Observer-Based Adaptive Fuzzy H^∞ Control Using A Fuzzy Algorithm To Increase Robustness

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Abstract : An adaptive fuzzy H^∞ controller is designed for the case where the plant's state vector is not fully measurable and has to be partially reconstructed via a state observer. An adaptive fuzzy system compensates the unknown part of the plant's model and at the same time forms the base of the state observer. Furthermore a fuzzy inference-based algorithm is developed in order to select the optimal H^∞ controller and assure maximum robustness. The stability of the closed-loop system is established for a category of plants with slow dynamics. Finally, the efficiency of the method is tested through simulations of the control of a dc-motor .

Keywords : Adaptive fuzzy H^∞ control, state observer, Lyapuvov stability, dc-motor

1 Introduction

In this paper a hybrid control architecture that combines the merits of adaptive fuzzy systems and H^∞ techniques is proposed and applied to the control of a dc-motor. The efficiency of the adaptive fuzzy controller is enhanced with the use of an H^∞ control element. Previous approaches to the design of neural and fuzzy adaptive H^∞ controllers can be found in [1] and [2] .

The adaptive fuzzy H^∞ controller proposed in this paper uses the observed state vector which is constructed with the aid of the fuzzy-based observer. The stability of the closed-loop system is proved via Lyapunov stability theory. Furthermore, to select an optimal H^∞ controller with maximum robustness, the fuzzy version of an existing algorithm is developed. Both the observer-based adaptive fuzzy H^∞ controller and the methodology for the selection of the optimal H^∞ controller were tested in the control problem of a dc-motor. Computer simulations verify the efficiency of the above techniques.

The paper is organized as follows. In Section 2 the principles of conventional H^∞ control are reviewed and the fuzzy logic algorithm for the computation of the optimal H^∞ controller is analyzed. In Section 3 the H^∞ control architecture for a class of nonlinear systems is presented and the stability of the closed-loop system is proved for the case where the adaptive fuzzy H^∞ controller uses the state vector reconstructed by a fuzzy estimator. In Section 4 the control problem of a dc-motor is investigated and simulation results are given to illustrate the satisfactory performance of the system under the proposed hybrid control scheme. Finally, some concluding remarks are provided in Section 5.

2 The conventional H^∞ control

2.1. Design of Mini-Max controllers

For the linear system described by

$$\dot{x}(t) = Ax(t) + Bu(t) + Ld(t) \qquad (1)$$
$$y(t) = Cx(t)$$

where $x(t) \in R^n$, $u(t) \in R^m$,
$d(t) \in R^q$ (disturbance vector),
and $y(t) \in R^p$,
the objective function is defined as

$$J(t) = \frac{1}{2}\int_0^T [y^T(t)y(t) + r\,u^T(t)u(t) - r^2 d^T(t)d(t)]dt$$

with $r, r > 0$,
where the weight r determines how much the control signal should be penalized while the weight r determines how much the disturbance influence should be rewarded. It is also assumed that :

i) the energy transfered by the disturbance signal $d(t)$ is bounded, i.e.

$$\int_0^\infty d^T(t)d(t)\,\mathrm{dt} < \infty$$

ii) $[A\ \ B]$ and $[A\ L]$ are stabilizable
iii) $[A\ C]$ is detectable .
The meaning of the negative signum of the disturbance term $d(t)$ in the above cost function is that the disturbance tries to maximize the value of the cost function $J(t)$, while the control signal $u(t)$ tries to minimize it. *The optimization goal is to find a control signal $u(t)$ which is able to compensate the worst possible disturbance imposed to the plant by its external environment.*

The optimal mini-max control law is [4] :

$$u(t) = -Kx(t) \quad \text{with} \quad K = \frac{1}{r} B^T P \tag{2}$$

where P is a positive definite symmetric matrix derived from the algebraic Riccati equation

$$A^T P + PA + Q - P(\frac{1}{r} BB^T - \frac{1}{r^2} LL^T) P = 0 \tag{3}$$

and Q is a positive definite symmetric matrix.

The worst-case disturbance is given by

$$d(t) = \frac{1}{r^2} L^T Px(t) \tag{4}$$

A crucial point in the design of mini-max controllers is the choice of the weighting parameter r in the cost function $J(u,d)$. The parameter r is an indication of the closed-loop system robustness and has to be selected such that the closed-loop system :
i) can reject the maximum-possible disturbance, and
ii) remains asymptotically stable.

