Stabilization of a Class of Discrete Time Feedforward Systems

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Abstract

In this paper we give a procedure to solve the stabilization problem of a class of discrete-time feedforward systems.

This result, based on Lyapunov techniques, can be viewed as a discrete version of the work of Jankovic, Sepulchre and Kokotovic.

Keywords: Nonlinear discrete-time, feedforward systems, stabilization.

1 Introduction

The stabilization problem of feedforward systems have been largely studied in the literature, see for instance [3] [19]. Recently, Sepulchre, Jankovic and Kokotovic have been developed an iterative procedure to stabilize the following class of nonlinear feedforward

$$\begin{aligned} \dot{x}_1 &= x_2 + f_1(x_2, \ldots, x_n) && (1)\\ \dot{x}_2 &= x_3 + f_2(x_3, \ldots, x_n) && (2)\\ \vdots &= \vdots && (3)\\ \dot{x}_n &= u. && (4) \end{aligned}$$

This algorithm is a *"bottom-up"* procedure. In order to stabilize the overall system we begin by stabilizing the sub-system $\dot{x}_n = u$. By considering this simple Lyapunov function $V_n = \frac{1}{2}x_n^2$, we derive the controller $u_n = -x_n$. After this, we consider the augmented the sub-system

$$\begin{aligned} \dot{x}_{n-1} &= x_{n-1} + f_{n-1}(x_{n-1}, x_n) && (5)\\ \dot{x}_n &= u_n + u && (6) \end{aligned}$$

and we construct a new Lyapunov function V_{n-1} by adding a new term in the above Lyapunov function. This term will ensure that $\dot{V}_{n-1}|_{u=0} \leq 0$, from this we deduce a controller $u = u_{n-1}$ which ensure the global asymptotic stability of the augmented sub-system. By repeating this procedure at each step, we deduce that the stabilizing controller have the folowing structure $u = \sum_{i=n}^{n} u_i$. The aim of this paper is to proof that by using the same idea in the discrete-time systems, we can derive a controller which ensure the global asymptotical stability of the systems

$$\begin{aligned} x_{k+1}^1 &= x_k^1 + f_1(x_k^2, \ldots, x_k^n)\\ x_{k+1}^2 &= x_k^2 + f_2(x_k^3, \ldots, x_k^n)\\ \vdots &= \vdots\\ x_{k+1}^n &= u_k, \end{aligned} \qquad (7)$$

where f_i are a smooth functions which satisfie the following growth condition

$$|f_i(x_i, \ldots, x_n)| \leq \alpha ||(x_i, \ldots, x_n|| \quad \alpha > 0. \qquad (8)$$

We also assume that the linear approximation is stabilizable.

1.1 Globally asymptotically stabilizing feedback

In this section, we recall some results on the stabilization problem. Let us consider the discrete-time nonlinear systems

$$x_{i+1} = f(x_i) + g(x_i, u_i)u_i \qquad (9)$$

with $f(0) = 0$ and $x_i \in \mathbf{R}^n$ and $u \in \mathbf{R}$ and introduce the following assumptions:

Assumption 1.1 *There exists a positive definite function V zero at the origin and such that $V(f(x)) - V(x) \leq 0$.*

Assumption 1.2 *The sets*

$$\begin{aligned}\Omega &= \{x \in \mathbf{R}^n : V(f^{i+1}(x)) = V(f^i(x)),\\ &\qquad i = 0, 1, 2, \ldots\}\end{aligned} \tag{10}$$

$$\begin{aligned}S &= \{x \in \mathbf{R}^n : \frac{\partial V}{\partial \alpha}|_{\alpha = f^{i+1}(x)} g(f^i(x), 0) = 0,\\ &\qquad i = 0, 1, 2, \ldots\}\end{aligned} \tag{11}$$

are such that

$$\Omega \bigcap S = \{0\}.$$

Remark. The systems (9) is a single-input system.

