A Fast Solution of the One-Variable and Two-Variable Lyapunov's Equation (By FFT)

NIKOS E. MASTORAKIS

Military Institutions of University Education (MIUE),
Hellenic Naval Academy, Chair of Computer Science,
Terma Hatzikyriakou, 18539, Piraeus, GREECE.

Tel-Fax: +301 777 5660

Abstract:- In this paper, a fast algorithm for the solution of one-variable and two-variable Lyapunov's Equation is presented. The algorithm is based on the use of the FFT (Fast Fourier Transform). The simplicity and efficiency of the method are illustrated by a numerical example.

Key-words: Fast Fourier Transform, Lyapunov's Equation CSCC'99 Proceedings, Pages:3991-3996

1. Introduction

Fast Fourier Transform (FFT), [1], has been applied in many problems in systems theory in order to facilitate or at least to speed up the relevant algebraic manipulation.

In [2], the FFT is used in order to determine the characteristic polynomial of a rectangular matrix and in [3] the same technique is used for the calculation of a Determinant Polynomial. The extension of this technique in 2-D systems is given in [4]. The use of FFT for arbitrary transformations of one-variable polynomials and rational functions is known [5].

In this paper, we consider the one-variable Lyapunov Equation: $A_1(s)XB_1(s)+\dots+A_n(s)XB_n(s)=C(s)$ as well as the two-variable Lyapunov Equation: $A_1(s_1,s_2)XB_1(s_1,s_2)+\dots+A_n(s_1,s_2)XB_n(s_1,s_2)=C(s_1,s_2)$. It is not difficult, based on the solution of the simple Lyapunov Equation: $A_1XB_1+\dots+A_nXB_n=C$, to formulate algorithms for the solution of one- and two-variable Lyapunov Equation via FFT.

In Section II, the algorithm is stated for the one-variable Lyapunov Equation whereas in Section III, the algorithm is described in the case of the two-variable Lyapunov Equation. Finally one can find some concluding remarks.

2. Solution of The One-Variable Lyapunov Equation

The Equation in question is:

$$A_1(s)XB_1(s)+\ldots+A_n(s)XB_n(s)=C(s) \qquad (1)$$

where $A_1,\ldots,A_n$ are $\mu\times\mu$ matrices, $B_1,\ldots,B_n$ are $\nu\times\nu$ matrices, X and $C(s)$ are $\mu\times\nu$ matrices. We also suppose that the various elements of A_i and B_i, $i=1,\ldots,n$ are polynomials of n_1 and n_2 degree correspondingly. The objective here is to determine the unknown matrix X. To this end, we transform (1) to the equivalent equation

$$G(s)x=c(s) \qquad (2)$$

where $G(s)=A_1(s)\otimes B_1^T(s)+\ldots+A_n(s)\otimes B_n^T(s)$, x is the associate vector to the matrix X (that is that $x=\left(x_{11}\ldots x_{1\nu}x_{21}\ldots x_{2\nu}\ldots x_{\mu 1}\ldots x_{\mu\nu}\right)^T$ where x_{ij} is the i,j element of the matrix X) as well as $c(s)$ is the associate vector to the matrix $C(s)$. Also, $\otimes$ denotes the Kronecker (or as also called direct) product of two matrices as well as T denotes the matrix transpose. Since $G(s)$ is a $(\mu\nu,\mu\nu)$ rectangular matrix, then x is given by

$$x=G^{-1}(s)c(s) \qquad (3)$$

or

$$x=\frac{adjG(s)\cdot c(s)}{det\,G(s)} \qquad (4)$$

The computation of x=$x(s)$ could be achieved via FFT as follows.

