Optimum Estimation of a Phase Modulating Wiener Process with Applications in Coherent Detection Systems

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Abstract: - In this work we apply the state variable approach in the synthesis of a maximum – likelihood estimator for a Wiener process modeling a phase noise modulation onto a high frequency carrier to be received by a coherent detector in the presence of additive noise.
We perform computer simulations on the general estimator and on simplified versions, such as the constant variance and the phase – locked – loop estimators.

Key – Words: - maximum – likelihood estimation, coherent reception, state variable estimation, homodyne and heterodyne detection, Wiener processes, phase modulation. *IMACS/IEEE CSCC'99 Proceedings***,** Pages:2801-2806

1 Introduction

In the synthesis of optimum estimators for random processes the state variable approach is a powerful technique since it allows the treatment of a wide variety of signal and noise statistics with nonlinear modulations, both for analog and digital processing. This technique has been widely used in the analog and digital communications theory, together with the maximum – likelihood principle, yielding estimators that perform good both for the transient and the steady – state behaviour.
These estimators are obtained by the mechanization of the so – called «estimator» and «variance» equations, resulting from the maximization of a suitable likelihood function.
Several works have been reported that use this approach for the coherent detection of a random process modulating a carrier inmersed in noise, and implementable estimator structures have been obtained for the case when the signal processing is performed in the intermediate frequency (heterodyne detection). This allows the filtering of unwanted higher order terms appearing in the maximum – likelihood estimator equations due to the quadratic nature of the likelihood functional.
In this work we develop the above mentioned technique for the more complex case of the base – band processing, this is the homodyne case, which is more attractive from the power and bandwidth economy point of view, especially for angle modulations, such as phase – shift - keying. However, in this case the higher order terms must be taken into account, which yields a more complicated estimator configuration.
We first cast our coherent detection channel model in a state variable formulation including the random process generator, the (nonlinear) angle modulation, the noise addition and the observation process, using the Ito representation.
We next develop the Fokker – Planck analysis for the posterior probability of the phase process given the observable, arriving at non – linear integro – differential equations for the estimator and variance, which allows us to mechanize an optimum estimator. Since the likelihood equation contains higher order terms, this estimator is highly complex, therefore we perform computer simulations on the general estimator and on simplified versions, such as the uncoupled variance and the phase – locked – loop systems.

2 The Ito's calculus for the coherent channel

Our goal is to obtain an optimum structure for the estimation of a Wiener process x(t) that constitutes a phase noise process on an information signal S(t) that is digitally modulated by a data train I_K

$$S(x,t,I_K) = \sqrt{2P_H}\,\sin(x + I_k) \qquad (1)$$

where P_H is the homodyne power.

Let $V_0(t)$ be an electrical observable resulting from the received signal inmersed in noise $n(t)$, with white spectral density:

$$S_n(f) = \frac{N_o}{2}\left[\frac{Volts^2}{Hz}\right] \qquad (2)$$

$$V_o(t) = \sqrt{2P_H}\,\sin(x + I_k) + n(t) \qquad (3)$$

Using the Ito's calculus with the following definition:

$$V_o(t) = r(t) = \frac{dy(t)}{dt} \qquad (4)$$

we can cast the expression:

$$dy(t) = V_o(t)dt = S(x,t,I_k)dt + n(t)dt \qquad (5)$$

Using the Stratonovich representation:

$$n(t)dt = \sqrt{\frac{N_o}{2}}du(t)\ \text{[Volts-sec]} \qquad (6)$$

where

$\dot{u}(t) = \frac{du(t)}{dt}$ is a white gaussian process with spectral density: $\frac{N_o}{2}$

$$dy(t) = S(x,t,I_k)dt + \sqrt{\frac{N_o}{2}}du(t) \qquad (7)$$

x(t) is a Wiener process that can be described in the Ito's notation:

$$dx(t)\ \text{[rads]} = \frac{1}{\sqrt{t_c}}dv'(t)\ \text{[rads]} \qquad (8)$$

with $x(0) = \theta$, θ being a uniformly distributed random variable and

t_c is the coherence time

related to the linewidth by:

$$t_c = \frac{1}{2\pi f_L}\ [s], \qquad (9)$$

and

$dv'(t)\left[\frac{rads}{\sqrt{Hz}}\right]$ is a standard Wiener process .

