Trajectory-based global and semi-global stability results [1]

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Abstract: - In this paper we are interested in (global) exponential stability results for ordinary differential equations. Stability results are often obtained within the framework of Lyapunov theory, but here we provide an alternative approach which is based on an analysis of trajectories. It is shown that, if trajectories of a given system are close enough to trajectories of another system that is known to be globally exponentially stable, then (global) exponential stability may be concluded for the original system.

This approach extends previous results in this context, and leads to global and semi-global stability results for fast time-varying systems —studied in averaging theory— and highly oscillatory systems —studied by Sussmann and Liu.

Key-Words: - stability, global, semi-global, averaging, highly oscillatory systems.

1 Introduction

Stability results are often obtained within a framework of Lyapunov theory. The aim of this paper is to propose an alternative approach: following the line of research initiated in [5, 4], we present trajectory-based techniques for the analysis of stability.

We assume that we are given two systems: a system whose stability properties are under investigation, and a *reference* system whose origin is known to be a globally exponentially stable equilibrium point. It is shown that, if trajectories of the first system are close enough to those of the reference system, then trajectories of the first system are attracted towards the origin along with those of the reference system, and hence the origin of the system under investigation is (globally) exponentially stable.

Based on this approach we obtain (semi-)global stability results for fast time-varying systems studied in averaging theory and highly oscillatory systems studied, e.g., in [8].

The paper differs from previous work in this area [5, 4] in two respects: (i) whereas [5] restricts attention to homogeneous systems and [4] considers practical stability, here we are interested in (semi-)global exponential stability for general dynamical systems, not necessarily homogeneous. (ii) In order to obtain these (semi-)global stability results, we have to adapt the approach from [5, 4] and incorporate ideas from [1].

This paper is organized as follows. Section 2 introduces some preliminary notions. Section 3 contains a general stability result. In Sections 4 and 5, this stability result is applied to fast time-varying and highly oscillatory systems. Section 6 concludes the paper.

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2 Preliminaries

The state space of all systems featuring in the present paper is $\mathbb{R}^n$ with $n \in \mathbb{N}$.

2.1 Homogeneity

Although the present theory does not concentrate on homogeneous systems, we introduce some notions related to homogeneity. This will be used to estimate the rate of convergence of the reference system.

Given an n-tuple $r = (r_1, \dots, r_n) \in ((0, \infty))^n$, we define the *dilation* δ^r to be the map

$$\begin{aligned}\delta^r : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n : \\ (\lambda, x) \mapsto \delta^r(\lambda, x) = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n).\end{aligned} \quad (1)$$

An *r-homogeneous norm* ρ is defined to be a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}$ that satisfies the following conditions:

$$\rho(x) > 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \rho(0) = 0, \quad (2)$$

$$\rho(\delta^r(\lambda, x)) = \lambda \rho(x). \quad (3)$$

Notice that an r-homogeneous norm is a radially unbounded function.

2.2 Dynamical systems and flows

All dynamical systems featuring in the present paper are described by ordinary differential equations

$$\dot{x} = f(t, x)$$

where, by assumption, (i) f is a continuous map from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$, and (ii) $\dot{x} = f(t, x)$ has the uniqueness property of solutions. Such dynamical systems will be referred to as *admissible dynamical systems on* $\mathbb{R}^n$. We do not assume completeness of solutions; that is, we do not exclude finite escape times.

Given an admissible dynamical system on $\mathbb{R}^n$, let $\phi(t, t_0, x_0)$ be the trajectory of this system passing through state x_0 at time t_0 evaluated at time t. We call the function $\phi : (t, t_0, x_0) \mapsto \phi(t, t_0, x_0)$ the *flow* of this system.

3 Stability result

The aim of this section is to investigate stability properties of a dynamical system by comparing its trajectories with those of an exponentially stable reference system.

Consider two admissible dynamical systems on $\mathbb{R}^n$

$$\dot{x} = f(t, x), \quad (4)$$

$$\dot{x} = g(t, x) \quad (5)$$

with respective flows ϕ and ψ.

