Symbolic Determination of Generalized State Equation for Singular System Modelled by Bond Graph

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Abstract: The aim of the paper is to show how a bond graph model of a singular system is a good tool for the symbolic calculus of the associated mathematical model. We propose a method based on causal path handling leading to the generalized state equation determination, and the calculation of the generalized characteristic polynomial.
Keys words: Descriptor systems, derivative causality, causal cycle, bond graph
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1. Introduction

Many synonymous terms are used to define the type of state space model we are working on: generalized state space system, implicit system, differential - algebraic equations, singular system, descriptor system or semi-state system.

The descriptor systems have attracted an increasing interest of many researchers in the last twenty years [1], [2], [5] and [11]. Descriptor systems appear in many applications, either during the study of regular systems (linearization of implicit systems, interconnection between sub-systems, inversion of systems, multi time scale systems,…), or as a process model (electrical networks, robotics, economics,…). Comparison of some specific characteristics between the regular and the singular cases points out that the latter is not a trivial extension of the regular case.

The bond graph methodology leads to unified representation and the same calculation procedure of the state equation, whatever the physical domain. Some problems occur when algebraic loops or derivative causalities appear in the bond graph model, which may induce manipulations on the equations to avoid simulation difficulties.

This paper is organized as follows: section 2 establishes the problem statement, in section 3 we present the variables in a bond graph. Then the main result, in the general case, is given section 4 and 5: the graphical determination of matrices constituting a new descriptor form on Bond Graph models, and the symbolic calculus of the coefficients of the generalized characteristic polynomial based on causal cycle families gains.

2. Problem Statement

Let us consider the general linear time invariant multivariable state model, described equation (**1**)

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \qquad (1)$$

with the state vector $x \in \Re^n$, the input vector $u \in \Re^m$ and the output vector $y \in \Re^p$.

If E is non singular, the system is said to be regular, equation (1) can be written as (**2**):

$$\begin{cases} \dot{x} = A^{'}x + B^{'}u \\ y = Cx + Du \end{cases} \qquad (2)$$

where $A^{'} = E^{-1}A, \quad B^{'} = E^{-1}B$

If E is a singular matrix, equation (1) represents a descriptor system.

The solvability of model (1) is defined as the existence of a unique solution for any given control function sufficiently differentiable *u(t)* and any given admissible initial condition *x(0).* In the frequency domain, the use of Laplace transform leads to write Eqn.(1) as Eqn (3), with necessary and sufficient conditions of existence and unicity for the solution.

The strongest condition is det $(sE - A) \neq 0$. for almost every $s \in \forall$

$$\begin{cases} (sE - A)X(s) = E.x(0) + B.U(s) \\ Y(s) = C.X(s) + D.U(s) \end{cases} \qquad (3)$$

In this case, the pencil (sE-A) is regular. The output and the input under null initial conditions *(Ex(0) = 0)* are related by the non strictly proper transfer function, as follows:

$$Y(s) = T(s).U(s)$$
with $$T(s) = C(sE\text{-}A)^{-1}B + D$$

In the generalized case, the choice of the initial vector *x(0)* must not be arbitrary and must verify some conditions of consistency. The response of the system can exhibit:

✦ d exponential modes.

where $d = deg\{det(sE-A)\} \leq r = rank(E)$

rank(E) is the actual order of system.

the equality $r = d$ occurs in the regular case.

✦ *(r-d)* impulsive (or distributional) modes corresponding to *(sE-A)* losing rank at s = ∞, hence to poles at infinity.

The solvability of descriptor system is one of the main structural properties which every study must begin with, their important role appears precisely when studying infinite structure from the transfer function of the system.

3. Bond Graph Approach

We study here the case where the bond graph model contains some *I*, *C*-elements in derivative causality when an integral causality assignment is performed.