If the values of $r > 0$ are excessively decreased with respect to the weighting parameter r, then the solution of the Riccati equation is no longer a positive definite matrix. Consequently there is a lower bound $r_{\min}$ of r for which the mini-max optimization problem has a solution. The acceptable values of r lie in the interval $[r_{\min}, \infty)$. If the value $r_{\min}$ is identified and used in the design of the H^∞ controller, then the closed-loop system will be supplied with increased robustness. Unlike this, if a value $r > r_{\min}$ is used in the design of the H^∞ controller, then an admissible stabilizing mini-max controller will be derived but it will be a suboptimal one.

The Riccati equation (3) is solved through spectral factorization of the associate Hamiltonian matrix

$$H = \begin{bmatrix} A & -(\frac{1}{r} BB^T - \frac{1}{r^2} LL^T) \\ Q & -A^T \end{bmatrix} \tag{5}$$

If $[A, B]$ is stabilizable and $[A, C]$ detectable, then the solution of the Riccati equation is a unique positive semi-definite symmetric matrix P, $P = P^T \geq 0$. *A necessary condition for the solution of the algebraic Ricatti equation to be a positive semi-definite symmetric matrix is that H has no imaginary eigenvalues* [3,4,5]. Therefore to calculate a value of the parameter r suitable for the design of an H^∞ controller one has to verify that :
i) the matrix H has no eigenvalues on the jw axis
ii) the corresponding matrix P is positive semi-definite.

In this paper an algorithm based on fuzzy inference is applied to determine the minimum value $r_{\min}$ and the associated matrix P that satisfies the above two conditions, thus resulting in an optimallly designed H^∞ controller.

2.2. Fuzzy computation of the parameter r of an H^∞ controller

The algorithm is based on the one proposed by Doyle et al. [3,4] . The novelty given in this paper is the substantiation with the use of fuzzy inference.

Step 1 : Select a random initial value for r .
Step 2: Calculate the eigenvalues of the matrix H given by (5) and define the magnitude of the permitted changes in the value of the parameter r .
Step 3: If the matrix H has imaginary eigenvalues then r should be increased and the algorithm should be repeated starting from Step 2 . If H does not have any imaginary eigenvalue then the solution P of the Algebraic Ricatti Equation (ARE) (3) must be calculated and the algorithm proceeds to Step 4.
Step 4 : The solution P of the ARE is tested to find out if it is positive semidefinite $(P \geq 0)$. If it is not, the parameter r should be further increased and the algorithm must return to Step 2. On the

other hand if $P \geq 0$ then the algorithm proceeds to Step 5 .

Step 5 : If the absolute difference between the current and the previous value of the parameter r is greater than a small positive constant $e > 0$, then the magnitude of the permitted changes of the values of r is reduced , the parameter r is decreased and the algorithm returns to Step 2. Otherwise the algorithm continues to Step 6 .

Step 6 : End .

The methodology proposed in this paper can be considered as a fuzzy-based modification of the bisection technique.

The fuzzy search over r :

To ensure that the parameter r *is increased* with the aid of the fuzzy inference , the following rules are employed [6]:

IF r_k is R_1 THEN r_{k+1} is R_2, . . . ,
IF r_k is R_{n-1} THEN r_{k+1} is R_n (6)

where the index k indicates the k-th iteration of the algorithm.

Similarly, to ensure that the parameter r *is decreased* via the fuzzy inference the rules that must be used are :

IF r_k is R_2 THEN r_{k+1} is R_1,. . . ,
IF r_k is R_n THEN r_{k+1} is R_{n-1} (7)

where $R_1, R_2, ..., R_{n-1}, R_n$ are the fuzzy subsets in which the fuzzy phase plane of the parameter r is divided.

In order to achieve convergence to the optimal value $r_{\min}$ *,the smaller the distance from* $r_{\min}$ becomes, the smaller *the size of the incremental or decremental changes of* r *should be*.

To satisfy the above requirement , the width of the membership functions should be modified at every crossing over the optimal value $r_{\min}$. The last two values of r , r_{k-1} and r_k, are taken into account, where : r_{k-1} is the last value that produces a matrix H with imaginary eigenvalues and r_k is the last value that results in a matrix H without imaginary eigenvalues and for which the solution of the algebraic Ricatti equation is a positive semi-definite matrix P .