Assume that the origin is a globally uniformly exponentially —with respect to an r-homogeneous norm— stable equilibrium point of $\dot{x} = g(t, x)$; i.e., there exist $\mu \geq 1$, $\nu > 0$ and an r-homogeneous norm ρ such that

$$\rho(\psi(t, t_0, x_0)) \leq \mu e^{-\nu(t - t_0)} \rho(x_0) \quad \forall t \geq t_0, \ \forall t_0, \ \forall x_0. \quad (6)$$

Let $T > 0$ be such that

$$\mu e^{-\nu T} = \beta < 1. \quad (7)$$

This implies that, after a time T, the function ρ has decreased at least with a factor $\beta < 1$ along trajectories of $\dot{x} = g(t, x)$:

$$\rho(\psi(t_0 + T, t_0, x_0)) \leq \beta \rho(x_0) \quad \forall t_0, \ \forall x_0. \quad (8)$$

Let $d > 0$ be such that

$$\left.\begin{aligned}\rho(x_1) \leq \beta \\ \rho(x_2 - x_1) \leq d\end{aligned}\right\} \Rightarrow \rho(x_2) \leq \alpha < 1 \quad \forall x_1, x_2 \in \mathbb{R}^n. \quad (9)$$

Based on (3) this implies that for any $x_0 \in \mathbb{R}^n$

$$\left.\begin{aligned}\rho(x_1) \leq \beta \rho(x_0) \\ \rho(x_2 - x_1) \leq d \rho(x_0)\end{aligned}\right\} \Rightarrow \rho(x_2) \leq \alpha \rho(x_0) \quad \forall x_1, x_2 \in \mathbb{R}^n. \quad (10)$$

Assume that trajectories of $\dot{x} = f(t, x)$ are close to those of $\dot{x} = g(t, x)$ in the following sense:

$$\rho(\phi(t, t_0, x_0) - \psi(t, t_0, x_0)) \leq d \rho(x_0) \quad \forall t \in [t_0, \, t_0 + T], \ \forall t_0, \ \forall x_0. \quad (11)$$

This implies that trajectories of $\dot{x} = f(t,x)$ are attracted towards the origin along with trajectories of $\dot{x} = g(t,x)$. Indeed, estimates (8) and (11) together with (10) yield

$$\rho(\phi(t_0+T, t_0, x_0)) \le \alpha\rho(x_0) \quad \forall t_0,\ \forall x_0. \tag{12}$$

That is, after a time T, the function ρ has decreased at least with a factor $\alpha < 1$ along trajectories of $\dot{x} = f(t,x)$.

We now study the trajectories of $\dot{x} = f(t,x)$ for intermediate times $t \in [t_0,\, t_0+T)$. Let $\kappa \ge 1$ be such that

$$\left.\begin{array}{r} \rho(x_1) \le \mu \\ \rho(x_2 - x_1) \le d \end{array}\right\} \Rightarrow \rho(x_2) \le \kappa \quad \forall x_1, x_2 \in \mathbb{R}^n. \tag{13}$$

Based on (3) this implies that for any $x_0 \in \mathbb{R}^n$

$$\left.\begin{array}{r} \rho(x_1) \le \mu\rho(x_0) \\ \rho(x_2 - x_1) \le d\rho(x_0) \end{array}\right\} \Rightarrow \rho(x_2) \le \kappa\rho(x_0) \qquad \forall x_1, x_2 \in \mathbb{R}^n. \tag{14}$$

Estimates (6) and (11) together with (14) yield

$$\rho(\phi(t, t_0, x_0)) \le \kappa\rho(x_0) \qquad \forall t \in [t_0,\, t_0+T),\ \forall t_0,\ \forall x_0. \tag{15}$$

That is, for intermediate times $t \in [t_0,\, t_0+T)$ we have a bound on trajectories of $\dot{x} = f(t,x)$.

Based on these estimates (12) and (15), we conclude that the origin is a globally uniformly exponentially —with respect to the r-homogeneous norm ρ— stable equilibrium point of $\dot{x} = f(t,x)$. Notice that, if assumption (11) is only satisfied for $\rho(x_0) \le \sigma$ for some $\sigma > 0$, we may conclude that the origin is a locally uniformly exponentially —with respect to the r-homogeneous norm ρ— stable equilibrium point of $\dot{x} = f(t,x)$ with region of attraction $\{x_0 \in \mathbb{R} : \rho(x_0) \le \sigma\}$. We have thus proven the following result:

Theorem 1. *Given admissible dynamical systems (4) and (5) with respective flows ϕ and ψ. Given an r-homogeneous norm ρ and real numbers $\mu \ge 1$, $\nu > 0$, $T > 0$, $d > 0$. Assume that ψ satisfies (6) and that T and d satisfy (7) and (9) for some $\alpha, \beta \in (0, 1)$. If*

$$\rho(\phi(t, t_0, x_0) - \psi(t, t_0, x_0)) \le d\rho(x_0) \qquad \forall t \in [t_0,\, t_0+T],\ \forall t_0,\ \forall x_0$$

then the origin is a globally uniformly exponentially —with respect to the r-homogeneous norm ρ— stable equilibrium point of $\dot{x} = f(t,x)$. If

$$\rho(\phi(t, t_0, x_0) - \psi(t, t_0, x_0)) \le d\rho(x_0) \qquad \forall t \in [t_0,\, t_0+T],\ \forall t_0,\ \forall x_0 \text{ with } \rho(x_0) \le \sigma$$

for some $\sigma \in (0, \infty)$, then the origin is a locally uniformly exponentially —with respect to the r-homogeneous norm ρ— stable equilibrium point of $\dot{x} = f(t,x)$ with region of attraction $\{x_0 \in \mathbb{R}^n : \rho(x_0) \le \sigma\}$.

