsymbol{x}_0\| \le 0.91$. In fact, if there is no uncertainty (i.e., $\gamma_1 = \gamma_2 = 0$), then application of the Theorem to the pole-placement design leads to the stability condition

$$\mu_2(\bar{\boldsymbol{A}}) + q\|\boldsymbol{A}_2\| + \tfrac{1}{3}\|\boldsymbol{B}\|\|\boldsymbol{F}\| + \gamma_1 = -0.84$$

Hence, the system is stable. Furthermore, the region of attraction in this case is given by $\|\boldsymbol{x}_0\| \le 1.41$. Clearly, the system without uncertainty enjoys a larger region of attraction, as expected.

Note that Longchamp [4] claims that for the particular example, the region of attraction $\|\boldsymbol{x}_0\| \le 0.255$ is considered adequate for practical applications; hence, the region of attraction $\|\boldsymbol{x}_0\| \le 0.91$ found by the approach proposed here could also be deemed appropriate.

4 Conclusions

This paper establishes sufficient conditions for the robust stabilization of uncertain bilinear systems with delay in the state variables. The results presented are applicable to continuous time models that include delayed states as well as general nonlinear uncertainty descriptions. The derived conditions are given in terms of succinct scalar inequalities. A characterization of the region of attraction is also given, and an example illustrates the results. The derived results are believed to be extendable to the case of constrained control and the latest findings will be reported at the conference. Research currently focuses on deriving conditions for the case where the states are not available and must be estimated via an appropriate extended Kalman filter.

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