The state vector $x = \begin{pmatrix} x_i \\ x_d \end{pmatrix}$ is composed of the energy variables associated with the I and C dynamical elements respectively in integral and derivative causality.

$$\dim x = \dim x_i + \dim x_d = n_i + n_d$$

The assumption is made that there is no signal bond in the bond graph model. For sake of simplicity, we will suppose in the following that there are no algebraic loops between *R*-elements in the studied bond graph model.

The different state, input and output vectors are represented by:

$$x_i = \begin{bmatrix} p_{I_{i1}} \\ p_{I_{i2}} \\ \vdots \\ q_{C_{i1}} \\ q_{C_{i2}} \\ \vdots \end{bmatrix} \quad x_d = \begin{bmatrix} p_{I_{d1}} \\ p_{I_{d2}} \\ \vdots \\ q_{C_{d1}} \\ q_{C_{d2}} \\ \vdots \end{bmatrix} \quad u = \begin{bmatrix} u_1 \\ \vdots \\ \vdots \\ \vdots \\ u_m \end{bmatrix} = \begin{bmatrix} S_{e_1} \\ S_{e_2} \\ \vdots \\ S_{f_1} \\ S_{f_2} \\ \vdots \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} D_{e_1} \\ D_{e_2} \\ \vdots \\ D_{f_1} \\ D_{f_2} \\ \vdots \end{bmatrix}$$

We recall here the definition of the causal path generalized length was introduced in [7] as an extension of the definition of the causal path length [8] and the zero-order causal path [12]

Definition 1

i) The generalized length, denoted L_k^g, of a causal path from an element e_i to an element e_j, where e_i belongs to the set $\{R, S_e, S_f, I, C\}$ and e_j belongs to $\{D_e, D_f\}$, is equal to *k*, given by (**4**):

$$k = (-1)^s (N_i - N_d) \qquad (4)$$

where,

s = 0 if an preferred integral causality assignment is performed on the bond graph model,

s = 1 if a preferred derivative causality assignment is performed on the bond graph model,

N_I (resp. N_D) is the number of *I* or *C*-elements in integral (resp. in derivative) causality met when following the causal path from e_i to e_j.

ii) If e_i belongs to $\{S_e, S_f, R, I, C\}$ and e_j belongs to $\{I, C\}$, the generalized length of the causal path between e_i and e_j is equal to (*k+1*) and noted L_{k+1}^g.

Remark

Here we will only consider the case where s = 0. The case s = 1 is used for the input –output decoupling .

Definition 2

A closed causal path which contains several {I,C} elements (with any causality) is called a causal cycle.

4. Symbolic Calculus of the state space representation

4.1 Representation of a Linear descriptor system from a Bond Graph

The descriptor system is represented by equation (1) with *E* a singular matrix

Equation (1) can be written as follows:

$$\begin{pmatrix} I_{n_i} & E_{id} \\ 0_{di} & 0_{dd} \end{pmatrix} \begin{pmatrix} \dot{x}_i \\ \dot{x}_d \end{pmatrix} = \begin{pmatrix} A_i & 0 \\ A_{di} & A_{dd} \end{pmatrix} \begin{pmatrix} x_i \\ x_d \end{pmatrix} + \begin{pmatrix} B_i \\ B_d \end{pmatrix} u \quad (5)$$

The expression of submatrices composing E, A, and B in equation (5), are deduced from the junction structure equation built from the bond graph model.

The different vectors involved in a bond graph are linked through the junction structure as:

$$\begin{pmatrix} \dot{x}_i \\ z_d \\ D_i \\ Y \end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & 0 & 0 & S_{24} \\ S_{31} & 0 & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \begin{pmatrix} z_i \\ \dot{x}_d \\ D_o \\ u \end{pmatrix} \quad (6)$$

S_{22} is a null matrix because two dependent variables {(I, C) in derivative causality } are not directly causally connected (it is possible to simultaneously change their causality to obtain two dynamical elements in integral causality). S_{23} *and* S_{32} are also null matrices because it is impossible to have a derivative causality on elements I or C causally connected to linear R- elements (it is possible to exchange the causalities).