Recalling the conventional bisection method , the optimal value $r_{\min}$ should be searched in the range [r_{k-1}, r_k]. The new fuzzy subsets $R_1, R_2, ..., R_{n-1}, R_n$ correspond to the division of the interval between these two values [r_{k-1}, r_k] in n equal segments.

The shrinkage of the fuzzy search interval [r_{k-1}, r_k] continues until the bounds r_{k-1} and r_k practically coincide . Then the optimal value $g_{\min}$ will have been found.

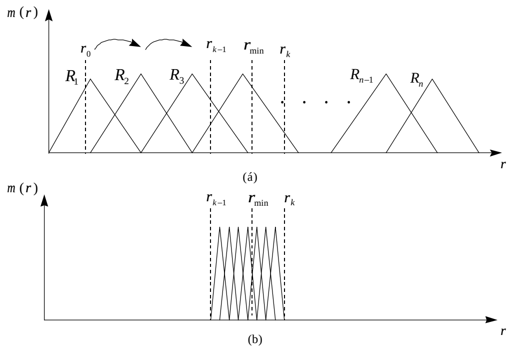

Fig. 1 (a) Fuzzy search over the universe of discourse of the parameter r , (b) reduction of the fuzzy sets' width around the optimal value $r_{\min}$ and new search

3 The fuzzy H^∞ control law for a class of non-linear systems

The H^∞ control methodology can be extended to nonlinear systems with non-fully measurable state vector .The system is first subject to feedback linearization with the use of an adaptive fuzzy function approximator. The following n-th order SISO dynamic system is assumed :

$$\frac{d}{dt}\hat{x}^{(n-1)} = \hat{f}(x, \dot{x}, ..., \hat{x}^{(n-1)}) + g \bullet u + d \qquad (8)$$
$$y = x$$

where the scalar u is the control input and $\hat{\mathbf{x}} = [x, \dot{x}, ..., \hat{x}^{(n-1)}]^T$ is the non-fully measurable state vector. The goal is to find the appropriate control signal u that will get the state $\mathbf{x}$ to track track a time-varying setpoint $x_d = [x_d, \dot{x}_d, ..., x_d^{(n-1)}]^T$.

The output tracking error is denoted by $e = y_d - y$. The observation error is $\mathbf{e_o} = \hat{\mathbf{x}} - \mathbf{x}$. Suppose that the system is free of external disturbances and that a fuzzy system could approximate, with infinite accuracy, the functions $\hat{f}(\hat{\mathbf{x}})$, i.e. $\hat{f}(\hat{\mathbf{x}}) = \hat{f}(\mathbf{x})$, and estimate correctly the missing state variable, i.e. $\hat{\mathbf{x}} = \mathrm{x}$ and $\hat{e}^{(n-1)} = e^{(n-1)}$. Then, the control law

$$u = \frac{1}{g}[-\hat{f}(\hat{\mathbf{x}}) + \mathbf{k}\,\hat{\mathbf{e}} + x_d^{(n)}] \qquad (9)$$

where $\hat{\mathbf{e}} = [\hat{e}^{(n-1)},\ldots..,\dot{e},e]$ and $k = [k_1,\ldots.,k_{n-1},k_n]^T$ results in the n-th order homogenous differential equation

$$e^{(n)} + k_1 e^{(n-1)} + \ldots.. + k_n e = 0\,.$$

If the elements of the vector $\mathbf{k}$ are selected such that the roots of the polynomial $s^n + k_1 s^{n-1} + \ldots.. + k_n s = 0$ are in the open left-half plane, then it is guaranteed that $\mathbf{e} \underset{t\to\infty}{\to} 0$.

In the application of this nonlinear inverse model control law three sources of error should be taken into account. The first is the noise caused by the external disturbances. The second is the approximate inaccuracies of the function f by the fuzzy system. The third is due to the observation error. The aggregate error signal is denoted by w and is assumed to be non-gaussian. To eliminate the impact of w a robust H^∞ control law will be applied in addition to the above described inverse model architecture. To assure the stability of the closed-loop system the following H^∞ performance index inequality has to be satisfied :

$$\int_0^T \hat{\mathbf{e}}^T Q\,\hat{\mathbf{e}}\,dt \le \hat{\mathbf{e}}^T(0) P\,\hat{\mathbf{e}}(0) + r\,\bar{q}^T(0)\bar{q}(0) + r^2\int_0^T w^T w\,dt \qquad (10)$$

and additionally to show that $\hat{\mathbf{x}} \underset{t\to\infty}{\to} \mathbf{x}$.

