On Multiobservers for Nonlinear Systems *

ZBIGNIEW BARTOSIEWICZ
Technical University of Białystok
Wiejska 45, Białystok
POLAND
e-mail: bartos@cksr.ac.bialystok.pl

MAŁGORZATA WYRWAS
Technical University of Białystok
Wiejska 45, Białystok
POLAND
e-mail: kosk@cksr.ac.bialystok.pl

Abstract: A new concept of multiobserver for a nonlinear analytic system is introduced. The multiobserver's output is a multivalued function which estimates the whole class of states that are indistinguishable from the current state of the system. Such states do appear because the only assumption about the system is local observability. If the system is restricted to a compact subset of the state space, the values of the output of the multiobserver are finite sets. Assuming rather local observability than global observability is a more realistic approach to the observer problem for nonlinear systems. Local observability is also easier to check than its global counterpart. The weak side of the weaker assumptions is loss of regularity of the (multi)observer. We can only claim existence of continuous multiobserver, losing analyticity on a small subset of the state space.

Key-Words: observer, multiobserver, local observability, multivalued function, analytic system.

1 Introduction

In a recent paper Jouan and Gauthier [5] have proposed a construction of an observer for a nonlinear, smooth or analytic, control system. The main assumptions about the system were global observability and Ascending Chain Property (ACP). The latter assumption allowed for computing higher derivatives of the output as smooth functions of a finite number of lower derivatives. This was crutial for constructing a new system (a dynamics of the observer) whose state estimated first derivatives of the output of the original system on the basis of the output itself.

We follow here the main ideas of Jouan and Gauthier. Our goal is to weaken their assumptions as much as possible. First of all we give up global observability of the system as this property is hard to check and there are many examples of systems which do not have this property and are only locally observable. There are many tests of local observability, from simple well known sufficient conditions [4] to recent, more sophisticated, characterications [1, 2, 6]. Assuming only local observability we allow for existence of indistinguishable states, but they form a discrete set which becomes finite if we restrict the system to a compact subset of the state space. This means that a potential observer cannot follow the true state of the system. The most we can get is an approximation of one of the states that are indistinguishable from the true state, or all such states. The first possibility gave rise to the concept of quasiobserver introduced in [3]. The second leads to the idea of multiobserver which is developed in this paper.

The main advantage of multiobserver over quasiobserver lies in a bigger regularity of the former. Though in both cases we lose analyticity of the original system and have to live with only continuous observers, in some cases quasiobservers can fail to be even continuous, so some special assumptions are necessary to save this prop-

*Supported by KBN under the Technical University of Białystok grant No. W/IMF/99

erty. This is not needed for multiobservers, since working with the entire class of indistinguishable states makes the output map continuous.

We also show that local observability implies a weaker version of ACP. We can compute higher derivatives of the output as *continuous* functions of lower derivatives. As one of such functions appear in the dynamics of the (multi)observer, the dynamics is again only continuous.

We study here only the simplest case of analytic system with one output and without control. The main result says that a locally observable system admits a continuous multiobserver. The construction of the multiobserver is similar as the construction of the observer given by Jouan and Gauthier. We provide an example, giving the explicite description of the multiobserver.

2 Local observability

Let us consider a system Σ defined on $\Omega \subset \mathbb{R}^n$

$$\dot{x} = f(x) \tag{1}$$

$$y = h(x). \tag{2}$$

For simplicity we will assume that the output y is scalar. The vector field f and the function h are assumed to be real analytic.

Let us recall that two states x_1 and x_2 are *indistinguishable* if

$$h(x(t, x_1)) = h(x(t, x_2))$$

for every $t \geq 0$ for which both sides of the equation are defined. Here $x(t, x_0)$ denotes the trajectory of f starting at x_0 evaluated at time t. Otherwise x_1 and x_2 are called *distinguishable.* It is known that x_1 and x_2 are indistinguishable iff $\varphi(x_1) = \varphi(x_2)$ for every φ of the form $L_f^k h$, where L_f is the Lie derivative with respect to the vector field f. These functions generate the *observation algebra* $H(\Sigma)$ of the system Σ. Indistinguishability is an equivalence relation so the state space can be decomposed into disjoint indistinguishability classes.