4 Fast time-varying systems

Here we give a first application of Theorem 1. For reasons of simplicity, we restrict attention to exponential stability with respect to the Euclidean norm $\|\cdot\|$. That is, we work with the standard dilation $r = (1, \ldots, 1)$ and take $\rho(x) = \|x\|$.

Consider a fast time-varying system; i.e., a system that depends on a real parameter $\varepsilon > 0$ as follows:

$$\dot{x} = f(\frac{t}{\varepsilon}, x) \tag{16}$$

where the map $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies three conditions: (i) f is continuous, (ii) f is locally Lipschitz in x uniformly with respect to its first argument, and (iii) f is periodic in its first argument with period $\tau > 0$ independent of x.

Associated to this system is the averaged system

$$\dot{x} = f_{av}(x) \tag{17}$$

where the map $f_{av} : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$f_{av}(x) = \frac{1}{\tau}\int_0^\tau f(t,x)\, dt. \tag{18}$$

The hypotheses on f imply that f_{av} is locally Lipschitz.

Assume that that the origin is an equilibrium point of system (16) and hence also of the associated averaged system (17). We are interested in the stability properties of the fast time-varying system (16). It is well known that, if the origin is a locally exponentially stable equilibrium point of the averaged system, it is a locally exponentially stable equilibrium point of the fast time-varying system (16) provided $\varepsilon > 0$ is sufficiently small; see, for example, [3, 1]. Here we present a (semi-) global version of this result.

Theorem 2. *Given a fast time-varying system* (16) *that satisfies the three conditions introduced above and whose origin is an equilibrium point. Assume that the origin is a globally exponentially stable equilibrium point of the associated averaged system* (17)*. Then, (i) for every $\sigma \in (0, \infty)$, there exists $\varepsilon^* \in (0, \infty)$ such that for all $\varepsilon \in (0, \varepsilon^*]$ the origin is a locally exponentially stable equilibrium point of* (16) *with region of attraction $\{x \in \mathbb{R}^n : \|x\| \leq \sigma\}$; and (ii) if in addition f is globally Lipschitz in x uniformly with respect to its first argument, then there exists $\varepsilon^* \in (0, \infty)$ such that for all $\varepsilon \in (0, \varepsilon^*]$ the origin is a globally exponentially stable equilibrium point of* (16)*.*

Proof. First of all, notice that we consider exponential stability with respect to the Euclidean norm, which corresponds to taking $\rho(x) = \|x\|$ in the previous section.

Since the origin is assumed to be a globally exponentially stable equilibrium point of the averaged system (17), there exist $\mu \geq 1$, $\nu > 0$ such that

$$\|\psi(t, t_0, x_0)\| \leq \mu \mathrm{e}^{-\nu(t-t_0)} \|x_0\| \qquad \forall t \geq t_0,\ \forall t_0,\ \forall x_0 \quad (19)$$

where ψ denotes the flow of the averaged system. Let $T > 0$ be large enough such that (7) is satisfied for some $\beta \in (0, 1)$, and let $d > 0$ be small enough such that (9) holds for some $\alpha \in (0, 1)$.

In order to prove part (i), take an arbitrary $\sigma > 0$. By Theorem 1 it suffices to show the existence of $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$

$$\|\phi^\varepsilon(t, t_0, x_0) - \psi(t, t_0, x_0)\| \leq d\|x_0\| \qquad \forall t \in [t_0,\, t_0 + T],\ \forall t_0,\ \forall x_0 \text{ with } \|x_0\| \leq \sigma \quad (20)$$

where ϕ^ε denotes the flow of (16).

We prove this based on the fact that trajectories of (16) converge uniformly on compact time intervals to trajectories of the averaged system (17) as $\varepsilon \downarrow 0$, and that the Gronwall Lemma yields explicit bounds on this convergence; cf. [6].

