Backward Procedures in Linear Differential Games of Small Dimension

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Abstract: The paper shortly describes an algorithm of backward constructions for linear differential games with geometric constraints on controls, fixed terminal time and convex terminal payoff function that depends on two components of the phase vector. The algorithm is based on ideas of the dynamic programming. Numerical simulation results are illustrated on the Pontryagin's test example.

Key-Words: dynamic programming, differential games, backward constructions, value function, numerical methods

1 Introduction

Ideas of the backward constructions of the dynamic programming method are widely applied in the theory of differential games [1, 2, 3, 4, 5]. Contemporary computational capabilities permit to elaborate effective systems for solving some classes of differential games and for investigation of solution dependence on parameters of a game problem.

In the paper, results of constructing the level sets of the value function are represented. Those are built by means of the backward construction. For games with fixed terminal time and convex terminal payoff function, the backward constructions schematically look as follows. A level set M_c (a Lebesgue set) of the payoff function is taken. A time step Δ of is defined. Making the step Δ back of the terminal time T, we find the set $W_c(T-\Delta)$ of all such points of the phase space for each of those, under arbitrary admissible control of the second player, an admissible control of the first player can be found that carries the system into the set M_c. Such order of choosing the players' controls in accordance with the Fleming terminology [2] corresponds to the minorant game, and in accordance with the Krasovskii and Subbotin terminology [4] corresponds to the concept of the u-stability. Having constructed the set $W_c(T-\Delta)$, we pass to constructing the set $W_c(T-2\Delta)$, and further to $W_c(T-3\Delta)$ and so on. The rule ensuring the pass from the set $W_c(t_i)$ to the set $W_c(t_i-\Delta)$ reflects the essence of the backward construction. It can be described in terms of procedures for union and intersection of sets [3, 4, 6, 7].

In the Institute of Mathematics and Mechanics (Ekaterinburg, Russia), the algorithms and programs were elaborated [8, 9, 10, 11, 12, 13] that realize such rule for differential games of sufficiently general types. But it is necessary to note that for differential games with nonlinear dynamics mentioned algorithms are very complicated even in problems of low dimension.

In the papers [3, 7], it was noted that the rule of the back pass can be significantly simplified for games with linear dynamics and convex payoff function. Under such conditions, the sets $W_c(t_i)$ are convex, and the rule of the pass can be reduced to construction of the convex hull of some positively-homogeneous function. For linear differential games, algorithms based on this fact were elaborated in 80's [11, 14, 15]. It was remarked that for constructing the convex hull, one can use the information about places of possible violations of the local convexity of the positively-homogeneous function to be convexed. Most effectively it can be done when the phase space, where the sets $W_c(t_i)$ are built, has the dimension equal to 2. Elaborated earlier software package was recently upgraded and supplemented with new programs. Its good performance and specially elaborated tools of visualization permit to calculate and qualitatively analyze level sets of the value function of rather complicated structure.

We consider a linear antagonistic differential game

$$\begin{gathered}\dot{x} = A(t)x + B(t)u + C(t)v,\\ x \in R^n, \quad u \in P, \quad v \in Q, \quad \varphi(x_i(T), x_j(T))\end{gathered} \tag{1}$$

with fixed terminal time T and convex payoff function φ, which depends on two coordinates x_i, x_j of the phase vector. The first (second) player governs the control $u \quad (v)$ choosing it from the convex compact $P \quad (Q)$ and minimizes (maximizes) the value of the function φ at the instant T.

It is known that the substitution $y(t) = X_{i,j}(T,t)x(t)$ where $X_{i,j}(T,t)$ is a matrix combined of two rows of the fundamental Cauchy matrix, provides the transformation to the equivalent differential game of the second order.

So, the original dynamics (1) can have a high order, but the equivalent game has the order equal to 2. Such reduction to small dimension permits to calculate practical problems with mechanical meanings. The following scheme is available: linearization of the original nonlinear problem, fixation of the terminal time of the game, determination of the payoff function depending on two coordinates of the phase vector at the terminal instant, finding the level sets of the value function for the obtained equivalent differential game, constructing the optimal strategies on the base of the collection of the value function level sets, application of the optimal strategies in the original nonlinear problem. Such scheme was used in the works [16, 17].