Elementary laws are associated with components:

$z_i = F_i\, x_i$
$x_d = (F_d)^{-1}\, z_d$ (7)
$D_{out} = L\, D_{in}$

z_i ,z_d are called complementary state vectors, and F_i , F_d are diagonal matrices composed of 1/I and 1/C coefficients. These matrices are always invertible for bond graph models without multiport elements.

D_{in} and D_{out} denote the vectors composed of variables flowing respectively into and out of the R-components ; L is a diagonal matrix composed of resistance and conductance parameters.

From (6) and (7) comes:

$$\begin{cases} (I - S_{33}L)\, D_i = S_{31}F_i\, x_i + S_{34}\, u \\ \dot{x}_i - S_{12}\, \dot{x}_d = S_{11}F_i\, x_i + S_{13}L\, D_i + S_{14}.u \\ F_d\, x_d = S_{21}F_i\, x_i + S_{24}\, u \end{cases}$$

which combined with (5) gives :

$$\begin{cases} E_{id} = - S_{12} \\ A_{di} = S_{21}\, F_i \\ A_{dd} = - F_d \\ B_d = S_{24} \end{cases} \qquad (8)$$

Both matrices A_i and B_i keep the same expression, as developed in [10]:

$A_i = [S_{11} + S_{13}L(I- S_{33}L)^{-1}\, S_{31}]F_i$
$B_i = [S_{14} + S_{13}L(I- S_{33}L)^{-1}\, S_{34}]$ (9)

(I = the identity matrix of appropriate dimension).

4.2 Symbolic calculus of Generalised state equation

The symbolic calculus of the descriptor form (5) for continuous linear singular systems directly, from their bond graph model aims to allow many structural properties study.

Determination of the E-matrix

Proposition 1

In the E_{id} - matrix, the element $(e_{id})_{jk}$ -term is obtained by expression (**10**):

$$(e_{id})_{jk} = -\sum_{p\in P} \left(\tilde{G}_{L_1^g} \left((x_d)_k, (x_i)_j \right) \right)_p \quad (10)$$

where $k \in \{1..n_d\}$ and $j \in \{1..n_i\}$

$\tilde{G}_{L_1^g}((x_d)_k,(x_i)_j)$ is the constant term of the gain of the causal path of generalized length L_1^g from $(x_d)_k$ to $(x_i)_j$, (N_i = 0, N_d = 0).

P is the list of all causal paths linking $(x_d)_k$ to $(x_i)_j$

Determination of the A-matrix

- A_i is calculated independently of the elements in derivative causality as in the regular case (see appendix).
- A_{dd} = $-F_d$ diagonal matrix composed of the parameters associated with I and C elements composing x_d , with negative sign.
- A_{di} represent causal relations between x_i and x_d .

Proposition 2

In the A_{di} -matrix, the $(a_{di})_{hj}$ -term is obtained by expression (**11**):

$$(a_{di})_{hj} = \sum_{p\in P} \left(\tilde{G}_{L_1^g} \left((x_i)_j, (x_d)_h \right) \right)_p * \tilde{g}((x_i)_j) \quad (11)$$

where $h \in \{1..n_d\}$ and $j \in \{1..n_i\}$

$\tilde{G}_{L_1^g}((x_i)_j,(x_d)_h)$ is the constant term of the gain of the causal path of generalized length L_1^g *(N_i = 0, N_d = 0)* from $(x_i)_j$ to $(x_d)_h$.

$\tilde{g}((x_i)_j)$ is the constant term of the jth transmittance of the I or C element in integral causality associated with $(x_i)_j$.

Determination of the B-matrix

- B_i is calculated by ignoring elements in derivative causality as in the regular case (see appendix).
- B_d is calculated using causal paths between the inputs *u* and x_d .