Consider the compact set $K_\sigma = \{x \in \mathbb{R}^n : \|x\| \leq (\mu + 1)\sigma\}$ and let k be a Lipschitz constant for f on K_σ; i.e.

$$\|f(t, x_1) - f(t, x_2)\| \leq k\|x_1 - x_2\| \qquad \forall t \in \mathbb{R},\ \forall x_1, x_2 \in K_\sigma. \quad (21)$$

Consider and initial state x_0 with $\|x_0\| \leq \sigma$ and let $K_{\|x_0\|} = \{x \in \mathbb{R}^n : \|x\| \leq (\mu + 1)\|x_0\|\}$. Of course $K_{\|x_0\|} \subset K_\sigma$. Since $f(t, 0) = 0$ for all t, $k(\mu + 1)\|x_0\|$ is an upperbound for f on $K_{\|x_0\|}$; i.e.

$$\|f(t, x)\| \leq k(\mu + 1)\|x_0\| \quad \forall t,\ \forall x \in K_{\|x_0\|}. \quad (22)$$

Based on the Gronwall Lemma —cf. [6]— the following may be proven. Let ε^* be the solution of

$$2k(\mu + 1)\tau\varepsilon^*(1 + kT)\mathrm{e}^{kT} = \min\{1, d\}. \quad (23)$$

Then

$$\|\phi^\varepsilon(t, t_0, x_0) - \psi(t, t_0, x_0)\| \leq d\|x_0\| \qquad \forall t \in [t_0,\, t_0 + T],\ \forall t_0, \qquad \forall x_0 \text{ with } \|x_0\| \leq \sigma,\ \forall \varepsilon \in (0,\, \varepsilon^*]. \quad (24)$$

Applying Theorem 1 completes the proof of part (i).

The proof of part (ii) is along the lines of the proof of part (i), except that now we consider all initial states x_0 and we let k be a global Lipschitz constant for f; i.e.

$$\|f(t, x_1) - f(t, x_2)\| \leq k\|x_1 - x_2\| \qquad \forall t \in \mathbb{R},\ \forall x_1, x_2 \in \mathbb{R}^n. \quad (25)$$

□

5 Highly oscillatory systems

As a second application of Theorem 1, we consider highly oscillatory systems of the form

$$\dot{x} = X_0(x) + \frac{1}{\sqrt{\varepsilon}} \cos\frac{t}{\varepsilon} X_1(x) + \frac{1}{\sqrt{\varepsilon}} \sin\frac{t}{\varepsilon} X_2(x) \quad (26)$$

where ε is a strictly positive real parameter and where X_0 (resp. X_1 and X_2) is a vectorfield on $\mathbb{R}^n$ of class C^1 (resp. of class C^2). Associated to system (26) is the extended system

$$\dot{x} = X_0(x) + \frac{1}{2}[X_1, X_2](x). \tag{27}$$

Assume that the origin is an equilibrium point of (26). We are interested in the stability properties of this equilibrium point. For reasons of simplicity, we restict attention to exponential stability with respect to the Euclidean norm, as in the previous section. Based on the linearization principle and Theorem 1 from [5], it may be proven that, if the origin is a locally exponentially stable equilibrium point of the extended system (27), it is a locally exponentially stable equilibrium point of the highly oscillatory system (26) provided $\varepsilon > 0$ is sufficiently small. Since the proof of this result is based on the linearization principle, it is intrinsically local. Based on the techniques introduced here in Section 3, we are able to present a (semi-) global version of this result; see the following theorem and Remark 1.

Theorem 3. *Given a highly oscillatory system* (26) *where X_0 (resp. X_1, X_2) is a vectorfield on $\mathbb{R}^n$ of class C^1 (resp. of class C^2), and whose origin is an equilibrium point. Assume that the origin is a globally exponentially stable equilibrium point of the associated extended system* (27)*. Then, for every $\sigma \in (0, \infty)$, there exists $\varepsilon^* \in (0, \infty)$ such that for all $\varepsilon \in (0, \varepsilon^*]$ the origin is a locally exponentially stable equilibrium point of* (26) *with region of attraction $\{x \in \mathbb{R}^n : \|x\| \leq \sigma\}$.*

Proof. First of all, notice that we consider exponential stability with respect to the Euclidean norm, which corresponds to taking $\rho(x) = \|x\|$ in Section 3.

Since the origin is assumed to be a globally exponentially stable equilibrium point of the extended system (27), there exist $\mu \geq 1$, $\nu > 0$ such that

$$\|\psi(t, t_0, x_0)\| \leq \mu e^{-\nu(t-t_0)}\|x_0\| \qquad \forall t \geq t_0,\ \forall t_0,\ \forall x_0 \tag{28}$$

where ψ denotes the flow of the extended system. Let $T > 0$ be large enough such that (7) is satisfied for some $\beta \in (0, 1)$, and let $d > 0$ be small enough such that (9) holds for some $\alpha \in (0, 1)$.