***Theorem* :**

Consider the nonlinear system

$$x^{(n)} = f(x,\dot{x},\ldots..,x^{(n-1)}) + g \bullet u + d \qquad (8)$$

where $f(x,\dot{x},\ldots..,x^{(n-1)})$ is an unknown function , g is a known constant and d represents the external disturbances. The state vector of the system is $\mathbf{x} = [x,\dot{x},\ldots.,x^{(n-1)}]$ and the state variable $x^{(n-1)}$ is not measurable .Using :

a) a fuzzy-based state observer for the estimation of the state variable $x^{(n-1)}$, i.e.

$$\hat{x}^{(n-1)}(t) = \int_0^t [\hat{f}(x,\dot{x},\ldots..,\hat{x}^{(n-1)}) + g \bullet u]\,dt \qquad (11)$$

where $\hat{\mathbf{x}} = [x,\dot{x},\ldots..,\hat{x}^{(n-1)}]$ is the estimated state vector and $\hat{f}(x,\dot{x},\ldots..,\hat{x}^{(n-1)})$ is the estimate of $f(x,\dot{x},\ldots.,\hat{x}^{(n-1)})$ found by an adaptive fuzzy system described by the parameters' update following relations :

$$\hat{f}(\hat{\mathbf{x}}) = x^T(\hat{\mathbf{x}})q_f \qquad (12)$$

$$\dot{q}_f = -g_1 \frac{x(\hat{\mathbf{x}})B^T P\hat{e}}{||x(\hat{\mathbf{x}})||^2} \qquad (13)$$

b) the adaptive fuzzy H^∞ control law

$$u_c = \frac{1}{g(\mathbf{x}')}[-x^T(\hat{\mathbf{x}})q_f + y_m^{(n)} + \mathbf{k^T}\hat{\mathbf{e}} - u_a] \qquad (14)$$

where $\mathbf{k} = [k_1,k_2,\ldots...,k_n]^T$

$$u_a = -\frac{1}{r}B^T P\hat{\mathbf{e}}\;,\; r > 0 \qquad (15)$$

$$\dot{q}_f = -g_1 \frac{x(\hat{\mathbf{x}})B^T P\hat{\mathbf{e}}}{||x(\hat{\mathbf{x}})||^2} \qquad (16)$$

and $P = P^T > 0$ is the solution of the Ricatti-like equation

$$PA + A^T P + Q - \frac{1}{r}PBB^T P + \frac{1}{r^2}PBB^T P = 0$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots. & \ldots. & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -k_1 & -k_2 & -k_3 & \ldots & -k_n \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \ldots \\ 1 \end{bmatrix}, \qquad (17)$$

then H^∞ tracking performance is achieved for the closed-loop system .

Proof :

Introducing the control law u_c in (8) the tracking error dynamic equation of the closed-loop system is derived

$$e^{(n)} = -\mathbf{k}^T\hat{\mathbf{e}} + [\hat{f}(\hat{\mathbf{x}}|q_f) - f(x)] + u_a - d \quad \text{, i.e.}$$

$$\hat{e}^{(n)} = -\mathbf{k}^T\hat{\mathbf{e}} + [\hat{f}(\hat{\mathbf{x}}|q_f) - f(x)] + u_a - d + e_{of} \qquad (18)$$

where $\mathbf{e}=\mathbf{x}-\mathbf{x_m}$, $\hat{\mathbf{e}}=\hat{\mathbf{x}}-\mathbf{x_m}$ and $\mathbf{e_{of}}$ is the first derivative of the observation error i.e.

$$\mathbf{e_{of}} = \hat{\mathbf{e}}^{(n)} - \mathbf{e}^{(n)} = \hat{f}(\hat{\mathbf{x}}|q_f) - f(\mathbf{x}) \\ = x^T(\hat{\mathbf{x}})q_f - f(\mathbf{x}) \quad (19)$$

Equation (18) can be written in the form

$$\hat{e}^{(n)} = -\mathbf{k}^T\hat{\mathbf{e}} + [\hat{f}(\hat{\mathbf{x}}|q_f) - \hat{f}(\hat{\mathbf{x}}|q_f^*)] \\ + [\hat{f}(\hat{\mathbf{x}}|q_f^*) - f(x)] + u_a - d + e_{of} \quad (20)$$

where :

$$q_f^* = \arg\min_{q_f \in \Omega_f}[\sup_{\hat{x} \in \Omega_x} \|\hat{f}(\hat{\mathbf{x}}|q_f) - f(\mathbf{x})\|]$$