First one has to compute $det\,G(s)$. To this end, we compute it numerically at $r+1$ points where $r=(n_1+n_2)\nu\mu$ equally spaced on the unit disc. In particular, we define the points $z(l)=w^{-l}$ where $w=e^{j2\pi/(r+1)}$ and $l=0,\ldots,r$. So, we compute the $r+1$ determinants $det\,G(z(l))$ where

$$det\,G(z(l))=det\left(A_1(z(l))\otimes B_1^T(z(l))+\ldots+A_n(z(l))\otimes B_n^T(z(l))\right) \qquad (5)$$

or simply $det\,G(l)$

$$det\,G(l)=det\left(A_1\left(e^{-j2\pi l/(r+1)}\right)\otimes B_1^T\left(e^{-j2\pi l/(r+1)}\right)+\ldots+A_n\left(e^{-j2\pi l/(r+1)}\right)\otimes B_n^T\left(e^{-j2\pi l/(r+1)}\right)\right) \qquad (6)$$

If one takes into account that $det\,G(s)=\sum_{k=0}^{r} g_k\cdot s^k$ where g_k are real coefficients, then g_k can be evaluated using the inverse FFT as follows

$$g_k=\frac{1}{r+1}\sum_{l=0}^{r} det\,G(l)\cdot w^{kl} \quad , \qquad k=0,1,\ldots,r \qquad (7)$$

or equivalently

$$g_k=\frac{1}{r+1}\sum_{l=0}^{r} det\,G(l)\cdot e^{j2\pi kl/(r+1)} \quad , \qquad k=0,1,\ldots,r \qquad (8)$$

So in (4), the denominator polynomial has been completely determined.

In order to evaluate the numerator matrix polynomial one has to compute the numerical value of it at the $r'+1$ points, $r'=(n_1+n_2)(\nu\mu-1)$ also equally spaced on the unit disc. Similarly, we define the points $z(l)=w^{-l}$ where $w=e^{j2\pi/(r'+1)}$ and $l=0,\ldots,r'$ .So, we also compute the $r'+1$ adjoint matrices $adjG(z(l))$ where

$$adjG(z(l)) = adj\left(A_1(z(l)) \otimes B_1^T(z(l)) + \ldots + A_n(z(l)) \otimes B_n^T(z(l))\right) \quad (9)$$

or simply $adjG(l)$

$$adjG(l) = adj\left(A_1\left(e^{-j2\pi l/(r+1)}\right) \otimes B_1^T\left(e^{-j2\pi l/(r+1)}\right) + \ldots + A_n\left(e^{-j2\pi l/(r+1)}\right) \otimes B_n^T\left(e^{-j2\pi l/(r+1)}\right)\right) \quad (10)$$

Taking into account that $adjG(s) = \sum_{k=0}^{r'} G_k \cdot s^k$ where G_k are numerical (non polynomial) matrices, then G_k can be evaluated using the inverse FFT as follows

$$G_k = \frac{1}{r'+1} \sum_{l=0}^{r'} adjG(l) \cdot w^{kl} \quad , \quad k = 0,1,\ldots,r' \quad (11)$$

or equivalently

$$G_k = \frac{1}{r'+1} \sum_{l=0}^{r'} adjG(l) \cdot e^{j2\pi kl/(r+1)} \quad , \quad k = 0,1,\ldots,r' \quad (12)$$

3. Solution of the Two-Variable Lyapunov Equation

Consider now the equation

$$A_1(s_1,s_2)XB_1(s_1,s_2) + \ldots + A_n(s_1,s_2)XB_n(s_1,s_2) = C(s_1,s_2) \quad (13)$$

where $A_1,\ldots,A_n$ are $\mu \times \mu$ matrices, $B_1,\ldots,B_n$ are $\nu \times \nu$ matrices, X and $C(s)$ are $\mu \times \nu$ matrices. We also suppose that the various elements of A_i and B_i, $i = 1,\ldots,n$ are polynomials of n_1, m_1 and n_2, m_2 degree in s_1, s_2 correspondingly. In order to determine the unknown matrix X, we transform (13) to the equivalent equation