In order to prove the theorem, take an arbitrary $\sigma > 0$. By Theorem 1 it suffices to show the existence of $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$

$$\|\phi^\varepsilon(t, t_0, x_0) - \psi(t, t_0, x_0)\| \leq d\|x_0\| \qquad \forall t \in [t_0,\ t_0 + T],\ \forall t_0,\ \forall x_0 \text{ with } \|x_0\| \leq \sigma \tag{29}$$

where ϕ^ε denotes the flow of (26).

We prove this based on the fact that trajectories of (26) converge uniformly on compact time intervals to trajectories of the extended system (27) as $\varepsilon \downarrow 0$, and that the Gronwall Lemma yields explicit bounds on this convergence; cf. [6].

Consider the compact set $K_\sigma = \{x \in \mathbb{R}^n : \|x\| \leq (\mu + 1)\sigma\}$ and let

- k be a common Lipschitz constant for X_0, X_1, X_2 and $[X_1, X_2]$ on K_σ.
- M be a common upper bound for DX_1 and DX_2 on K_σ; i.e.[3]

$$\|DX_i(x)\|_u \leq M \quad \forall x \in K_\sigma,\ \forall i \in \{1, 2\}. \tag{30}$$

- M' be a common upper bound for $D(DX_i \cdot X_j)$ $(i, j \in \{1, 2\})$ on K_σ; i.e.[4]

$$\|D(DX_i \cdot X_j)(x)\|_u \leq M' \quad \forall x \in K_\sigma,\ \forall i, j \in \{1, 2\}. \tag{31}$$

The existence of these constants is garanteed by the assumed smoothness properties of X_0, X_1 and X_2.

Consider an initial state x_0 with $\|x_0\| \leq \sigma$ and let $K_{\|x_0\|} = \{x \in \mathbb{R}^n : \|x\| \leq (\mu + 1)\|x_0\|\}$. Of course $K_{\|x_0\|} \subset K_\sigma$. Since $X_0(0) = X_1(0) =$

[3] $DX_i(x)$ denotes the Jacobian of X_i at x. $\|\cdot\|_u$ denotes the uniform norm of an $n \times n$-matrix corresponding to the Euclidean norm on $\mathbb{R}^n$; see, for example, [2].

[4] "$\cdot$" indicates the matrix product.

$X_2(0) = 0$, $k(\mu+1)\|x_0\|$ is an upperbound for X_0, X_1 and X_2 on $K_{\|x_0\|}$; i.e.

$$\|X_i(x)\| \le k(\mu+1)\|x_0\| \quad \forall x \in K_{\|x_0\|}, \qquad \forall i \in \{0,1,2\}. \quad (32)$$

Based on integration by parts and the Gronwall Lemma —cf. [6]— the following may be proven. Let ε^* be the solution of

$$k(\mu+1)\Big(\sqrt{\varepsilon}(4+2(M+M')T) + \varepsilon(2M+M'T)\Big)\mathrm{e}^{\frac{3}{2}kT} = \min\{1,d\}. \quad (33)$$

Then

$$\|\phi^\varepsilon(t,t_0,x_0) - \psi(t,t_0,x_0)\| \le d\|x_0\| \quad \forall t \in [t_0,\, t_0+T],\ \forall t_0, \quad \forall x_0 \text{ with } \|x_0\| \le \sigma,\ \forall \varepsilon \in (0,\, \varepsilon^*]. \quad (34)$$

Applying Theorem 1 completes the proof. □

Remark 1. As in the case of fast time-varying systems it is also possible to obtain global instead of semi-global results. Assume that there exist a Lipschitz constant k and upperbounds M and M' as introduced in the proof above but that are globally valid. Then, under the assumptions of Theorem 3, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0,\, \varepsilon^*]$ the origin is a globally exponentially stable equilibrium point of the highly oscillatory system (26).

Remark 2. In [7] an explicit design tool is presented for local exponential stabilization of nonholonomic control systems (global if the control system has some homogeneity property). The stabilizing effect of the resulting feedback laws may be proven based on the stability results from [5]. Similarly, the stability results from [4] give rise to an explicit design tool for practical stabilization of nonholonomic control systems. The present work extends the stability results from [5, 4] to a context of (semi-)global exponential stability. Accordingly we expect that this may lead to an explicit design tool for (semi-)global exponential stabilization of nonholonomic control systems.

6 Conclusion

We have analysed stability properties of a dynamical system by comparing its trajectories with those of a globally exponentially stable system. This gives rise to (semi-)global exponential stability results for fast time-varying systems and highly oscillatory systems.

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