Ω_f is the set of suitable bounds for q_f, $\Omega_{\hat{x}}$ is the set of suitable bounds for $\hat{\mathbf{x}}$ and it is assumed that q_f and $\hat{x}$ never reach the boundaries of Ω_f and $\Omega_{\hat{x}}$

The minimum approximation error is defined as

$$w_e = \hat{f}(\hat{\mathbf{x}}|q_f^*) - f(\mathbf{x}) \quad (21)$$

consequently (20) becomes

$$\hat{e}^{(n)} = -\mathbf{k}^T\hat{\mathbf{e}} + [\hat{f}(\hat{\mathbf{x}}|q_f) - \hat{f}(\hat{\mathbf{x}}|q_f^*)] \\ + w_e + u_a - d + e_{of} \quad (22)$$

or equivalently in the form of state-space equations

$$\dot{\hat{\mathbf{e}}} = A\hat{\mathbf{e}} + B[\hat{f}(\hat{\mathbf{x}},q_f) - \hat{f}(\hat{\mathbf{x}},q_f^*)] \\ + B[w_e - d] + Be_{of} + Bu_a(\hat{\mathbf{e}}) \quad (23)$$

where

$$A=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots. & \dots. & \dots \\ \dots & \dots & \dots & \dots & \dots \\ -k_1 & -k_2 & -k_3 & \dots & -k_n \end{bmatrix}, B=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 1 \end{bmatrix} \quad (17)$$

The following Lyapunov function is introduced

$$V = \frac{1}{2}\hat{\mathbf{e}}^T P\hat{\mathbf{e}} + \frac{\|x(\hat{\mathbf{x}})\|^2}{2g_1}\bar{q}_f^T\bar{q}_f + \frac{1}{2g_1}e_{of}^2 \quad (24)$$

where $\bar{q}_f = q_f - q_f^*$

$$e_{of} = \dot{e}_o = \hat{f}(\hat{\mathbf{x}}|q_f) - f(\mathbf{x}) = x^T(\hat{\mathbf{x}})q_f - f(\mathbf{x})$$

Assumption 1 : The following assumptions are made

$$\frac{d}{dt}x(\hat{\mathbf{x}}) = 0 \ , \ \frac{d}{dt}x(\mathbf{x}) = 0 \ \text{and} \ \frac{d}{dt}f(\mathbf{x}) = 0$$

which are reasonable if the rate of change of $\hat{\mathbf{x}}$ and $\mathbf{x}$ is small (i.e. the system under control has slow dynamics).

Using Assumption 1 one gets

$$\dot{e}_{of} \approx x^T(\hat{\mathbf{x}})\dot{q}_f = x^T(\hat{\mathbf{x}})(\frac{-g_1 x(\hat{\mathbf{x}})B^T P\hat{\mathbf{e}}}{\|x(\hat{\mathbf{x}})\|^2}) \\ = \frac{-g_1\|x(\hat{\mathbf{x}})\|^2 B^T P\hat{\mathbf{e}}}{\|x(\hat{\mathbf{x}})\|^2}$$

i.e.

$$\dot{e}_{of} \approx -g_1 B^T P\hat{\mathbf{e}} \quad (25)$$

Differentiating (21) yields

$$\dot{V} = \frac{1}{2}\dot{\hat{e}}^T P\hat{e} + \frac{1}{2}\hat{e}^T P\dot{\hat{e}} + \frac{\|x(\hat{x})\|^2}{g_1}\dot{\bar{q}}_f^T\bar{q}_f + \frac{1}{g_1}e_{of}\dot{e}_{of} \quad (26)$$

The following equalities hold

$$\dot{\bar{q}}_f = \dot{q}_f \quad (27)$$

$$\dot{\hat{\mathbf{e}}} = A\hat{\mathbf{e}} + B[x^T(\hat{x})(q_f - q_f^*)] + Bw + Be_{of} + Bu_a(\hat{\mathbf{e}}) \quad (28)$$