$$G(s_1,s_2)x = c(s_1,s_2) \quad (14)$$

where $G(s_1,s_2) = A_1(s_1,s_2) \otimes B_1^T(s_1,s_2) + \ldots + A_n(s_1,s_2) \otimes B_n^T(s_1,s_2)$, x is the associate vector to the matrix X and c is the associate vector to the matrix C. x is also given by

$$x = G^{-1}(s_1,s_2)c(s_1,s_2) \quad (15)$$

or

$$x = \frac{adjG(s_1,s_2) \cdot c(s_1,s_2)}{det\, G(s_1,s_2)} \quad (16)$$

The computation of $x = x(s_1,s_2)$ could be achieved via FFT as follows.

First one has to compute $det\, G(s_1,s_2)$. To this end, we compute it numerically at $(r_1+1)(r_2+1)$ points, where $r_1 = (n_1+n_2)\nu\mu$ and $r_2 = (m_1+m_2)\nu\mu$, equally spaced on the unit bidisc. In particular, we define the points $z_1(l_1) = w_1^{-l_1}$ where $w_1 = e^{j2\pi/(r_1+1)}$ and $l_1 = 0,\ldots,r_1$ and the points $z_2(l_2) = w_2^{-l_2}$ where $w_2 = e^{j2\pi/(r_2+1)}$ and $l_2 = 0,\ldots,r_2$. So, we compute the $(r_1+1)(r_2+1)$ determinants $det\, G(z_1(l_1), z_2(l_2))$ where

$$det\,G(z_1(l_1),z_2(l_2)) = det\Big(A_1(z_1(l_1),z_2(l_2))\otimes B_1^T(z_1(l_1),z_2(l_2))$$
$$+\ldots+A_n(z_1(l_1),z_2(l_2))\otimes B_n^T(z_1(l_1),z_2(l_2))\Big)$$
(17)

or simply $det\,G(l_1,l_2)$

$$det\,G(l_1,l_2) = det\Big(A_1\left(e^{-j2\pi l_1/(r_1+1)},e^{-j2\pi l_2/(r_2+1)}\right)\otimes$$
$$B_1^T\left(e^{-j2\pi l_1/(r_1+1)},e^{-j2\pi l_2/(r_2+1)}\right)+\ldots+$$
$$+A_n\left(e^{-j2\pi l_1/(r_1+1)},e^{-j2\pi l_2/(r_2+1)}\right)\otimes B_n^T\left(e^{-j2\pi l_1/(r_1+1)},e^{-j2\pi l_2/(r_2+1)}\right)\Big)$$
(18)

If one takes into account that $det\,G(s_1,s_2)=\sum_{k_1=0}^{r_1}\sum_{k_2=0}^{r_2} g_{k_1,k_2}\cdot s_1^{k_1}s_2^{k_2}$ where g_{k_1,k_2} are real coefficients, then g_{k_1,k_2} can be evaluated using the double inverse FFT as follows

$$g_{k_1,k_2}=\frac{1}{r_1+1}\cdot\frac{1}{r_2+1}\sum_{l_1=0}^{r_1}\sum_{l_2=0}^{r_2} det\,G(l_1,l_2)\cdot w^{k_1 l_1}w^{k_2 l_2}\quad ,$$
$$k_1=0,1,\ldots,r_1,\; k_2=0,1,\ldots,r_2 \quad (19)$$

or equivalently

$$g_{k_1,k_2}=\frac{1}{r_1+1}\cdot\frac{1}{r_2+1}\sum_{l_1=0}^{r_1}\sum_{l_2=0}^{r_2} det\,G(l_1,l_2)\cdot e^{j2\pi k_1 l_1/(r_1+1)}e^{j2\pi k_2 l_2/(r_2+1)}$$
$$,k_1=0,1,\ldots,r_1,\; k_2=0,1,\ldots,r_2$$
(20)

So in (16), the denominator polynomial has been completely determined.