System Σ is *locally observable* at x_0 if there is a neighborhood U of x_0 such that for every $x \in U$, x and x_0 are distinguishable. Σ is *locally observable* if it is locally observable at every point. Necessary and sufficient conditions for local observability were developed in [1, 2, 6]. A simple sufficient rank condition of Hermann and Krener [4] was the first important achievement in nonlinear observability theory.

Let us recall that local observability is a weaker property than *observability* which means that any two distinct points are distinguishable. Though, from a system-theoretic point of view, observability is a better property than local observability, it is much harder to check it. In fact no efficient algorithm for checking it is known (for $n > 1$).

A system that is locally observable may still have indistinguishable states, but they form a discrete set, i.e. each point is isolated from the others. If the set Ω is compact (closed and bounded) then any indistinguishability class in Ω is finite and there is a common bound on the number of elements in a class.

The criterion of local observability given in [2, 6] is based on the sequence $(L_f^i h)_{i \geq 0}$. It is enough to compute only finitely many of these derivatives to check local observability on a compact set. Also finitely many is needed to distinguish states that are distinguishable. So let us assume that the first N functions from the sequence, $h, L_f h, \ldots, L_f^{N-1} h$, determine the indistinguishability relation. It means that if $Y = (y_0, \ldots, y_{N-1}) = (h(x), \ldots, L_f^{N-1} h(x))$, then the indistinguishability class of x, denoted by $[x]$, consists exactly of those $\tilde{x}$ that $(h(\tilde{x}), \ldots, L_f^{N-1} h(\tilde{x})) = Y$.

From now on let us assume that Ω is compact and Σ is locally observable on Ω. Let $\Phi : \Omega \to \mathbb{R}^N$ be given by $\Phi(x) = (h(x), \ldots, L_f^{N-1} h(x))$. Denote by $\widetilde{\Omega}$ the quotient space of Ω with respect to the indistinguishability relation. Then Φ may be redefined on $\widetilde{\Omega}$ as $\widetilde{\Phi}([x]) = \Phi(x)$. We impose the quotient topology on $\widetilde{\Omega}$. Then $\widetilde{\Omega}$ is compact and the map $\widetilde{\Phi} : \widetilde{\Omega} \to \mathbb{R}^N$ is continuous and injective. Then $\Psi := \widetilde{\Phi}^{-1}$ is a continuous map (in fact homeomorphism) from $\widetilde{\Phi}(\tilde{\Omega})$ to $\widetilde{\Omega}$. The map Ψ assigns to the extended output Y the class of indistinguishable states that produced Y. From this we get:

Theorem 2.1. *For every $k \geq N$ there is a continuous function $\varphi_k : \mathbb{R}^N \to \mathbb{R}$ such that $L_f^k h = \varphi_k(h, \ldots, L_f^{N-1})$.*

To achieve exponential (multi)observers we would have to assume that the map φ_N is Lipschitz when restricted to a subset containing $\tilde{\Phi}(\tilde{\Omega})$.

3 Multiobservers

By a *multiobserver* S of the system Σ given by (1, 2) we shall mean a system

$$\dot{z} = F(z, y), \tag{3}$$
$$\hat{x} = g(z), \tag{4}$$

where the input y is the output of system Σ and g is a multivalued map with values in $\mathbb{R}^n$, such that the Hausdorff distance between $\hat{x}(t)$ and the indistinguishability class of $x(t)$ tends to 0 when t goes to $+\infty$. We assume that F and g are continuous in appropriate topologies and values of g are finite subsets of $\mathbb{R}^n$.

Thus multiobserver gives simultaneously finitely many estimations of the state of the original system. They estimate the state of Σ modulo the indistinguishability relation for system Σ. Though one could describe $g(z)$ as a finite sequence of elements from $\mathbb{R}^n$ rather then a subset, this would lead to complications. First of all, the number of elements in $g(z)$ may depend on z. Secondly, in general, g as a sequence may not be continuous because its components may lose continuity. On the other hand, in the subset we do not number the elements and this allows to preserve continuity of g as a multivalued map.

In fact, continuity is the most we can get. Analyticity is important in our considerations, but it is lost during the process of recovering the state (or, more precisely, the indistinguishability class of the state) from the output. One has to expect root-type functions which are not differentiable. However this lack of differentiability is restricted only to small subsets of measure zero. Besides such a subset all the data will be analytic.