Introducing (27) and (28) into (26) and after some operations one finally gets

$$\dot{V} = \frac{1}{2}\hat{\mathbf{e}}^T(-Q - \frac{1}{r^2}PBB^T P)\hat{\mathbf{e}} + B^T P\hat{\mathbf{e}}e_{of} \\ + \hat{\mathbf{e}}^T PBx^T(\hat{\mathbf{x}})\bar{q}_f + \hat{\mathbf{e}}^T PBw - \hat{\mathbf{e}}^T PBx^T(\hat{\mathbf{x}})\bar{q}_f$$

i.e.

$$\dot{V} = -\frac{1}{2}\hat{\mathbf{e}}^T Q\hat{\mathbf{e}} - \frac{1}{2r^2}\hat{\mathbf{e}}^T PBB^T P\hat{\mathbf{e}} + \hat{\mathbf{e}}^T PBw, \ \text{i.e.}$$

$$\dot{V} = -\frac{1}{2}\hat{\mathbf{e}}^T Q\hat{\mathbf{e}} - \frac{1}{2}(\frac{1}{r}B^T P\hat{\mathbf{e}} - r\,w)^T(\frac{1}{r}B^T P\hat{\mathbf{e}} - r\,w) \\ + \frac{1}{2}r^2 w^T w$$

where $\frac{1}{2}(\frac{1}{r}B^T P\hat{\mathbf{e}} - rw)^T(\frac{1}{r}B^T P\hat{\mathbf{e}} - rw) \geq 0$

This implies that

$$\dot{V} \leq -\frac{1}{2}\hat{\mathbf{e}}^T Q\hat{\mathbf{e}} + \frac{1}{2}r^2 w^T w \quad (29)$$

which means that if $\frac{1}{2}\hat{\mathbf{e}}^T Q\hat{\mathbf{e}} \geq \frac{1}{2} r^2 w^T w$ then $\dot{V} < 0$. Thus, the following conditions will hold

$$V(t) \geq 0 \quad \forall t > 0, \quad V(0)=0$$

$$\dot{V}(t) \leq 0 \quad \forall t > 0$$

and therefore $\hat{e} \underset{t\to\infty}{\rightarrow} 0$.

Having proved that $\hat{\mathbf{e}} \underset{t\to\infty}{\rightarrow} 0$ one gets

$$\hat{e} = e \underset{t\to\infty}{\rightarrow} 0, \; \dot{\hat{e}} = \dot{e} \underset{t\to\infty}{\rightarrow} 0$$

$$\ldots\ldots$$

$$\hat{e}^{(n-2)} = e^{(n-2)} \underset{t\to\infty}{\rightarrow} 0, \; \hat{e}^{(n-1)} \underset{t\to\infty}{\rightarrow} 0$$

Since $e^{(n-1)} = \frac{d}{dt} e^{(n-2)} \Rightarrow e^{(n-1)} \underset{t\to\infty}{\rightarrow} 0$

The inequality (29) can be written in a form that satisfies the H^∞ tracking performance criterion . Integrating in the interval $[0,T]$ yields

$$V(T) - V(0) \leq -\frac{1}{2}\int_0^T \hat{e} Q \hat{e}\, dt + \frac{1}{2} r^2 \int_0^T w^T w dt$$

Since $V(T) \geq 0$, one gets

$$\frac{1}{2}\int_0^T \hat{e}^T Q \hat{e}\, dt \leq V(0) + r^2 \int_0^T w^T w dt$$

where

$$V(0) = \frac{1}{2}\hat{\mathbf{e}}^T(0) P \hat{\mathbf{e}}(0)$$

$$+ \frac{||x(\hat{\mathbf{x}}(0))||^2}{2g_1} \bar{q}_f^T(0) \bar{q}_f(0) + \frac{1}{2g_1} e_{of}^2(0)$$

Thus

$$\frac{1}{2}\int_0^T \hat{\mathbf{e}}^T Q \hat{\mathbf{e}}\, dt \leq \frac{1}{2}\hat{\mathbf{e}}^T(0) P \hat{\mathbf{e}}(0)$$

$$+ \frac{||x(\hat{\mathbf{x}}(0))||^2}{2g_1} \bar{q}_f^T(0) \bar{q}_f(0) + \frac{1}{2g_1} e_{of}^2(0)$$

$$+ \frac{1}{2} r^2 \int_0^T w^T w dt$$

This is the H^∞ tracking performance condition sought.

Remark : The critical point in the proof of the H^∞ tracking behavior is Assumption 1 which implies that the system's dynamics are slow enough (all the components in the system dynamics show slower variations with respect to the loop sampling speed) and thus the randomly initialized fuzzy system can follow the changes of f .