In order to evaluate the numerator matrix polynomial one has to compute the numerical value of it at the $(r_1'+1)(r_2'+1)$ points, $r_1'=(n_1+n_2)(\nu\mu-1)$ and $r_1'=(m_1+m_2)(\nu\mu-1)$ also equispaced on the unit bidisc. Similarly, we define $z_1(l_1)=w_1^{-l_1}$ where $w_1=e^{j2\pi/(r_1'+1)}$ and $l_1=0,\ldots,r_1'$ and the points $z_2(l_2)=w_2^{-l_2}$ where $w_2=e^{j2\pi/(r_2'+1)}$ and $l_2=0,\ldots,r_2'$. So, we compute the $(r_1'+1)(r_2'+1)$ adjoint matrices $adjG(z_1(l_1),z_2(l_2))$ where

$$adjG(z_1(l_1),z_2(l_2)) = adj\Big(A_1(z_1(l_1),z_2(l_2))\otimes B_1^T(z_1(l_1),z_2(l_2))+$$
$$\ldots+A_n(z_1(l_1),z_2(l_2))\otimes B_n^T(z_1(l_1),z_2(l_2))\Big)$$
(21)

or simply $adjG(l_1,l_2)$

$$adjG(l_1,l_2) = adj\Big(A_1\left(e^{-j2\pi l_1/(r_1'+1)},e^{-j2\pi l_2/(r_2'+1)}\right)\otimes$$
$$B_1^T\left(e^{-j2\pi l_1/(r_1'+1)},e^{-j2\pi l_2/(r_2'+1)}\right)+\ldots+$$
$$+A_n\left(e^{-j2\pi l_1/(r_1'+1)},e^{-j2\pi l_2/(r_2'+1)}\right)\otimes B_n^T\left(e^{-j2\pi l_1/(r_1'+1)},e^{-j2\pi l_2/(r_2'+1)}\right)\Big)$$
(22)

If one takes into account that $adjG(s_1,s_2)=\sum_{k_1=0}^{r_1'}\sum_{k_2=0}^{r_2'} G_{k_1,k_2}\cdot s_1^{k_1}s_2^{k_2}$ where G_{k_1,k_2} are numerical (non polynomial) matrices, then G_{k_1,k_2}

can be evaluated using the double inverse FFT as follows

$$G_{k_1,k_2} = \frac{1}{r_1'+1}\cdot\frac{1}{r_2'+1}\sum_{l_1=0}^{r'_1}\sum_{l_2=0}^{r_2'} adjG(l_1,l_2)\cdot w^{k_1 l_1} w^{k_2 l_2},$$

$$k_1 = 0,1,\ldots,r_1',\ k_2 = 0,1,\ldots,r_2' \qquad (23)$$

or equivalently

$$G_{k_1,k_2} = \frac{1}{r_1'+1}\cdot\frac{1}{r_2'+1}\sum_{l_1=0}^{r_1'}\sum_{l_2=0}^{r_2'} adjG(l_1,l_2)\cdot e^{j2\pi k_1 l_1/(r_1+1)} e^{j2\pi k_2 l_2/(r_2+1)}$$

$$k_1 = 0,1,\ldots,r_1',\ k_2 = 0,1,\ldots,r_2' \qquad (24)$$

Thus, we compute $adjG(s_1,s_2)$ and $detG(s_1,s_2)$ from the relations

$$det\,G(s_1,s_2) = \sum_{k_1=0}^{r_1}\sum_{k_2=0}^{r_2} g_{k_1,k_2}\cdot s_1^{k_1} s_2^{k_2}$$ and

$$adjG(s_1,s_2) = \sum_{k_1=0}^{r_1}\sum_{k_2=0}^{r_2} G_{k_1,k_2}\cdot s_1^{k_1} s_2^{k_2}.$$ Furthermore, by using (16), x is evaluated.