2 Backward constructing level sets of the value function

Assume that the transfer from the game (1) with the payoff function φ depending on two coordinates of the phase vector to the equivalent game

$$\begin{gathered}\dot{y} = D(t)u + E(t)v,\\ y \in R^2, \quad \varphi(y(T)), \quad u \in P, \quad v \in Q,\\ D(t) = X_{i,j}(T,t)B(t), \quad E(t) = X_{i,j}(T,t)C(t)\end{gathered} \tag{2}$$

is already done. Let on the interval $[0,T]$ the sequence of instants $t_i : \ t_N = T, \ldots, t_i = t_{i+1} - \Delta, \ldots, t_0 = 0$ dividing the interval with the step Δ is given. The interest is to find the time sections $W_c(t_i) = \{y \in R^2 : V(t_i, y) \leqslant c\}$ of the level set $W_c = \{(t,y) \in [0,T] \times R^2 : V(t,y) \leqslant c\}$ of the value function V for the given value of the parameter c.

Replace the dynamics (2) by the piecewise-constant dynamics

$$\begin{gathered}\dot{y} = \mathbf{D}(t)u + \mathbf{E}(t)v,\\ \mathbf{D}(t) = D(t_i), \quad \mathbf{E}(t) = E(t_i), \quad t \in [t_i, t_{i+1}).\end{gathered} \tag{3}$$

Instead of the sets P and Q, let us consider their polyhedral approximations $\mathbf{P}$, $\mathbf{Q}$. Let $\hat{\varphi}$ be the approximating payoff function. For any c, its level set $\mathbf{M}_c = \{y : \hat{\varphi}(y) \leqslant c\}$ is a convex polygon close to $M_c = \{y : \varphi(y) \leqslant c\}$.

The approximating game (3) is taken so that, for each step $[t_i, t_{i+1})$ of the backward procedure, we deal with the game with simple motions [1], polyhedral convex control constraints and the convex polygonal target set. On the base of $\mathbf{W}_c(t_N) = \mathbf{M}_c$ the game solvability set $\mathbf{W}_c(t_{N-1})$ can be computed. Further, starting from $\mathbf{W}_c(t_{N-1})$, the set $\mathbf{W}_c(t_{N-2})$ can be built, and so on. As a result, the collection of convex polygons is obtained, which approximate sections $W_c(t_i)$ of the the level set W_c of the value function in the game (2) in the Hausdorf metric.

Let $\mathcal{P}(t_i) = D(t_i)\mathbf{P}$, $\mathcal{Q}(t_i) = E(t_i)\mathbf{Q}$. The support function $l \to \rho(l, \mathbf{W}_c(t_i))$ of the polygon $\mathbf{W}_c(t_i)$ is [7] the convex hull of the function

$$\begin{aligned}\gamma(l, t_i) = \ & \rho(l, \mathbf{W}_c(t_{i+1})) + \\ & \Delta\rho(l, -\mathcal{P}(t_i)) - \Delta\rho(l, \mathcal{Q}(t_i)).\end{aligned}$$

The function $\gamma(\cdot, t_i)$ is positively-homogeneous and piecewise-linear. The property of local convexity of the function γ can be violated only at the boundary of the linearity cones of the function $\rho(\cdot, \mathcal{Q}(t_i))$, i.e. at the boundary of the cones generated by the normals to the edges of the polygon $\mathcal{Q}(t_i)$, which have the common vertex.

Multi-step process of convexing begins with checking the local convexity of the function γ near the mentioned normals of the polygon $\mathcal{Q}(t_i)$. If the local convexity is violated, the function should be corrected. After finite number of corrections, we obtain the convex hull. In the algorithm, the ordering of the vectors, which determine the piecewise-linear structure of the function γ, is used essentially.

3 The Pontryagin's test example

Let the dynamics of the first and second players is described by the differential equations

$$\begin{aligned} &\ddot{x} + a\dot{x} + bx = u, \\ &\ddot{y} + c\dot{y} + dy = v, \\ &x, y \in R^2, \quad u \in P, \quad v \in Q, \\ &\varphi(x(T), y(T)) = ||x(T) - y(T)||. \end{aligned} \tag{4}$$

Here x is a geometric position of the first player, y is that of the second player, u and v are players' controls, P and Q are convex compacts in the plane. The payoff function φ is the Euclidean distance between the players at the terminal instant T. The game of the type (4) is known as "the Pontryagin's test example" [18]. If the sets P and Q are circles, then for some variants of the dynamic coefficients, the value function can be described in the explicit form [18, 19]. In general case, it is impossible.

