On the Discrete Time - Varying JLQ Problem

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Abstract: Systems, whose parameters or working conditions are subject to abrupt changes can be naturally modelled as jump linear systems. Because of their numerous applications in tracking, fault-tolerant control, manufacturing processes and robotics, such systems have drawn extensive attention. This paper is concerned with the optimal control of time - varying, discrete - time linear systems whose parameters are dependent on time and finite-state Markov processes which is directly observed. The cost functional to be minimized is the infinite-time horizonte quadratic cost. The solution of this time-varying jump linear quadratic control problem relies on study of nonnegative definite global and bounded solution of coupled difference Riccati equation. Necessary and sufficient conditions for existence of such a solution are obtained in terms of optimizability and detectability. Moreover the condition of the optimal close-loop system are established. In the time invariant case this results generalize the existing results about discrete JLQ on infinite time interval.. The more realistic case in which we have only partial observation of the Markovian parameter is objective of further researches.

Key-Words: Discrete time-varying systems; linear systems; optimal control; coupled Riccati equation; Markovian jumps.
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1. Introduction

Consider discrete-time linear system with Markovian jumps, modeled by

(1) $x(k+1) = A(k, r(k))x(k) + B(k, r(k))u(k)$,

where the coefficient matrix functions $A : \mathrm{N} \times S \to R^{n\times n}$, $B : \mathrm{N} \times S \to R^{n\times m}$ are such that $A(\cdot, i)$ and $B(\cdot, i)$ are bounded for each $i \in S$, $x(k)$ denotes the state vector, $u(k)$ is the control input, and the abrupt changes are incorporated into the model via the Markov chain $r(k)$ taking values in a finite set S and constant probability matrix $P = (p_{ij})_{i, j\in S}$,where

$$p_{ij} = P(r(k+1) = j | r(k) = i).$$

Subject to (1) we consider the minimization of

(2) $$J(x_0, i_0, u) = E\left(\sum_{k=1}^{\infty} \left(\langle Q(k, r(k))x(k), x(k)\rangle + \right.\right.$$
$$\left.\left. + \langle R(k, r(k))u(k), u(k)\rangle \right| x(0) = x_0, r(0) = i_0\right)$$

with $Q : \mathrm{N} \times S \to R^{n\times n}$ and $R : \mathrm{N} \times S \to R^{n\times n}$ such that $Q(k, i) = Q'(k, i)$, $Q(k, i) \geq 0$, $R(k, i) = R'(k, i)$, $R(k, i) > 0$ for all $(k, i) \in \mathrm{N} \times S$ and $Q(\cdot, i)$, $R(\cdot, i)$ and $R^{-1}(\cdot, i)$ are bounded for each $i \in S$. In (2) x is the solution of (1) with control u belonging to U such that $u(k) = j\,(k, x(k), r(k))$. Together with the problem (1), (2) we also consider the optimal control problem on the finite time-interval $[0, N]$. In this case the cost functional takes the form

(3)
$$J_{k_0, N}(x_0, i_0, u) = E\left(\sum_{k=k_0}^{N} \left(\langle Q(k, r(k))x(k), x(k)\rangle + \right.\right.$$
$$+ \langle R(k, r(k))u(k), u(k)\rangle + \langle F(r(N))x(N), x(N)\rangle +$$
$$\left.\left. | x(0) = x_0, r(0) = i_0 \right)\right.,$$

where $F : S \to R^{n\times n}$ is such that $F(i) = F'(i)$, $F(i) \geq 0$ for all $i \in S$.

Discrete time invariant version of JLQ problem was solved for finite time interval in [2]. In [1] the case where matrix *A* is not dependent on the Markov process is examined. Necessary and sufficient conditions were given for the existence of steady-state solutions with finite expected cost for the discrete time invariant JLQ problem in [3] and [6]. Finally in [4] the most general solution of discrete time invariant JLQ problem is presented when the space of values of the Markov chain is countable infinite under the assumption of stochastic stabilizability and stochastic detectability. To our knowledge the general time varying JLQ problem has not been treated elsewhere. Its significance results from the fact that linear model of the system is usually found basing on linearization of the real world nonlinear process along the given trajectory and therefore its parameters are time-varying.