3. Example

We consider Equation (1) with n=2 and

$\mu = n = 2$, $\quad A_1 = \begin{bmatrix} s+1 & s^2-1 \\ s & s^2+3+1 \end{bmatrix}$,

$B_1 = \begin{bmatrix} s-1 & s \\ 2 & 3s+2 \end{bmatrix}$, $\quad A_2 = \begin{bmatrix} s+1 & s \\ 2 & 0 \end{bmatrix}$,

$B_2 = \begin{bmatrix} s & -1 \\ s+1 & s+2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2s+5 \\ 1 & s \end{bmatrix}$,

Then, following the procedure described in Section II, we find the solution $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$

where

$x_1 = (-18 - 211 s - 739 s^2 - 1002 s^3 - 371 s^4 + 254 s^5 + 237 s^6 + 116 s^7 + 36 s^8 + 3 s^9) /$

$(14 + 97 s + 279 s^2 + 523 s^3 + 831 s^4 + 923 s^5 + 396 s^6 - 214 s^7 - 238 s^8 - 48 s^9 - 3 s^{10})$,

$x_2 = (-2 + s - 40 s^2 - 310 s^3 - 613 s^4 - 372 s^5 + 116 s^6 + 178 s^7 + 44 s^8 + 3 s^9) /$

$(-14 - 97 s - 279 s^2 - 523 s^3 - 831 s^4 - 923 s^5 -396 s^6 + 214 s^7 + 238 s^8 + 48 s^9 + 3 s^{10})$,

$$x_3 = (54 + 293 s + 586 s^2 + 605 s^3 + 331 s^4 + 76 s^5 + 30 s^6 + 28 s^7 + 3 s^8) /$$

$$(-14 - 97 s - 279 s^2 - 523 s^3 - 831 s^4 - 923 s^5 - 396 s^6 + 214 s^7 + 238 s^8 + 48 s^9 + 3 s^{10}),$$

and

$$x_4 = (-22 - 80 s - 54 s^2 + 20 s^3 + 9 s^4 + 32 s^5 + 66 s^6 + 25 s^7 + 2 s^8) /$$

$$(14 + 97 s + 279 s^2 + 523 s^3 + 831 s^4 + 923 s^5 + 396 s^6 - 214 s^7 - 238 s^8 - 48 s^9 - 3 s^{10}),$$

IV. CONCLUSION

A new fast algorithm for one-variable and two-variable Lyapunov's Equation is proposed. The algorithm is based on the DFFT (Discrete Fast Fourier Transform). The simplicity and efficiency of the method are illustrated by a numerical example

REFERENCES

[1] J.W. Cooley and J.W.Tukey, "An algorithm for machine calculation of complex Fourier series", Math. Comput., Vol.19, pp.297-301, April 1965.

[2] T.Lee. "A Simple Method to Determine the Characteristic Function $f(\lambda) = |\lambda I - A|$ by Discrete Fourier Series and Fast Fourier Transform", *IEEE Trans. on Circuits and Systems*, Vol.23, p.242, 1976.

[3] L.Paccagnella and G.Pierobon, "FFT Calculation of a Determinantal Polynomial", *IEEE Trans. on Circuits and Systems*, Vol.21, pp.401-402, June 1976.

[4] G.Antoniou, S.Mentzelopoulou and G.Glentis. "Determination of Transfer Function of Two-Dimensional Generalised Systems Using the Discrete Fourier Transform", *IEE Proceedings-D*, Vol.38, No.4, pp.327-330, July 1991.

[5] S.C.Chan and K.L.Ho. "A New Algorithm for Arbitrary Transformation of Polynomial and Rational Functions", *IEEE Trans. on Signal Processing*, Vol.40, No.2, pp.456-460, Febr.1992.

[6] N.Mastorakis. "Recursive Algorithms for Two-Dimensional Filters' Spectral Transformations", *IEEE Trans. on Signal Processing*, Vol.44, No.10, Octob. 1996, pp.2647-2651.