3 The maximum – likelihood estimator.

We use the estimator and variance equations: [1]:

$$d\hat{x}(t) = \frac{2}{N_o}E\left\{(x-\hat{x})S(x,t,I_j)\right\}\left[dy(t) - ES(x,t,I_j)dt\right]$$

(10)(estimator)

where $\hat{x}(t)$ is the MMSE estimate of the process x(t) given the observation process, and assuming that I_j was sent

$$\hat{x}(t) = E\left[x(t)/r_{0,t}, I_j\right] \qquad (11)$$

the term $(x(t) - \hat{x}(t))$ is the estimation error and will be noted as e(t) , the mathematical expectance E[.] is with respect to the conditional density of x(t) given the observation and assuming that I_j was sent.

The variance equation is:

$$dv(t) = \frac{1}{t_c}dt + \frac{2}{N_o}E\left\{(x-\hat{x})^2\left[S(x,t,I_j) - ES(x,t,I_j)\right)\right.$$

$$\times\left[dy(t) - ES(x,t,I_j)dt\right]$$

$$-\frac{2}{N_o}E^2\left[(x-\hat{x})S(x,t,I_j)\right]dt \qquad (12)$$

where v(t) is the variance in the estimation of x(t):

$$v(t) = E\left[(x-\hat{x})^2\right] \qquad (13)$$

the signal estimate is then:

$$\hat{S}(x,t,I_j) = E\left[S(x,t,I)/r_{o,t}, I_j\right] j=0,1 \qquad (14)$$

The time evolution of the conditional density of the phase process x(t) is described by the following (Fokker – Planck) partial differential equation :

$$\frac{\partial p(x,t/r_{o,t},I_j)}{\partial t} = \frac{1}{2t_c}\frac{\partial^2 p(x,t/r_{o,t},I_j)}{\partial x^2} +$$

$$\frac{2}{N_o} p\left(x, t / r_{o,t}, I_j\right)\left[S\left(x, t, I_j\right) - ES\left(x, t, I_j\right)\right] \times$$
$$\left[r(t) - ES\left(x, t, I_j\right)\right] \qquad (15)$$

where E denotes the mathematical expectancy with respect to $p(x, t / r_{o,t}, I_j)$. This equation is not resolvable or implementable because the mathematical expectance requires the knowledge of the conditional density. As a first approximation, the (mmse) estimator of the signal $S\left(x, t, I_j\right)$ is expressed in terms of the mmse estimate of the phase error as follows:

$$\hat{S}\left(x, t, I_j\right) = E\left[S(x, t, I) / r_{o,t}, I_j\right]$$

$$\hat{S}\left(x, t, I_j\right) = \sqrt{2P_H}\, \mathrm{Im}\left[\exp\left(j\left(I_j + \hat{x}(t)\right)\right) E \exp\left(j(x(t) - \hat{x}(t))\right)\right] \qquad (16)$$

with

$$e(t) = x(t) - \hat{x}(t) \qquad (17)$$

where $M_e(\omega) = E\left[\exp\left(j\omega e(t)\right)\right]$ is the characteristic function of $e(t)$:

After some algebra we arrive at the estimator and variance equations:

$$d\hat{x}(t) = -\frac{2}{N_o}\sqrt{2P_H}\cos\left(\hat{x} + I_j\right)\dot{M}_e(1)dy(t) +$$
$$\frac{2P_H}{N_o}\sin\left(2\left(\hat{x} + I_j\right)\right)\dot{M}_e(1)M_e(1)dt \qquad (18)$$

(estimator equation)

$$dv(t) = \frac{1}{t_c}dt - \frac{2P_H}{N_o}dt \times$$
$$\dot{M}_e^2(1)\left(1 + \cos\left(2\left(\hat{x} + I_j\right)\right)\right) -$$
$$M_e(1)\left(1 - \cos\left(2\left(\hat{x} + I_j\right)\right)\right)\left(\ddot{M}_e(1) + v(t)M_e(1)\right)$$
$$-\frac{2}{N_o}S\left(\hat{x}, t, I_j\right)dy(t)\left[M_e(1)v(t) + \ddot{M}_e(1)\right]$$