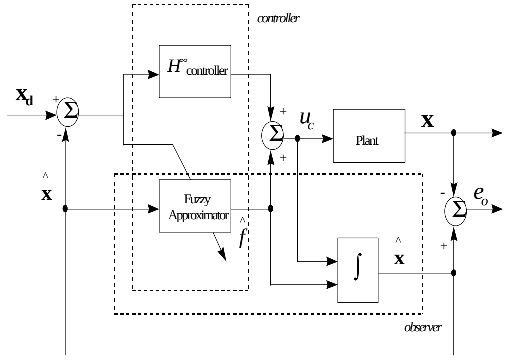

Fig. 2 The observer-based adaptive fuzzy H^∞ control architecture

4 Simulation results

4.1. The control problem of a dc-motor

The transfer-function of the dc-motor is given by

$$\frac{q_m(s)}{v_f(s)} = \frac{K}{s(1+st_f)(1+st_m)} \tag{30}$$

where $K = K_f \,/\, b R_f$, and $t_f = L_f \,/\, R_f$, $t_m = J \,/\, b$ are time constants. The corresponding state-space equations are

$$\begin{bmatrix} \dot{q}_m \\ \dot{w}_m \\ \dot{g}_m \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{t_f t_m} & -\frac{(t_f + t_m)}{u_f tm} \end{bmatrix} \begin{bmatrix} q_m \\ w_m \\ g_m \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{K}{t_f t_m} \end{bmatrix} v_f(s) \tag{31}$$

where q_m is the angular position , w_m is the angular speed and g_m is the angular acceleration. However under normal operation conditions some of the parameters of the above motor model , specifically the moment of inertia J or the friction coefficient b are not known or can be time varying. Thus a robust control scheme is required in order to compensate this parametric uncertainty.

The above model can be viewed as a sub-case of the general nonlinear model :

$$x^{(n)} = f(x, \dot{x}, \ldots.., x^{(n-1)}) + g(x, \dot{x}, \ldots.., x^{(n-1)}) \bullet u + d$$
$$y = x$$

where

i) f is an unknown but bounded function ,and

ii) g is the system's gain which is assumed to be a known constant (or , equivalently it is assumed that the uncertainty concerning g is included in disturbance term d) .

Furthermore it is assumed that the last state variable g_m of the motor model is not directly measurable, whereas the second state variable w_m can be calculated via the derivative $w_m \approx \frac{dq_m}{dt}$.

Therefore the problem reduces to that of controlling the system

$$\frac{d}{dt}\hat{x}^{(n-1)} = f(x, \dot{x}, \ldots\ldots, \hat{x}^{(n-1)}) + g \bullet u + d\,,$$

$$y = x \qquad (8)$$

Thus, the adaptive fuzzy H^{∞} control methodology developed for systems described by (6) can also be applied to control the dc-motor.

4.2. Simulation tests

The simulation code was written in C++. The fuzzy rules of the fuzzy approximator are of the form :

IF x is PS AND $\dot{x}$ is NS and $\ddot{x}$ is NS

THEN $\hat{f}$ is G^i

where G^i is one of the fuzzy subsets in which the output universe of discourse is divided, i.e. {NL,NM,NS,ZE,PS,PM,PL}* . All G^i s are initially chosen to be ZE (zero) and through the adaptation phase the fuzzy system finds the appropriate G^i s for each rule. Each variable in the antecedent part of the rule is analyzed in three fuzzy subsets. Taking all the possible combinations between the input fuzzy sets, 27 rules are derived.

The solution of the algebraic Ricatti equation for the calculation of the matrix P was done with the use of Matlab's *aresolv*( ) function. The matrix Q was selected to be the unity matrix I_3. The learning rate of the adaptive fuzzy system was set to $g_1 = 0.1$ and the parameter r in the Ricatti equation was set equal to 1. To increase the system's robustness the fuzzy inference-based algorithm described in Section 2.2 was used and the resulting r was found to be 1.1312 . The random choice of the parameters g_1 and r might cause intensive oscillations at the first stages of the training process. To minimize this effect, the selection of the above parameters by the application of genetic algorithms can be useful .