Let us recall a result which is an immediate consequence of [8, Lemma II.4] or of the feedback design of [20, Section 2].

Theorem 1.3 *Consider the discrete-time systems (9). Assume that Assumptions (1.1) and (1.2) are satisfied. Then for all function $\mu(x) > 0$, there exists a smooth function $\phi(x)$ such that the following feedback control*

$$\overline{u}(x) = -\phi(x) h(x, 0), \qquad 0 < \phi(x) \leq \mu(x) \tag{12}$$

$$h(x, u) = \int_0^1 \frac{\partial V}{\partial \alpha}|_{\alpha = f(x) + g(x,u)u\theta} g(x, u) d\theta \tag{13}$$

globally asymptotically stabilizes the system (9).

2 Design procedure

Step n

Firstly, let us consider the sub-system $x_{k+1}^n = u_k$. We can see that the stabilization is easily achieved by the controller $u_n(x_k^n) = a x_k^n$ avec $a < 1$ and the Lyapunov function $V_n(x_k^n) = {x_k^n}^2$.

Step $n-1$

Now, let us introduce the term v_{n-1} in the above control law

$$u_{n-1}(x_k^{n-1}, x_k^n) = u_n(x_k^n) + v_{n-1}.$$

Then, the augmented sub-system will be rewritten as follows:

$$\begin{aligned}x_{k+1}^{n-1} &= x_k^{n-1} + f_{n-1}(x_k^n)\\ x_{k+1}^n &= u_n(x_k^n) + v_{n-1}.\end{aligned} \tag{14}$$

In order to derive v_{n-1} which ensure the stabilization of the above sub-system, we firstly derive a Lyapunov function in the case where $v_{n-1} = 0$, and after this we derive a stabilizing control law. In order to do this, let us consider the following change of coordinates

$$z_k^{n-1} = x_k^{n-1} + \sum_{i=0}^{\infty} f_{n-1}(\tilde{x}_i^n) = x_k^{n-1} + h_{n-1}(x^n), \tag{15}$$

where $\tilde{x}_i^n$ is the solution of the sub-system $x_{k+1}^n = u_n(x_k^n)$ by considering x_k^n as initial conditions. The existence of this change of coordinates is linked to the convergence of the series

$$\sum_{i=0}^{\infty} f_{n-1}(\tilde{x}_i^n).$$

In order to guarantee this convergence, we assume that the function f_n satisfies a growth condition and from the fact that x_n converge exponentially, the existence of this series is guaranteed. In the other hand we can see that in the case where $v_{n-1} = 0$, we have $z_{k+1}^{n-1} = z_k^{n-1}$. This means that we can write the sub-system (14) in this case as follows:

$$\begin{aligned}z_{k+1}^{n-1} &= z_k^{n-1}\\ x_{k+1}^n &= a x_k^n.\end{aligned} \tag{16}$$

Now we can say that $V_{n-1} = V_n + (z^{n-1})^2$ is a Lyapunov function of the sub-system (16). In order to derive the stabilizing controller, let us substitute u_n by u_{n-1}. Then the sub-system (14) will be rewritten in the new coordinates as follows:

$$\begin{aligned}z_{k+1}^{n-1} &= z_k^{n-1} + k_1^{n-1}(x_k^n, v_{n-1}) v_{n-1}\\ x_{k+1}^n &= a x_k^n + v_{n-1}\end{aligned} \tag{17}$$

where

$$k_1^{n-1}(x_k^n, v_{n-1}) = \int_0^1 \frac{\partial h_{n-1}}{\partial \alpha}|_{\alpha = a x^n + \theta v_{n-1}} d\theta.$$

Let us compute $\Delta V_{n-1}(x^n, z^{n-1})$.