(19) (variance equation)

with the $\mathrm{SNR} = \frac{2P_H}{N_o}$

4 An optimum estimator for gaussian phase noise.

Using the gaussian statistics approximation for the phase noise:

$$M_e(\omega) = \exp\left[-\frac{1}{2}v(t)\omega^2\right] \qquad (20)$$

After some algebra, the estimator equation becomes:

$$dx^*(t) = \frac{2}{N_o}v^*\exp\left(-\frac{v^*}{2}\right)\sqrt{2P_H}\cos\left(x^* + I_j\right)dy(t)$$
$$-\frac{2P_H}{N_o}v^*\exp\left(-v^*\right)\sin\left(2\left(x + I_j\right)\right)dt \qquad (21)$$

and the variance equation results:

$$dv^*(t) = \frac{dt}{t_c} - \frac{2P_H}{N_o}\left(2v^{*2}\exp\left(-v^*\right)\right)\cos\left(2\left(x^* + I_j\right)\right)dt$$
$$-\frac{2}{N_o}v^{*2}\exp\left(-\frac{1}{2}v^*\right)\sin\left(x^* + I_j\right)dy(t)\sqrt{2P_H} \qquad (22)$$

* is used to indicate an approximation to the optimum value for Gaussian statistics

and the signal estimate:

$$S^*\left(x, t, I_j\right) = \exp\left(-\frac{1}{2}v^*(t)\right)S\left(x^*, t, I_j\right) \qquad (23)$$

In steady state, $v^* \approx 0$, and:

$$x^*(t) = \int v^*\frac{\sqrt{8P_H}}{N_o}\cos\left(x^* + I_j\right)r(t)dt$$
$$-\int v^*\frac{2P_H}{N_o}\sin\left(2\left(x^* + I_j\right)\right)dt \qquad (24)$$

and:

$$v^*(t) = \int \frac{1}{t_c} - \frac{\sqrt{8P_H}}{N_o} v^{*2} \sin\left(x^* + I_j\right) r(t) dt$$
$$- \int \frac{4P_H}{N_o} v^{*2} \cos\left(2\left(x^* + I_j\right)\right)\Big] dt \qquad (25)$$

5 The maximum likelihood receiver.

The estimator and variance equations are used for the implementation of the maximum likelihood estimator - correlator receiver. From the input observable we perform the estimation of the phase process x(t). Since we have two possibilities for the transmitted data (logic «1» or logic «0», this is, Ij= 0 or π), two blocks of estimators are required. The output from these estimators $S^*\left(x, t, I_j\right)$ is used for the correspondent maximum likelihood calculator $\lambda\left(I_j\right)$ described by the following equation:

$$\lambda\left(I_j\right) = \int_0^T \hat{S}\left(t, I / r_{o,t}, I_j\right) V_o(t) dt - \frac{1}{2}\int_0^T \hat{S}^2\left(t, I / r_{o,t}, I_j\right) dt$$

(26)(maximum likelihood)

the outputs $\lambda\left(I_j\right)$ are compared to decide if the received data was a logic «1» or a logic «0», obtaining the estimated data $\hat{I}$.

6 Simulation

Figure 1 is the mechanized block diagram of the coupled equations. By making several simplifications we also implemented the constant variance and PLL estimators.

In figure 2 we observe the signals corresponding to the phase noise for a linewidth of 2 Hz and the corresponding phase estimators.
In figure 3 we plot only the coupled equations and constant variance estimators together with the phase noise.
Finally in figure 4 we plot the difference between the three estimators and the corresponding phase noise.

7 Conclusion

We have applied the state variable approach to the problem of finding an optimum estimator for a phase noise modeled by a Wiener process in a coherent detection configuration. We arrived at an estimator structure mechanized by maximizing a suitable likelihood functional and performed computer simulations on this structure and simplified ones. We found that the more complex structure («coupled-equations») performs better, and the «constant variance» version performance is acceptable and can be readily implemented (no need of real time variance calculation). However the PLL version does not lead to acceptable results, this justifying the use of the more complicated structures that we arrived at in this work.