Proposition 3

In the control matrix B_d , the term $(b_d)_{jk}$ is obtained by (**12**):

$$(b_d)_{jk} = \sum_{p\in P} \left(\tilde{G}_{L_1^g} \left(u_k, (x_d)_j \right) \right)_p \quad (12)$$

where $k \in \{1 \dots m\}$ and $j \in \{1 \dots n_d\}$

$\tilde{G}_{L_1^g}(u_k, (x_d)_j)$ is the constant term of the gain of the causal path of generalized length L_1^g *(N_i = 0, N_d =0)* .from the (S_e or S_f) associated with u_k to dynamical element *(I,C)* in derivative causality associated with $(x_d)_j$.

Example

Consider the electrical system (figure **1**) and the associated bond graph model (figure **2**).

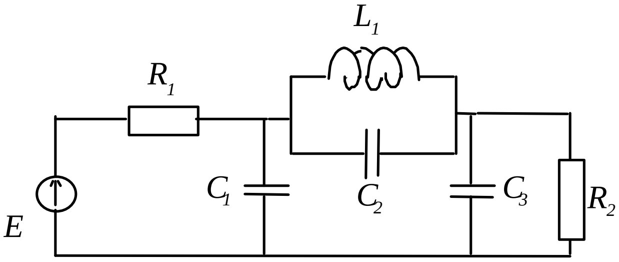

Figure **1:** electronic circuit

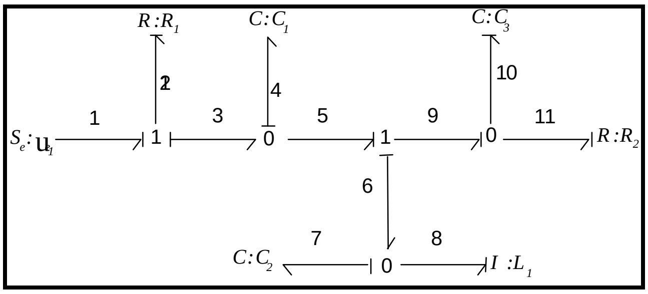

Figure **2:** Bond graph model

The I_1, C_1 and C_2 are in integral causality, C_3 is in derivative causality.

The different vectors are given by:

$$x = \begin{pmatrix} x_i \\ x_d \end{pmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} p_{I_1} \\ q_{C_1} \\ q_{C_2} \\ q_{C_3} \end{bmatrix} \text{ and } u = [u_1] = [E]$$

The matrices in equation (1) are obtained as:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & +1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} B_i \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/R_1 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & \frac{1}{C_2} & 0 \\ 0 & \frac{-1}{R_1C_1} + \frac{-1}{R_2C_1} & \frac{1}{R_2C_2} & 0 \\ \frac{-1}{L_1} & \frac{1}{C_1R_2} & \frac{-1}{R_2C_2} & 0 \\ 0 & \frac{1}{C_1} & \frac{-1}{C_2} & \frac{-1}{C_3} \end{bmatrix}$$

Remark

The last row of *A* represents the algebraic equation showing the dependancy between the independant and dependent state variables: $\frac{q_{c_1}}{C_1} - \frac{q_{c_2}}{C_2} - \frac{q_{c_3}}{C_3} = 0$

5. Bond Graph Interpretation of det [sE -A]

The characteristic polynomial is equal to the denominator of the transfer function $T(s) = C(sE - A)^{-1}B$, expressed as: $p_{(E,A)}(s) = |sE - A| = \sum_{i=0}^{n} p_i s^{n-i}$.

where $n = \dim x_i + \dim x_d = n_i + n_d$

In the following, a graphical interpretation of the polynomial coefficients is presented. This method is based exclusively on causality handling and causal cycle families concept. Because the bond graph model contains some elements in derivative causality, it requires a new definition of the order of a causal cycle.