Change of variables

$$\begin{cases} z_1 = x - y \\ z_2 = \dot{x} - \dot{y} \\ z_3 = y \\ z_4 = \dot{y} \qquad\qquad z_i \in R^2 \end{cases} \tag{5}$$

ensures the pass to the dynamics of the type (1) with the payoff $\varphi(z(T)) = ||z_1(T)|| = \sqrt{z_{11}^2(T) + z_{12}^2(T)}$.

Problem 1. Let $a = 2$, $b = 0$, $c = 0.2$, $d = 1$. In this case, the first player controls the motion of the material point in the plane, and the second player governs the two-dimensional mathematical pendulum. In both dynamics, a friction depending on the velocity is present. Suppose that the sets P and Q are ellipses, which are given by the following formulae:

$$P = \left\{ u \;\middle|\; u^T \begin{bmatrix} 0.8 & 0 \\ 0 & 0.4 \end{bmatrix} u \leqslant 1 \right\},$$

$$Q = \left\{ v \;\middle|\; v^T \begin{bmatrix} 1.5 & 0 \\ 0 & 1.05 \end{bmatrix} v \leqslant 1 \right\}.$$

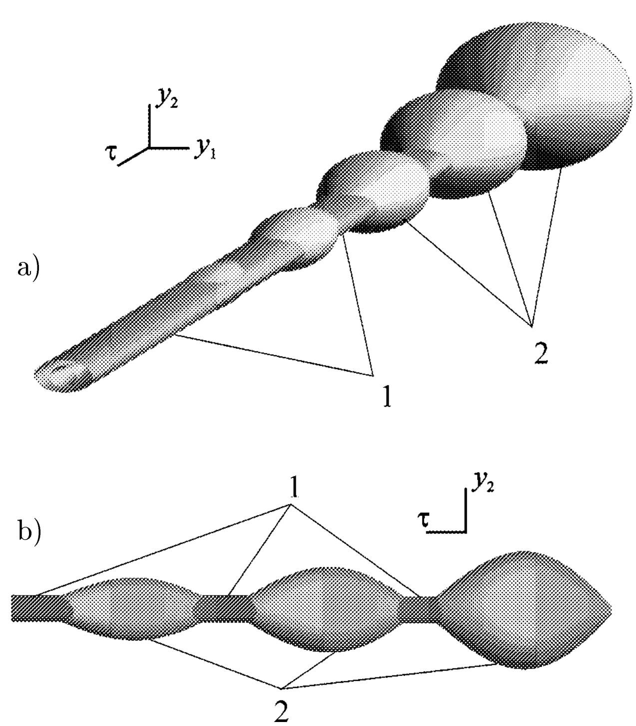

Fig. 1: Problem 1. The elliptical vectogram tubes for the first (1) and second (2) players: a) general view; b) zoomed fragment, side-view

In Figure 1a), the vectograms $\mathcal{P}(t) = D(t)\mathbf{P}$ and $\mathcal{Q}(t) = E(t)\mathbf{Q}$ of the first and second players are shown in the backward time in the coordinates of the equivalent game. The axis $\tau = T - t$ of the backward time goes from the right to the left. Because of the oscillatory dynamics of the second player, the character of the "advantage" of one player over the another changes in time. For the initial instants of the backward time, we have full advantage of the second player, but for sufficiently large τ, the first player obtains the complete advantage. The zoomed fragment of these sets (side-view sight) is given in Figure 1b).

In Figure 2, the level set W_c for $c = 2.45098$ is represented in coordinates of the equivalent game. This set has finite length in time. Before the instant of the break, orientation of the "stretching" is changed. Namely, till the last "throat", the sections are stretched vertically, but after it, their orientation becomes horizontal. It happens because of delicate interaction of the vectograms $\mathcal{P}$ and $\mathcal{Q}$. Detection of such phenomena has demanded some efforts. Existence of such effects in other problems

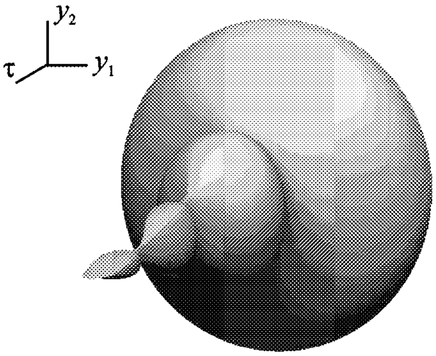

Fig. 2: Problem 1. A broken level set of the value function, $c = 2.45098$

had been detected by analytically in the paper [20]. For calculating such phenomena in numerical simulations, rather exact computation is necessary.