$$\begin{aligned}\Delta V_{n-1}(x^n, z^{n-1}) &= (a^2 - 1)(x^n)_k^2 + 2 a x_k^n v_{n-1}\\ &+ v_{n-1}^2 k_1^{n-1}(x_k^n, v_{n-1}) v_{n-1} z_k^{n-1}\\ &+ [k_1^{n-1}(x_k^n, v_{n-1})]^2 v_{n-1}^2.\end{aligned} \tag{18}$$

By choosing the following controller

$$v_{n-1} = -\phi_{n-1}(x_k^n)\big(2 a x_k^n + 2 k_1^{n-1}(x_k^n, 0) z_k^{n-1}\big) \tag{19}$$

where ϕ_{n-1} is defined as in [19] and if we assume that the sub-system satisfies (14) the Theorem (1.3) ,and from the fact that the linear approximation is stabilzable, we can say that the system (17) is globally asymptotically and locally exponentially stable in closed loop.

Step $i = n-2, \ldots, 1.$

Let us define

$$u_i(x^i, \ldots, x^n) = u_{i+1}(x^{i+1}, \ldots, x^n) + v_i. \quad (20)$$

As previously, we assume that the controller $u_{i+1}(x^{i+1}, \ldots, x^n)$ ensure the global asymptotic and local exponential stability of the sub-system

$$\begin{aligned} x^{i+1}_{k+1} &= x^{i+1}_k + f_{i+1}(x^{i+2}_k, \ldots, x^n_k) \\ \vdots &= \vdots \\ x^n_{k+1} &= u_{i+1}(x^{i+1}_k, \ldots, x^n_k) \end{aligned}$$

which is denoted by

$$X^{i+1}_{k+1} = F^{i+1}(X^{i+1}_k). \quad (21)$$

In order to derive v_i which ensure the global asymptotic and local exponential stability of the sub-system

$$\begin{aligned} x^i_{k+1} &= x^i_k + f_i(x^{i+1}_k, \ldots, x^n_k) \\ x^{i+1}_{k+1} &= x^{i+1}_k + f_{i+1}(x^{i+2}_k, \ldots, x^n_k) \\ \vdots &= \vdots \\ x^n_{k+1} &= u_{i+1}(x^{i+1}, \ldots, x^n) + v_i. \end{aligned} \quad (22)$$

at first we must find a lyapunov function for the sub-system (22) in the case $v_i = 0$. For that let us rewrite the sub-system (22) as follows:

$$\begin{aligned} x^i_{k+1} &= x^i_k + f_i(x^{i+1}_k, \ldots, x^n_k) \\ X^{i+1}_{k+1} &= F^{i+1}(X^{i+1}_k). \end{aligned} \quad (23)$$

Let us consider the following change of coordinate

$$z^i_k = x^i_k + \sum_{j=0}^{\infty} f_i(\tilde{x}^{i+1}_j, \ldots, \tilde{x}^n_j) = x^i_k + h_i(X^{i+1}_k), \quad (24)$$

where $\tilde{x}^{i+1}_j, \ldots, \tilde{x}^n_j$ is the solution of the sub-system $X^{i+1}_{k+1} = F^{i+1}(X^{i+1}_k)$ by considering that $(\tilde{x}^{i+1}_0, \ldots, \tilde{x}^n_0) = (x^{i+1}_k, \ldots, x^n_k)$.

As in the second step , the convergence of the series

$$\sum_{j=0}^{\infty} [f_i(\tilde{x}^{i+1}_j, \ldots, \tilde{x}^n_j)]$$

is ensured by the exponentially convergence of the sub-system

$$X^{i+1}_{k+1} = F^{i+1}(X^{i+1}_k)$$

and the fact that f_i satisfies a growth condition. Note that, like previously, $z^i_{k+1} = z^i_k$, this means that (21) will be

$$\begin{aligned} z^i_{k+1} &= z^i_k \\ X^{i+1}_{k+1} &= F^{i+1}(X^{i+1}_k). \end{aligned} \quad (25)$$