The solution of discrete time varying JLQ on finite interval is given by the following theorem.

Theorem 1 The optimal control for the control problem (1), (3) is given by

(4) $\tilde{u}(k) = -L(k, r(k))x(k)$,

where

$$L(k,i) = \left(R(k,i) + B'(k,i)F(k+1,i)B(k,i)\right)^{-1} B'(k,i)F(k+1,i)A(k,i),$$

$$F(k,i) = \sum_{j \in S} p_{ij} P_N(k,j),\ k = N,\dots,k_0, i \in S$$

and $P_N(k,i),\ k = N,\dots,k_0, i \in S$ are given by the following coupled difference Riccati equation

$$P_N(N,i) = F(i),$$

(5) $$P_N(k-1,i) = A'(k,i)F(k,i)A(k,i) - A'(k,i)B(k,i)L(k,i),$$

$k = N,\dots,k_0, i \in S$. Moreover

(6) $J_{k_0,N}(x_0, i_0, \tilde{u}) = \langle P_N(k_0, i_0)x_0, x_0 \rangle$.

2. Optimizability and Existence of Solution of Coupled Difference Riccati Equation

The primary concern in this section is to establish sufficient and necessary conditions for the existence of optimal control for the problem (1)-(2). For this purpose we need the following Lemma whose counterpart for analogue for processes without jumps is well known [5].

Lemma 1 If $P_N^{(1)}(k,i)$ and $P_N^{(2)}(k,i)$ are solution of (5) such that $0 \le P_N^{(1)}(N,i) \le P_N^{(2)}(N,i),\ i \in S$ then $P_N^{(1)}(k,i) \le P_N^{(2)}(k,i),\ k = N,\dots,k_0, i \in S$.

Proof. Fix $k \in \{N,\dots,k_0\}, i \in S$, $x_0 \in R^n$ and denote by $J_{k_0,N}^{(1)}(x_0,i_0,u)$ and $J_{k_0,N}^{(2)}(x_0,i_0,u)$cost functional (4) with $F(i) = P_N^{(1)}(k,i)$ and $F(i) = P_N^{(2)}(k,i)$, $i \in S$, respectively. Using $P_N^{(1)}(N,i) \le P_N^{(2)}(N,i),\ i \in S$ it follows easily that $J_{k_0,N}^{(1)}(x_0,i_0,u) \le J_{k_0,N}^{(2)}(x_0,i_0,u)$. so by (6) we conclude that

$$\left\langle P_N^{(1)}(k_0,i)x_0, x_0 \right\rangle \le \left\langle P_N^{(2)}(k_0,i)x_0, x_0 \right\rangle.$$

Because k_0 is arbitrary the proof is complete.

Definition The system (1) with cost functional (2) is called optimizable if, for every fixed $i_0 \in S$ and $x_0 \in R^n$ there exists control *u* such that $J(x_0,i_0,u) < \infty$.

Theorem 2 If the system (1), (2) is optimizable then the limits

(7) $\lim_{N \to \infty} P_N(k,i) = P(k,i)$

exists for all $k = 1,2,\dots,\ i \in S$, where $P_N(k,i)$ is the solution of (5) with terminal condition $P_N(N,i)$ =0, $i \in S$, $P(k,i)$ satisfies the equation (5). Moreover $P(k,i) = P'(k,i),\ P(k,i) \ge 0$ for all $k = 1,2,\dots,\ i \in S$ and $P(k,i)$ is the minimal nonnegative definite global and bounded solution of (5) and the optimal control is given by

(8) $\tilde{u}(k) = -L(k, r(k))x(k)$,

where

$$L(k,i) = \left(R(k,i) + B'(k,i)F(k+1,i)B(k,i)\right)^{-1} B'(k,i)F(k+1,i)A(k,i),$$

$$F(k,i) = \sum_{j \in S} p_{ij} P(k,j),\ k = 1,2,\dots, i \in S$$

and

$$J(x_0,i_0,\tilde{u}) = \langle P(0,i_0)x_0, x_0 \rangle.$$

On the other hand, if there exists nonnegative definite solution of (5) global and bounded than (1), (2) is optimizable.