References:

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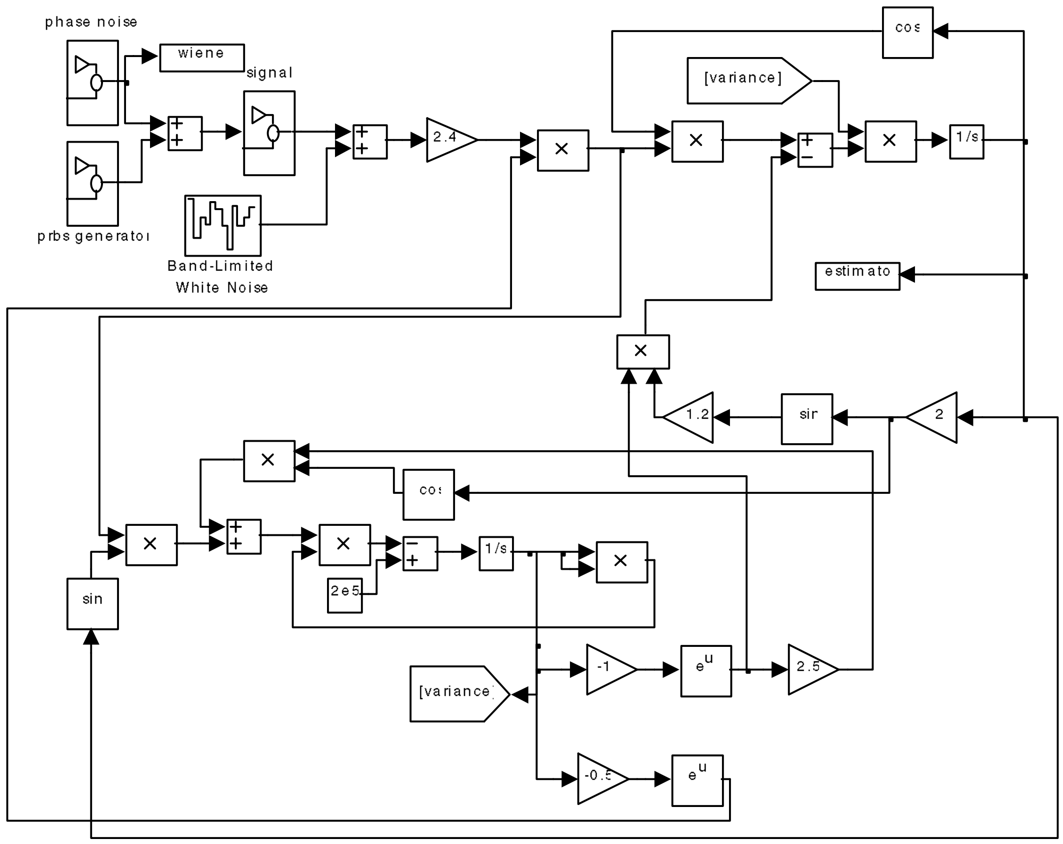