The main result of the paper may be stated as follows:

Theorem 3.1. *If system Σ is locally observable on a compact set Ω then there exists a continuous multiobserver of Σ.*

The construction of a multiobserver is based on the idea of Jouan and Gauthier [5]. They first construct a system whose state approximates the time derivatives of the output y of Σ. To achieve this they assume the Ascending Chain Property (ACP) which says that higher order derivatives of the output may be expressed as smooth functions of a finite number N of first its derivatives. We do not assume ACP. We show that local observability gives something similar, but the functions are no longer smooth. This is however not a big loss, as the output function (or multifunction) of the observer is not differentiable in general either.

Example 3.2. Let Σ be the system:

$$\dot{x}_1 = x_1^2$$
$$\dot{x}_2 = x_1 \cdot x_2$$
$$y = x_1^2 + x_2^2.$$

The observation algebra $H(\Sigma)$ is generated by: $y_0 = x_1^2+x_2^2$, $y_1 = 2x_1(x_1^2+x_2^2)$, $y_2 = 6x_1^2(x_1^2+x_2^2)$, $\ldots, y_i = (i+1)!x_1^i(x_1^2+x_2^2)$, $\ldots$, where $y_i = L_f^i h(x)$, $i = 0, 1, 2, \ldots$ and $h(x) = x_1^2 + x_2^2$. It is not finitely generated, but Σ is locally observable.

The following relation between functions in $\mathcal{H}(\Sigma)$ holds:

$$y_2 = \frac{3y_1^2}{2y_0}. \tag{5}$$

The function $\varphi(y_0, y_1) = \frac{3y_1^2}{2y_0}$ is not smooth, so ACP does not hold for the system Σ. But it is continuous on the set $y_1^2 \leq 4y_0^3$ (the image of Φ_Σ^2 defined below), so for this system we can construct a continuous multiobserver on a compact subset $\Omega \subset \mathbb{R}^2$.

Let $\Omega \subset \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0\}$ be a compact subset of $\mathbb{R}^2$. Let us consider

$$\Phi_\Sigma^2(x_1, x_2) = (x_1^2 + x_2^2, 2x_1(x_1^2 + x_2^2))$$

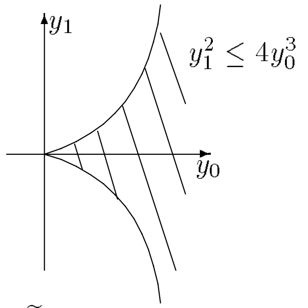
(as in [5]). The map Φ_Σ^2 is not injective, because Σ is not globally observable.

Let us consider indistinguishable states x and $\tilde{x}$ (writen as: $x \sim \tilde{x}$). Since Σ is an analytic system and the condition (5) holds, x and $\tilde{x}$ are indistinguishable iff $y_0(x) = y_0(\tilde{x})$ and $y_1(x) = y_1(\tilde{x})$.

Let $\widetilde{\Phi} : \Omega/_\sim \to \mathbb{R}^2$ and $\widetilde{\Phi}([x]) := (y_0(x), y_1(x))$, where $[x] \in \Omega/_\sim$ and $[x] = \{\tilde{x} \in \Omega : x \sim \tilde{x}\}$. Then

$\widetilde{\Phi}$ is injective and

$$\operatorname{im}\widetilde{\Phi} = \operatorname{im}\Phi^2_\Sigma|_\Omega \subset \operatorname{im}\Phi^2_\Sigma = \{(y_0, y_1) \in \mathbb{R}^2 : y_1^2 \leq 4y_0^3\}.$$



Let $g := \widetilde{\Phi}^{-1} : \operatorname{im}\widetilde{\Phi} \to \Omega/_\sim$ and

$$g(y_0, y_1) = [(\frac{y_1}{2y_0}, \frac{\sqrt{4y_0^3 - y_1^2}}{2y_0})] = \{(\frac{y_1}{2y_0}, \frac{\sqrt{4y_0^3 - y_1^2}}{2y_0}), (\frac{y_1}{2y_0}, -\frac{\sqrt{4y_0^3 - y_1^2}}{2y_0})\}.$$

We can extend g to a continuous function g^* on the entire $\mathbb{R}^2$. We obtain the multifunction g^* defined as follows $g^* : \mathbb{R}^2 \to \Omega/_\sim$ and

$$g^*(y_0, y_1) := \begin{cases} g(y_0, y_1) & \text{if } |y_1| < 2y_0^{\frac{3}{2}} \\ (\sqrt[3]{\frac{y_1}{2}}, 0) & \text{if } |y_1| \geq 2y_0^{\frac{3}{2}} \end{cases}.$$

This multifunction g^* is the output of the multi-observer $S_{\Sigma,\theta,\Omega}$.