Since $\Delta V_i(X^{i+1}, z^i) = \Delta V_{i+1}(X^{i+1})$, we can easily see that $V_i = V_{i+1} + z_i^2$ is a lyapunov function of the sub-system (25). Now, by replacing u_{i+1} by u_i, we will have

$$\begin{aligned} z^i_{k+1} &= z^i_k + k^i_1(X^{i+1}_k, v_i)v_i \\ X^{i+1}_{k+1} &= F^{i+1}(X^{i+1}_k) + b_i v_i \end{aligned} \quad (26)$$

or

$$k^i_1(X^{i+1}_k, v_i) = \int_0^1 [\frac{\partial h_i}{\partial \alpha}|_{\alpha = F^{i+1}(X^{i+1}_k) + b_i v_i \theta}] b_i \quad d\theta$$

Remarks

i From the structure of the system (7) we can say that his linear approximation is controllable, this means that $k^i_1(0,0) \neq 0$.

ii The change of coordinate $z = T(x)$ is a global diffeomorphism because $\frac{\partial T}{\partial x}$ is an upper triangular matrix which all it's diagonal elements are equal to one.

Proposition I The Assumptioms (1.1) and (1.2)are satisfied by the sub-system (26).

Proof
Since the system $X^{i+1}_{k+1} = F^{i+1}(X^{i+1}_k)$ converge globally asymptotically and $z^i_{k+1} = z^i_k$ in the case where $v_i = 0$ then the set Ω is defined by

$$\Omega = \{(X^{i+1}, z^i) = (0, z^i) \qquad z^i \in \mathbf{R}\}.$$

In the same way, the sub-system is defined by

$$\begin{aligned} S &= \{(X^{i+1}, z^i)/ \quad 2z^i k^i_1([F^{i+1}]^j(X^{i+1}), 0) \\ &+ \frac{\partial V_{i+1}}{\partial \alpha}|_{\alpha = [F^{i+1}]^j(X^{i+1})} b_i \\ & \quad j = 0, 1, \ldots\} \end{aligned} \quad (27)$$

By using the fact $k^i_1(0,0) \neq 0$, we conclude that $\Omega \cap S = \{(X^{i+1}, z^i) = (0,0)\}$.

Furthermore,

$$\begin{aligned} \Delta V_i(X^{i+1}, z^i) &= V_{i+1}(F^{i+1}(X^{i+1}_k) + b_i v_i) \\ &- V_{i+1}(X^{i+1}_k) z_i^2(k+1) - z_i^2(k). \end{aligned}$$

Then,

$$\begin{aligned} \Delta V_i(X^{i+1}, z^i) &= V_{i+1}(F^{i+1}(X_k^{i+1})) - V_{i+1}(X_k^{i+1}) \\ &+ l(X_k^{i+1}, z_k^i, v_i) v_i \end{aligned} \tag{28}$$

with

$$\begin{aligned} l(X_k^{i+1}, z_k^i, v_i) &= \int_0^1 [\frac{\partial V_{i+1}}{\partial \alpha}|_{\alpha = F^{i+1}(X_k^{i+1}) + b_i v_i \theta}] b_i d\theta \\ &+ 2 z_k^i k_1^i(X_k^{i+1}, v_i) + [k^i{}_1(X_k^{i+1}, v_i)]^2 v_i. \end{aligned}$$

By using the Theorem (1.3), and from the fact that the linear approximation is stabilizable, we can see that the controller

$$v_i = -\phi_i(X_k^{i+1}, z_k^i) l(X_k^{i+1}, z_k^i, 0) \tag{29}$$

where ϕ_i is defined as in the work of Mazenc [19] ensure the local exponential and global asymptotic stability (26).

Summarizing, we can give the following theorem

Theorem 2.1 *The controller* $u = ax^n + \sum_{i=n-1}^{1} v_i$ *where* $a < 1$ *ensure the global asymptotic and local exponential stability of the class of system (7) in closed loop.*

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