The first setpoint function used in the simulations was the sinusoidal signal

$$x_d(t) = 5 + 0.5\sin(2p\frac{k}{500}),\ \dot{x}_d(t) = p\frac{k}{500}\cos(2p\frac{k}{500})$$

and $$\ddot{x}_d(t) = -2(p\frac{k}{500})^2\sin(2p\frac{k}{500})$$

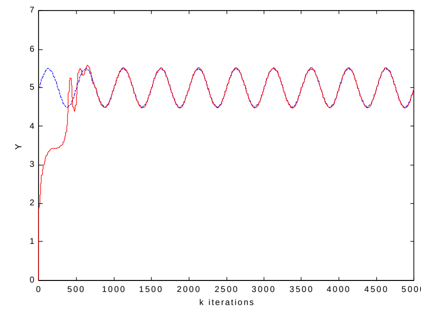

Fig. 3 Tracking of a sinusoidal setpoint by output y (solid line: actual value, dashed line: reference)

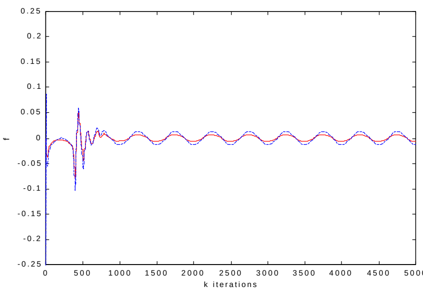

Fig 4. Estimation of parameter f by the adaptive fuzzy system (solid line : $\hat{f}$, dashed line f)

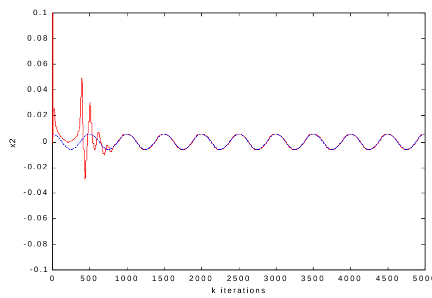

Fig. 5 State variable x_2 converges to the reference signal $x_{2\ \mathrm{des}}$ (solid line : actual value, dashed line : reference)

* NL = Negative Large, PS = Positive Small , etc.

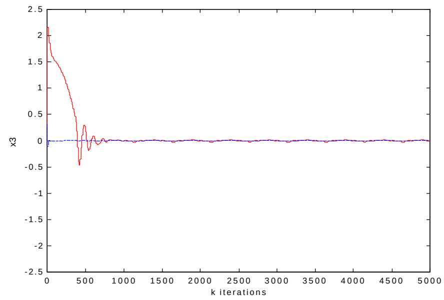

Fig. 6 Observed state variable $\hat{x}_3$ converges to the actual state variable x_3 (solid line : observed state , dashed line : actual state)

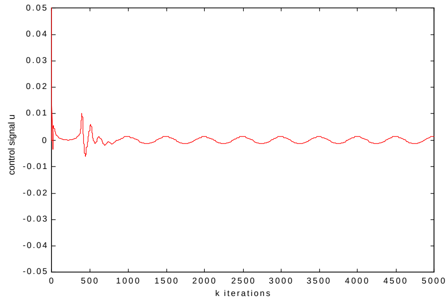

Fig. 7 The control signal u

The simulations tests showed a very satisfactory trajectory tracking (see Fig. 3) . The control signal and state variables fluctuation was quite smooth except from the initial stage of adaptation (see Fig. 5-7). However this is normal because no prior information about the plant's model was taken into account in the initialization of the fuzzy rule base.

5 Conclusions

In this study the control of a class of nonlinear systems was examined, namely.

$$x^{(n)} = f(x, \dot{x}, \dots, \hat{x}^{(n-1)}) + g \bullet u + d$$

Furthermore the last state of the state vector $(x, \dot{x}, \dots\dots, x^{(n-1)})$ was assumed not directly accessible and had to be reconstructed via a state observer. The observer used is of the integral type

$$\hat{x}^{(n-1)}(t) = \int_0^t [\hat{f}(x, \dot{x}, \dots, \hat{x}^{(n-1)}) + g \bullet u] dt$$

where the approximation $\hat{f}$ of the function f is produced by an adaptive fuzzy system. Using Lyapunov's stability theory, under the assumption that the plant has slow dynamics , it was shown that the closed-loop system (which consists of the plant, the adaptive fuzzy H^∞ controller and the state observer) satisfies the H^∞ tracking condition. Additionally, an algorithm based on fuzzy logic was developed for the selection of the optimal H^∞ controller . The algorithm performs a fuzzy search to find the value of the attenuation level r that provides the maximum robustness.

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