Proof. For $0 \le N_1 \le N_2$ fix $k_0 \in \{1,\dots,N_1\}$, $i_0 \in S$, $x_0 \in R^n$ and consider the cost functionals $J_{k_0,N_1}(x_0,i_0,u)$ and $J_{k_0,N_2}(x_0,i_0,u)$ both with $F(i) = 0$, $i \in S$. Then it follows easily from the form of the cost functional that

$$J_{k_0,N_1}(x_0,i_0,u) \le J_{k_0,N_2}(x_0,i_0,u)$$

and from (6) we conclude that

(9) $\langle P_{N_1}(k_0,i_0)x_0,x_0\rangle \le \langle P_{N_2}(k_0,i_0)x_0,x_0\rangle$.

By (6) and the optimizability condition we conclude that there is a constant $c > 0$ such that

(10) $$\|P_N(k,i)\| < c$$

for all N, $i \in S$ and $k = 1,...,N$. From (9) and (10) we conclude that the limit in (7) indeed exists and $P(k,i) = P'(k,i)$, $P(k,i) \ge 0$ for all $i \in S$ and $k = 1,2,\ldots$.Moreover because the constant in (10) does not depend on k, $P(k,i)$ is bounded for all $i \in S$. From (7) we see that $P(k,i)$ indeed satisfies (5). Next we shall show that $P(k,i)$ is the minimal nonnegative definite solution of (5) which is bounded. Let $L(k,i)$ be another bounded nonnegative definite solution of (5), and denote by $\overline{P}_N(k,i)$, $k = 1,...,N$, $i \in S$ the solution of (5) with $\overline{P}_N(N,i) = L(N,i)$. Since the solution is unique and, since $L(k,i)$ satisfies (5) we have $\overline{P}_N(k,i) = L(k,i)$ for all $k = 1,...,N$, $i \in S$. Furthermore, it follows from Lemma 1, together with

$$0 = P_N(N,i) \le \overline{P}_N(N,i) = L(N,i),$$

that

(11) $$P_N(k,i) \le \overline{P}_N(k,i) = L(k,i)$$

for all $k = 1,...,N$, $i \in S$. Combining (7) with (11) gives

$$P(k,i) \le L(k,i).$$

To solve the optimal control problem fix N, $i_0 \in S$, $x_0 \in R^n$ and consider the cost functional $J^{(1)}_{0,N}(x_0,i_0,u)$ with $F(i) = 0$, $i \in S$ and $J^{(2)}_{0,N}(x_0,i_0,u)$ with $F(i) = P(T,i)$, $i \in S$. Then apply to (1) the control (8) and use the fact that $J^{(1)}_{0,N}(x_0,i_0,u) \le J^{(2)}_{0,N}(x_0,i_0,u)$ and that $\tilde{u}$ is optimal for $J^{(2)}_{0,N}(x_0,i_0,u)$. We see that

$$J^{(1)}_{0,N}(x_0,i_0,\tilde{u}) \le J^{(2)}_{0,N}(x_0,i_0,\tilde{u}) = \langle P(0,i_0)x_0,x_0\rangle,$$

but the right hand side does not dependent on N, so

(12) $$J(x_0,i_0,\tilde{u}) = \lim_{N\to\infty} J^{(1)}_{0,N}(x_0,i_0,\tilde{u}) = \langle P(0,i_0)x_0,x_0\rangle.$$

On the other hand we have

(13) $$J(x_0,i_0,\tilde{u}) = \lim_{N\to\infty} J^{(1)}_{0,N}(x_0,i_0,\tilde{u}) \ge \lim_{N\to\infty}\langle P_N(0,i_0)x_0,x_0\rangle = \langle P(0,i_0)x_0,x_0\rangle.$$

(12) together with (13) shows the optimality of $\tilde{u}$.