The function φ which discribes the relation between the functions of $\mathcal{H}(\Sigma)$ is well defined on $\operatorname{im}\Phi^2_\Sigma$ and it is smooth on $\operatorname{int}(\operatorname{im}\Phi^2_\Sigma)$.

Let $\varphi^* : \mathbb{R}^2 \to \mathbb{R}$ and

$$\varphi^*(y_0, y_1) := \begin{cases} \varphi(y_0, y_1) & \text{if } |y_1| < 2y_0^{\frac{3}{2}} \\ \frac{3}{2}\sqrt[3]{4}y_1^{\frac{4}{3}} & \text{if } |y_1| \geq 2y_0^{\frac{3}{2}} \end{cases}.$$

Then the function φ^* is a continuous extension of the function φ. It is not smooth. Partial derivatives of φ^* exist and they are as follows:

$$\frac{\partial\varphi^*}{\partial y_0}(y_0, y_1) = \begin{cases} -\frac{3y_1^2}{2y_0^2} & \text{if } |y_1| < 2y_0^{\frac{3}{2}} \\ 0 & \text{if } |y_1| > 2y_0^{\frac{3}{2}} \end{cases};$$

$$\frac{\partial\varphi^*}{\partial y_1}(y_0, y_1) = \begin{cases} -\frac{3y_1}{y_0} & \text{if } |y_1| < 2y_0^{\frac{3}{2}} \\ 2\sqrt[3]{4}y_1^{\frac{1}{3}} & \text{if } |y_1| > 2y_0^{\frac{3}{2}} \end{cases}.$$

Since Ω is compact, then $V = \Phi^2_\Sigma(\Omega)$ is also compact, as well as $\varphi^*(V)$. Since partial derivatives of the function φ^* on the compact set V are restricted, then there is a Lipschitz constant L for the function φ^* on V.

There is no smooth observer for the system Σ, but we have the following continuous observer:

$$S_{\Sigma,\theta,\Omega} : \begin{cases} \dot{z}_1 = -k_1\theta z_1 + z_2 + k_1\theta y \\ \dot{z}_2 = -k_2\theta^2 z_1 + k_2\theta^2 y + \varphi^*(z_1, z_2) \end{cases},$$

where $k_1, k_2 \in \mathbb{R}$ and $\begin{pmatrix} -k_1 & 1 \\ -k_2 & 0 \end{pmatrix}$ is Hurwitz and $\theta \in \mathbb{R}_+$ ($\theta \geq 1$). The observer system $S_{\Sigma,\theta,\Omega}$ gives an (exponentional) estimation of successive derivatives of the output based on the known output $y(t)$, which is "observed" (similarly as in [5]).

Let us consider $[\hat{x}] = g^*(z_1, z_2)$. Then $[\hat{x}(t)]$ is an estimation of the equivalence class $[x(t)] \in \Omega/_\sim$, which contains the states indistinguishable from the state $x(t)$.

References

[1] Z. Bartosiewicz, Local observability of nonlinear systems, *System & Control Letters* 25, 1995, 295–298.

[2] Z. Bartosiewicz, Real analytic geometry and local observability, in: *Differential Geometry and Control*, Eds. G. Ferreyra *et al.*, Americam Mathematical Society, Providence 1998.

[3] Bartosiewicz, M. Kosk, D. Mozyrska, Local observability and quasi-observers for nonlinear systems, to appear in Proceedings of MTNS'98, Padova, July 1998.

[4] R. Hermann and A. Krener, Nonlinear controllability and observability, *IEEE Trans. Autom. Control* AC-22, 1977, 728–740.

[5] P. Jouan and J.-P. Gauthier, Finite singularities of nonlinear systems. Output stabilization, observability and observers, *J. Dynamical and Control Systems* 2, 1996, 255–288.

[6] D. Mozyrska and Z. Bartosiewicz, Algebraic criteria for stable local observability of analytic systems on $\mathbb{R}^n$, in: *Proceedings of European Control Conference ECC-97*, Brussels, July 1997.