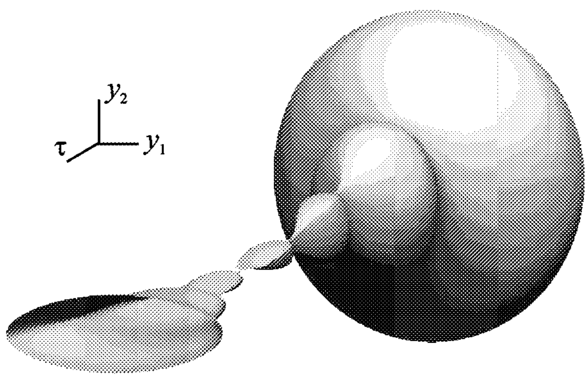

Fig. 3: Problem 1. The value function level set with a narrow "throat", $c = 2.45100$

When c is increased a little, the level set W_c increases by a burst. Such set for $c = 2.45100$ is shown in Figure 3. Figure 4 contains zoomed outlook of the set W_c near the thin "throat".

For sufficiently large c, the structure of the level sets is simplified. In Figure 5, the level set for $c = 3.87300$ is shown.

Problem 2. Let us show how the structure of the level sets changes, when the set Q changes from the ellipse to a rectangle. The coefficients of the system (4) are the same: $a = 2$, $b = 0$, $c = 0.2$, $d = 1$. The set P is retained as the ellipse

$$P = \left\{ u \;\middle|\; u^T \begin{bmatrix} 0.8 & 0 \\ 0 & 0.4 \end{bmatrix} u \leqslant 1 \right\}.$$

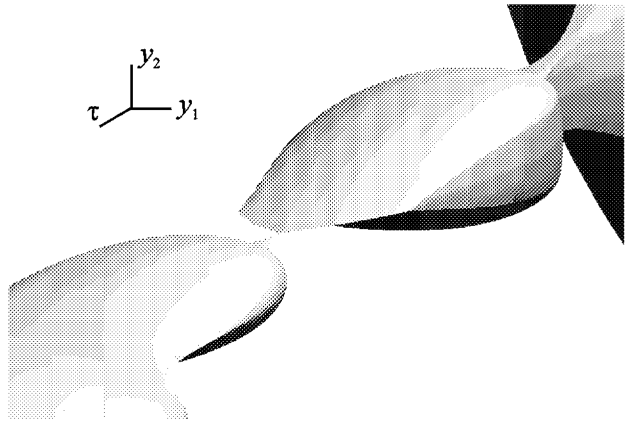

Fig. 4: Problem 1. Zoomed fragment with the "throat", $c = 2.45100$

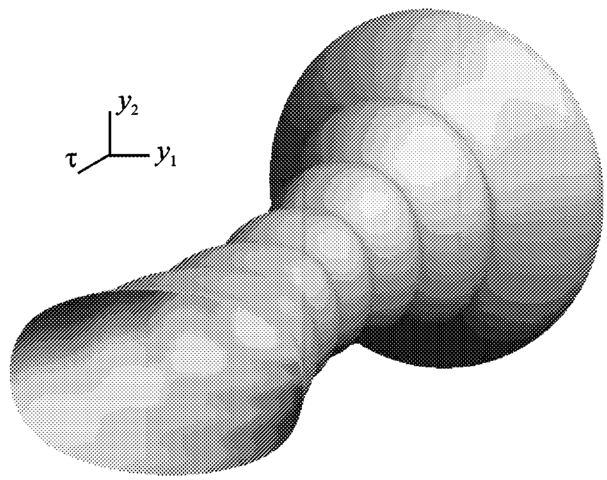

Fig. 5: Problem 1. The level set of the value function for $c = 3.87300$

The set $Q = [-1.5, 1.5] \times [-1.05, 1.05]$ is a rectangle circumscribed round the ellipse Q from the Problem 1.

In Figure 6, the vectograms $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ unrolled in time are represented. In Figures 7 and 8, the level sets W_c of the value function for $c = 4.58258$ and $c = 4.60435$ are shown in the situation similar to one in Figures 2 and 3.