Now suppose that there exists nonnegative definite and bounded solution $P(k,i)$ of (5). Fix $i_0 \in S$, $x_0 \in R^n$. Then apply the control

$$\overline{u}(k) = -L(k,r(k))x(k),$$

where

$$L(k,i) = (R(k,i) + B'(k,i)F(k+1,i)B(k,i))^{-1} B'(k,i)F(k+1,i)A(k,i),$$

$$F(k,i) = \sum_{j\in S} p_{ij}P(k,j),\ k = 1,2,...,i \in S$$

to (1) and use the facts that

$$J^{(1)}_{0,N}(x_0,i_0,\overline{u}) \le J^{(2)}_{0,N}(x_0,i_0,\overline{u})$$

and that $\overline{u}$ is optimal for $J^{(2)}_{0,N}(x_0,i_0,u)$ we have

$$J^{(1)}_{0,N}(x_0,i_0,\overline{u}) \le J^{(2)}_{0,N}(x_0,i_0,\overline{u}) = \langle P(0,i_0)x_0,x_0\rangle$$

but the right hand side does not depend on N, so

$$J(x_0,i_0,\overline{u}) = \lim_{N\to\infty} J^{(1)}_{0,N}(x_0,i_0,\overline{u}) = \langle P(0,i_0)x_0,x_0\rangle$$

that means that the system (1), (2) is optimizable. The proof is now complete.

3 Detectability and Uniqueness of Solution of Coupled Difference Riccati Equation

The main objective of this section is to find the sufficient conditions for the existence and uniqueness of nonnegative definite global and bounded solution of coupled difference Riccati equation (5) and sufficient conditions for stability of the optimal system. To formulate such a condition we need the following definitions.

Definition 2 The jump linear system

$$x(k+1) = A(k,r(k))x(k)$$

is stable, if for any $(x_0,i_0) \in R^n \times S$

$$E\left(\sum_{l=0}^{\infty}\|x(l)\|^2 \middle| x(0) = x_0, r(0) = i_0\right) < \infty.$$

In this case we call the function $A: \mathrm{N}\times S \to R^{n\times n}$ stable.

Definition 3 The jump linear system is stabilizable if for any $(x_0,i_0) \in R^n \times S$, there exists a linear feedback control $u(k) = F(k,r(k))x(k)$ such that the function $F(\cdot,i)$ is bounded for all $i \in S$ and the close loop system is stable. In this case we call the pair (A,B) of functions $A:\mathrm{N}\times S\to R^{n\times n}$ and $B:\mathrm{N}\times S\to R^{n\times m}$ stabilizable.

Definition 4 The jump linear system

$$x(k+1)=A(k,r(k))x(k)$$
$$y(k)=C(k,r(k))x(k)$$

is detectable, if the pair (A',C') is stabilizable. . In this case we call the pair (A,C) detectable.

Definition 5 The nonnegative global and bounded solution $P(\cdot,i)$, $i\in S$ of (5) is called stabilizable solution if the system (1) with control

$$u(k)=-L(k,r(k))x(k),$$

where

$$L(k,i)=\left(R(k,i)+B'(k,i)F(k+1,i)B(k,i)\right)^{-1} B'(k,i)F(k+1,i)A(k,i),$$

and

$$F(k,i)=\sum_{j\in S}p_{ij}P(k,j),\ k=1,2,\dots,i\in S$$

is stable.

The proofs of the following two lemmas are based on simple arithmetic transformations and we omit them.