Figure 1 Block diagram of the coupled-equations estimator

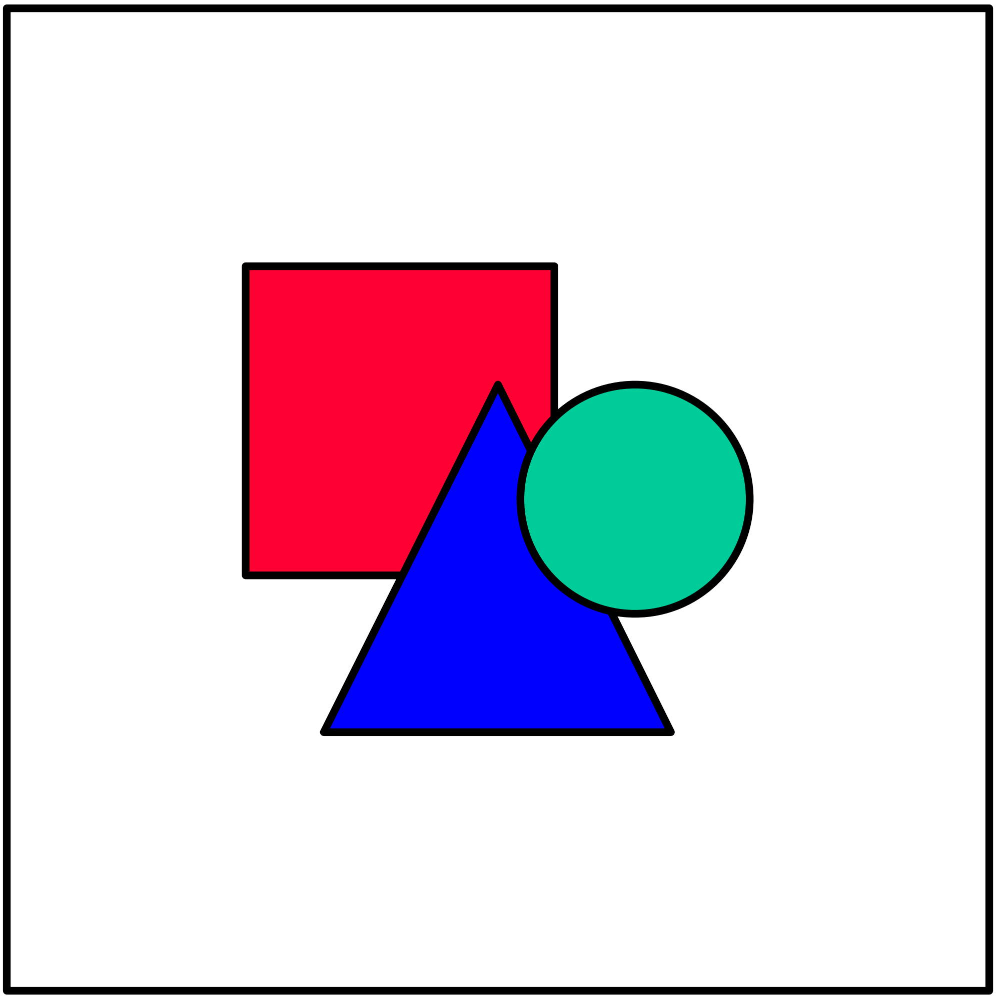

Figure 2 Plot of phase noise and three estimators

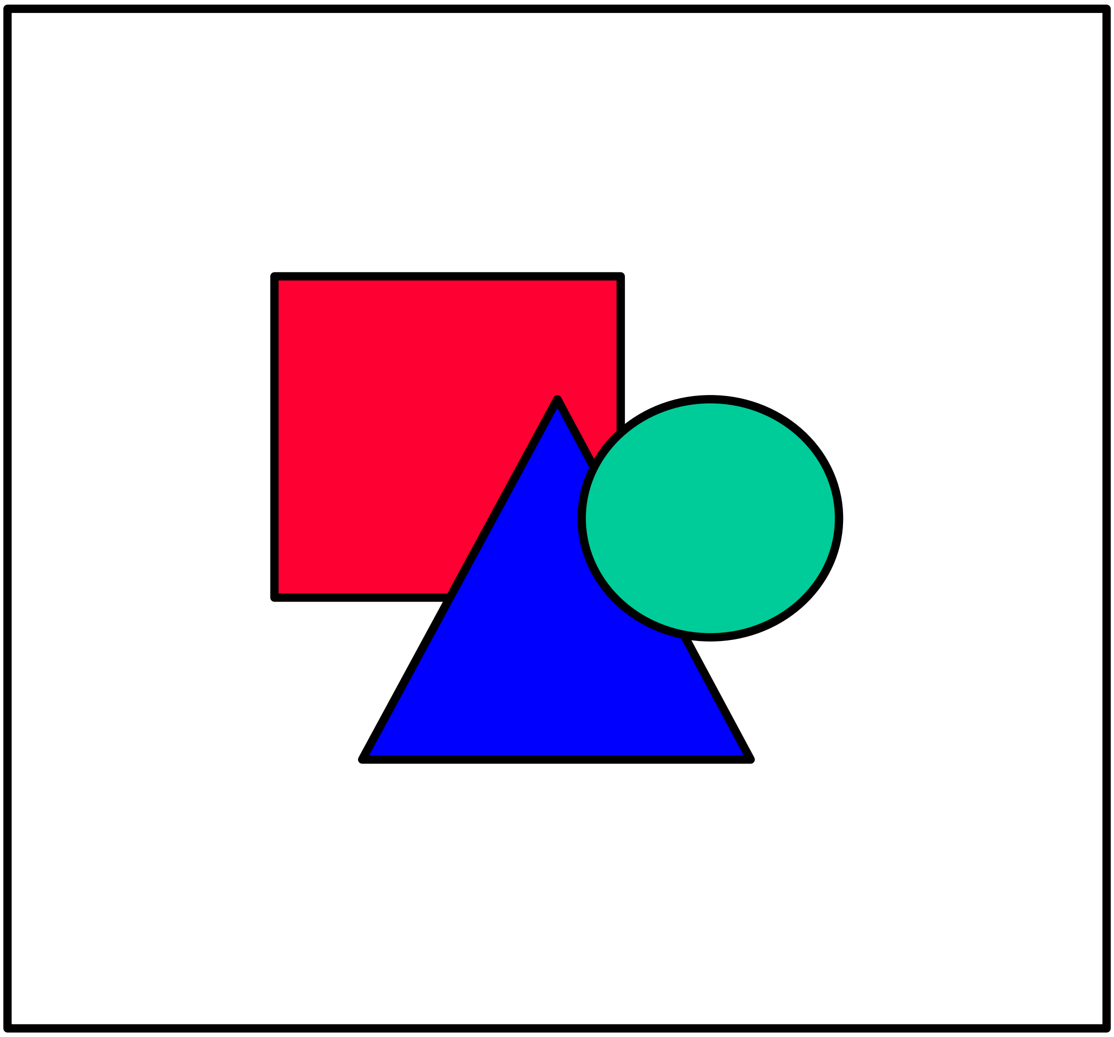