Figure 9 clarifies the work of the procedure for construction the convex hull of the positively-homogeneous function $\gamma(\cdot, t_i)$ when the support function $\rho(\cdot, W_c(t_i))$ of the next section $W_c(t_i)$ is constructed. The level lines of the function $\gamma(\cdot, t_i)$ and the level lines of the resultant support function are shown. The function $\gamma(\cdot, t_i)$ has four directions

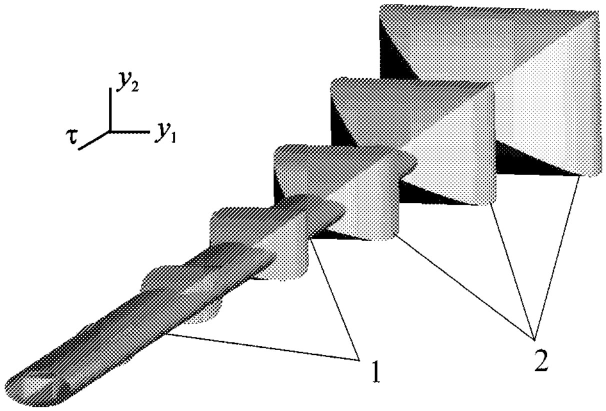

Fig. 6: Problem 2. Vectogram tubes for the first (1) and second (2) players

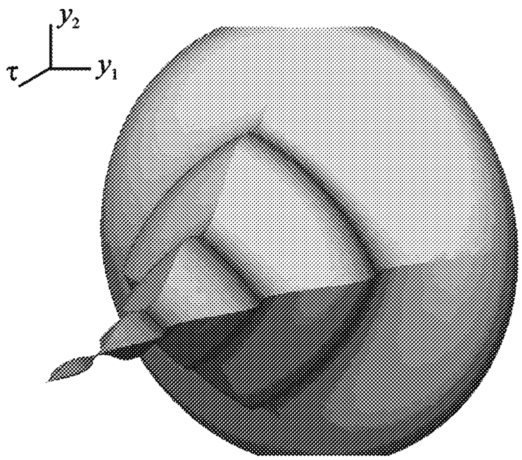

Fig. 7: Problem 2. The broken level set of the value function, $c = 4.58258$

of violation of the local convexity. For building the convex hull, the correction of the function $\gamma(\cdot, t_i)$ begins with namely these directions. The picture is made at the instant $t_i = 8$ $(\tau_i = T - t_i = 12)$.

The examples were calculated in the interval of the backward time $\tau = [0, 20]$ with the step $\Delta = 0.05$. The level set of the value function and the ellipses of constraints on the players controls were approximated by polygons with 100 vertices.

The pictures of the level sets were obtained by means of special visualization program elaborated by V.L.Averbukh and D.A.Yurtaev, scientific researchers of the Department of System Software of the Institute of Mathematics and Mechanics.

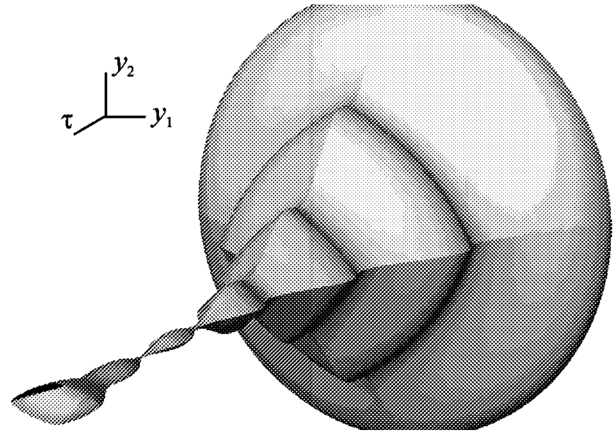

Fig. 8: Problem 2. The value function level set with a "throat", $c = 4.60435$

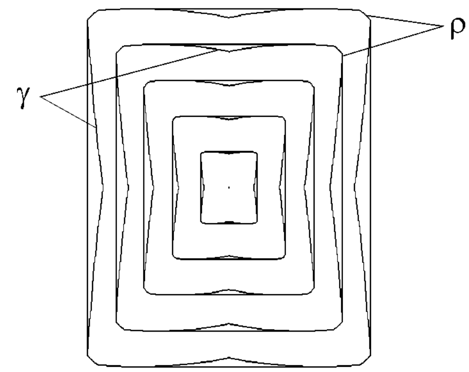

Fig. 9: Problem 2. Level contours of the function $\gamma(\cdot, t_i)$ and its convex hull $\rho(\cdot, W_c(t_i))$, $t_i = 8$

4 Conclusion

The level sets of the value function can have very complicated structure even in linear differential games with fixed terminal time. In the paper, the examples of such sets are represented for the well-known differential game "the Pontryagin's test example". The algorithm for constructing the level sets is shortly described. The algorithm can be regarded as a realization of the dynamic programming method in application to the concrete class of the differential games.

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