Lemma 2 If the function $f:\mathrm{N}\times S\to R^m$ is such that

$$E\left(\sum_{k=0}^{\infty}\|f(k,r(k))\|^2\middle|x(0)=x_0,r(0)=i_0\right)<\infty,$$

for each $(x_0,i_0)\in R^n\times S$ and $A:\mathrm{N}\times S\to R^{n\times n}$ is stable then the solution of equation

$$x(k+1)=A(k,r(k))x(k)+f(k,r(k)),$$

with initial value $x(0)=x_0$ satisfies

$$E\left(\sum_{k=0}^{\infty}\|x(k)\|^2\middle|x(0)=x_0,r(0)=i_0\right)<\infty.$$

Lemma 3 Suppose that $P(\cdot,i)$, $i\in S$ is the nonnegative definite global and bounded solution of (5). Then for every u and for all $(x_0,i_0)\in R^n\times S$ the following holds

$$J_{0,N}(x_0,i_0,u)=\langle P(0,i_0)x_0,x_0\rangle- E\left(\langle P(N,i_0)x(N),x(N)\rangle\middle|x(0)=x_0.r(0)=i_0\right)+ E\left(\sum_{k=0}^{N}\langle R(k,r(k))(u(k)-L(k.r(k))), (u(k)-L(k.r(k)))\rangle\middle|x(0)=x_0,r(0)=i_0\right),$$

where

$$L(k,i)=\left(R(k,i)+B'(k,i)F(k+1,i)B(k,i)\right)^{-1} B'(k,i)F(k+1,i)A(k,i),$$

$$F(k,i)=\sum_{j\in S}p_{ij}P(k,j),\ k=1,2,\dots,i\in S,$$

and $J_{0,N}(x_0,i_0,u)$ is given by (3) with $F(i)=0$, $i\in S$.

The next Theorem contains the main result of this section and is one of the most important contribution of this paper.

Theorem 3 Assume that the system (1), (2) is optimizable and the pair $\left(A,\sqrt{Q}\right)$ is detectable. Then the Riccati equation (5) has a unique nonnegative definite and bounded solution. Moreover this solution is a stabilizable one.

Proof By Theorem 2 we know that there exists nonnegative and bounded solution $P(\cdot.i),i\in S$ which is the minimal solution. We first show that under the control given by (8) the close loop system is stable. Let $F:\mathrm{N}\times S\to R^{n\times m}$ be such that $F(\cdot,i)$ is bounded for all $i\in S$ and $A+FC$ is stable. Then the solution $\tilde{x}$ of (1) which corresponds to the control given by (8) satisfies

(14)

$$\tilde{x}(k+1)=\left(A(k,r(k))+F(k,r(k))\right)\tilde{x}(k)+f(k),$$

where

$$f(k)=B(k,r(k))\tilde{u}(k)-F(k,r(k))C(k,r(k))\tilde{x}(k).$$

It follows that

$$\|f(k)\|^2\le\|B(k,r(k))\|^2\|\tilde{u}(k)\|^2+ \|F(k,r(k))\|^2\|C(k,r(k))\|^2\|\tilde{x}(k)\|^2\le$$
$$\frac{m}{s}\langle R(k,r(k))\tilde{u}(k),R(k,r(k))\tilde{u}(k)\rangle+ m\left\langle\sqrt{Q(k,r(k))}\tilde{x}(k),\sqrt{Q(k,r(k))}\tilde{x}(k)\right\rangle\le$$
$$d\left(\langle R(k,r(k))\tilde{u}(k),R(k,r(k))\tilde{u}(k)\rangle+\right.$$
$$\text{(15)}\qquad \left.\left\langle\sqrt{Q(k,r(k))}\tilde{x}(k),\sqrt{Q(k,r(k))}\tilde{x}(k)\right\rangle\right),$$

where $d=\max\left(\frac{m}{s},m\right)$, $\|B(k,i)\|^2<m$, $\|F(k,i)\|^2<m$, and $sI<R(k,i)$, $(k,i)\in\mathrm{N}\times S$.