Figure 3 Phase noise,coupled equations and constant variance estimators

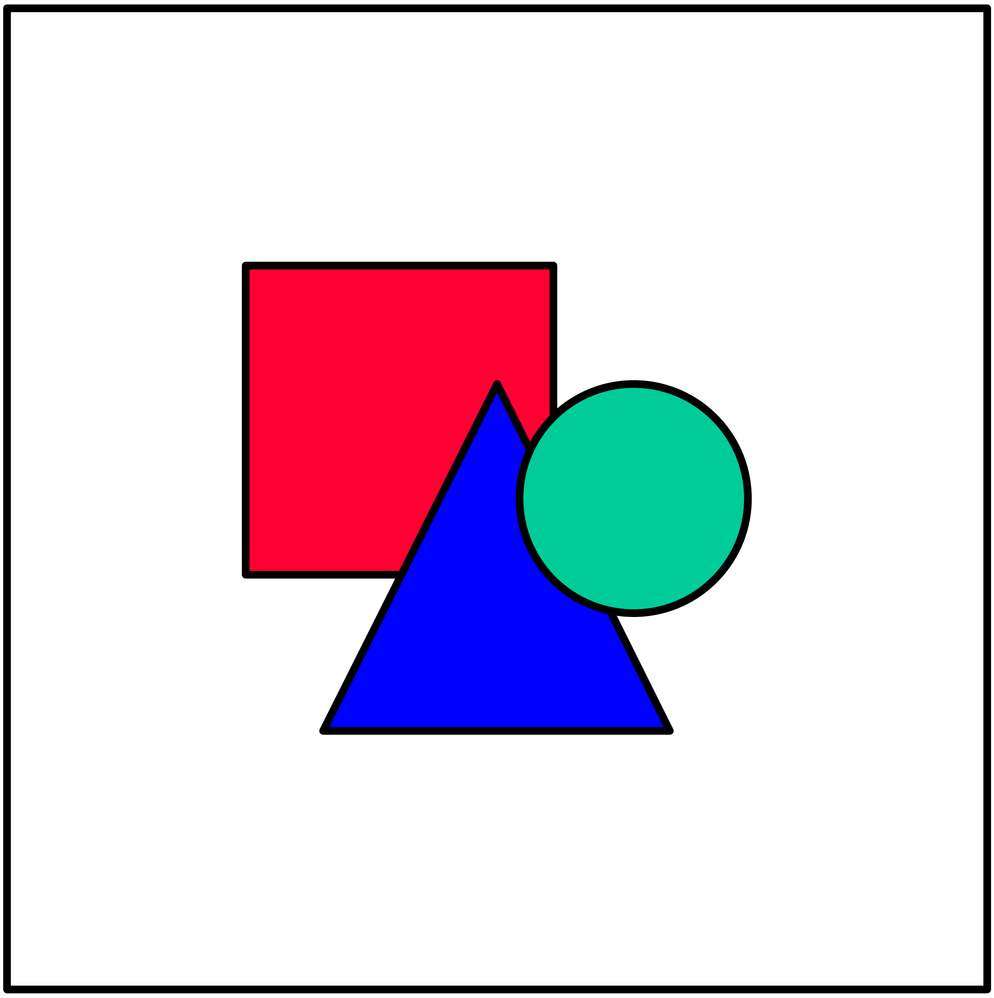

Figure 4 Difference between three estimators and phase noise