Definition 3

A causal cycle family is a set of disjoint causal cycles. This family is said to be of order t, if this family contains N_i independent dynamical elements and N_d statically dependent elements with relation ***$t = N_i - N_d$.***

Remark

Contrary with the generalized length of a causal path, any possible combination of disjoint causal cycles to obtain a precise order is accepted .

Proposition 4

Each coefficient p_i of $p_{(E,A)}(s)$ is equal to the sum of the constant terms of the gain of causal cycle families of order $i - N_d$ in the bond graph model.
Each causal cycle family gain has the sign $(-1)^d$ if the family under consideration has d disjoint causal cycles.

* *for* $i = 0,1,\dots,n$ *and* i ¹ N_d

$$p_i = \frac{\sum_j (-1)^{d_j} * \tilde{G}_j^{i-N_d}}{\Pi \tilde{g}(x_d)} \qquad (13)$$

** If $i = N_d$, p_{n_d} is related to causal cycle families of order zero (t = 0). Its expression is given as:*

$$p_{n_d} = \frac{\left[1 + \sum_j (-1)^{d_j} * \tilde{G}_j^0\right]}{\Pi \, \tilde{g}(x_d)} \qquad (14)$$

where

j : the jth. causal cycle family of order $i - N_d$.

d_j: number of disjoint causal cycles constituting the jth. family.

$\tilde{g}(x_d)$: the constant term in transmittance of elements {I, C} with a derivative causality in the bond graph model. $\tilde{g}(I_d) = I_d$ and $\tilde{g}(C_d) = C_d$.

$\tilde{G}_j^{i-N_d}$: the gain constant term of jth. causal cycle family of i- N_d order.

Illustrative Example

The bond graph model of figure (2) has three dynamical elements in integral causality *(C_1,C_2,I_1)* and one dependent element C_3 (n = 4, $n_i = 3$, $n_d = 1$).

So the characteristic polynomial can be written as:

$$P_{(E,A)}(s) = p_0 s^4 + p_1 s^3 + p_2 s^2 + p_3 s^1 + p_4$$

The bond graph model does not contain any causal cycle family of order (-1) (causal boucle between resistive element and dependent element), $p_0 = 0$.
The p_1 is calculated using formula (14) *($i - N_d = 0$)*. but *p_2, p_3 and p_4 are* determined by formula (13). For sake of simplicity, only the calculation of coefficients *p_1 and p_4* of characteristic polynomial is developed. The table 1 gathers the causal cycle families required for the calculation of these coefficients.

$$p_1 = \left(\frac{1}{\tilde{g}(C_3)}\right) * \left[1 + (-1)^1\left(-\frac{C_3}{C_1}\right) + (-1)^1\left(-\frac{C_3}{C_2}\right)\right] = \left(\frac{1}{C_3}\right) * \left[1 + \frac{C_3}{C_1} + \frac{C_3}{C_2}\right]$$

$$p_2 = \left(\frac{1}{C_3}\right) * \left[\frac{1}{R_2 C_2} + \left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) + \left(\frac{1}{R_1 C_1} * \frac{C_3}{C_2}\right)\right]$$

$$p_3 = \left(\frac{1}{C_3}\right) * \left[\frac{1}{R_2 C_2 R_1 C_1} + \frac{1}{L_1 C_2} + \frac{C_3}{L_1 C_2 C_1}\right]$$

$$p_4 = \left(\frac{1}{C_3}\right) * \left[\left(\frac{1}{L_1 C_2}\right) * \left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right)\right]$$

Coefficient p_i	**Corresponding Causal Cycle Families in the bond graph model**	**Gains**
p_1 two causal cycles families of zero order *($i - N_d = 0$).*	**(1)**. (C_3 - C_1) : f_{10}-f_9-f_5-f_4-e_4-e_5-e_9-e_{10} → (d = 1) **(2)**. (C_3 - C_2) : f_{10}-f_9- f_6 -f_7- e_7- e_6- e_9- e_{10} → (d = 1)	$\tilde{G}_1^0 = \frac{-C_3}{C_1}$ $\tilde{G}_2^0 = \frac{-C_3}{C_2}$
p_4 causal cycles families of order 3. *($i - N_d = 3$).*	**1.** cycle causal family: {(R_1 ,C_1); (I_1 ,C_2)} → (d = 2) (R_1 ,C_1) : f_2-f_3-f_4-e_4-e_3-e_2 (I_1 ,C_2) : f_8-f_7-e_7-e_8 **2**. cycle causal family: :{(R_2 ,C_1); (I_1 ,C_2)} → (d = 2) (R_2 ,C_1) : f_{11}-f_9-f_5-f_4-e_4-e_5-e_9-e_{11} (I_1 ,C_2) : f_8-f_7-e_7-e_8	$\tilde{G}_1^3 = \frac{+1}{R_1 C_1 C_2 L_1}$ $\tilde{G}_2^3 = \frac{+1}{R_2 C_1 C_2 L_1}$

6. CONCLUSION

In this paper we have proposed a representation of singular system deducted from the bond graph model. Then, the symbolic calculus of this representation is given. It is followed by a method, based exclusively on causality handling and concept of causal cycle families allowing the symbolic determination of characteristic polynomial. These results are necessary for the structural properties analysis like controllability, observability…

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APPENDIX

Definition

The gain of a causal path in a bond graph model is given by (**A1**):

$$G(s) = (-1)^{n_0+n_1} * \prod_{i=1}^{a} (m_i)^{h_i} * \prod_{j=1}^{b} (r_j)^{k_j} * T_r(s)$$

with

n_0: the total number of arrow reversals at 0-junctions, when the path follows the flow variable,

n_1: the total number of arrow reversals at 1-junctions, when the path follows the effort variable,

a: : the number of TF-elements in the path,

b : the number of GY-elements in the path,

m_i and r_j are respectively the *i*th TF and the *j*th GY modulus,

$h_i = \pm 1$ depending on the *i*th TF causality assignment,

$k_j = \pm 1$ depending on the *j*th GY causality assignment,

$T_r(s)$: the product of transmittances of the (*I, C,* or *R*)-elements involved in the causal path. If no (*I, C,* or *R*)-elements are involved in the causal path, then T_r (s) = 1.

The constant term of *G(s)*, when it depends on the Laplace operators is denoted $\tilde{G}$ obtained by suppressing the *s*-operator in *G(s).*

Symbolic determination of the A and *B* for linear regular system directly from the bond graph model.

Property 1

In the state matrix A_i, the elements a_{jh} is obtained by equation (**A2**):

$$a_{jh} = \sum_{p \in P} \left(\tilde{G}_{L_1}(x_h, x_j)\right)_p * \tilde{g}(x_h) \qquad \text{(A2)}$$

where $h \in \{1 \dots n\}$ and $j \in \{1 \dots n\}$

$\tilde{G}_{L_1}(x_h, x_j)$ is the constant term in the gain of the causal path of length L_1 from the *I* or *C* elements associated with x_h to those associated with x_j. $\tilde{g}(x_h)$ is the constant term in the gain of x_h, $\tilde{g}(I) = \frac{1}{I}$ and $\tilde{g}(C) = \frac{1}{C}$.

Property 2

In the control matrix B_i, the term b_{jk} is obtained by (**A3**):

$$b_{jk} = \sum_{p \in P} \left(\tilde{G}_{L_1}(u_k, x_j)\right)_p \qquad \text{(A3)}$$

where $k \in \{1 \dots m\}$ and $j \in \{1 \dots n\}$

$\tilde{G}_{L_1}(u_k, x_j)$ is the constant term in the gain of the causal path of length L_1 from the (S_e or S_f) associated with u_k to the *I* or *C* elements associated with x_j.