From (15) we have

$$E\left(\sum_{k=0}^{\infty}f(k)\middle|\tilde{x}(0)=x_0,r(0)=i_0\right)\le dJ(x_0,i_0,\tilde{u})= d\langle P(0,i_0)x_0,x_0\rangle\le dn\|x_0\|^2,$$

where $0\le P(k,i)\le nI$, $(k,i)\in\mathrm{N}\times S$. By this inequality and applying the Lemma 2 to (14) we have

$$E\left(\sum_{k=0}^{\infty}\left\|\tilde{x}(k)\right\|^2\middle|\tilde{x}(0)=x_0, r(0)=i_0\right)<\infty,$$

that proves the stability of the close loop system. Next we show that under the detectability condition the stabilizable solution $P(\cdot,i)$ of (5) is a maximal solution. Fix $(x_0,i_0)\in R^n\times S$ and denote by U_{stab} the subset of U consisting of such control u, that the corresponding solution x of (1) satisfies

$$\lim_{k\to\infty}E\left(\left\|x(k)\right\|^2\middle|x(0)=x_0, r(0)=i_0\right)=0.$$

Let $\tilde{P}(\cdot,i)$ be any nonnegative definite and bounded solution of (5). From Lemma 3, we have

$$J_{0,N}(x_0,i_0,u)=\left\langle\tilde{P}(0,i_0)x_0,x_0\right\rangle-$$
$$E\left(\left\langle\tilde{P}(N,i_0)x(N),x(N)\right\rangle\middle|x(0)=x_0.r(0)=i_0\right)+$$
$$E\left(\sum_{k=0}^{N}\langle R(k,r(k))(u(k)-L(k.r(k))),\right.$$
$$\left.(u(k)-L(k.r(k)))\rangle\middle|x(0)=x_0, r(0)=i_0\right),$$

where

$$L(k,i)=(R(k,i)+B'(k,i)F(k+1,i)B(k,i))^{-1}$$
$$B'(k,i)F(k+1,i)A(k,i),$$
$$F(k,i)=\sum_{j\in S}p_{ij}\tilde{P}(k,j),\ k=1,2,...,i\in S.$$

Hence for $u\in U_{stab}$ it follows that

$$J(x_0,i_0,u)=\lim_{N\to\infty}J_{0,N}(x_0,i_0,u)=$$
$$\left\langle\tilde{P}(0,i_0)x_0,x_0\right\rangle-$$
$$E\left(\sum_{k=0}^{\infty}\langle R(k,r(k))(u(k)-L(k.r(k))),\right.$$

(16) $$\left.(u(k)-L(k.r(k)))\rangle\middle|x(0)=x_0, r(0)=i_0\right).$$

So

(17) $$J(x_0,i_0,u)\geq\left\langle\tilde{P}(0,i_0)x_0,x_0\right\rangle.$$

Now consider

(18) $$\tilde{u}(k)=L(k,r(k)),$$

where

$$L(k,i)=(R(k,i)+B'(k,i)F(k+1,i)B(k,i))^{-1}$$
$$B'(k,i)F(k+1,i)A(k,i),$$
$$F(k,i)=\sum_{j\in S}p_{ij}P(k,j),\ k=1,2,...,i\in S.$$

Substituting this in (16), we obtain

(19) $$J(x_0,i_0,\tilde{u})=\langle P(0,i_0)x_0,x_0\rangle.$$

Combining (17) with (19) gives

$$\left\langle\tilde{P}(0,i_0)x_0,x_0\right\rangle\leq\langle P(0,i_0)x_0,x_0\rangle$$

that in turn implies

$$\left\langle\tilde{P}(k,i_0)x_0,x_0\right\rangle\leq\langle P(k,i_0)x_0,x_0\rangle$$

by Lemma 1. Since the solution $P(\cdot,i)$ is simultaneously maximal and minimal, so it is unique. The proof is now complete.
From this theorem follows that if the system (1), (2) is optimizable and the pair $(A,\sqrt{Q})$ is detectable then the pair (A,B) is stabilizable. Having in mind this fact and combining Theorem 2 and Theorem 3 we have the following corollary.
Corollary If the pair (A,B) is stabilizable and the pair $(A,\sqrt{Q})$ is detectable then the coupled difference Riccati equation has unique nonnegative and bounded solution, the optimal control is given by (8) and the close loop system is stable.

4. Conclusions

In this paper we have solved the discrete time-varying JLQ problem on infinite time interval. We have shown that the solution exists if and only if the coupled difference Riccati equation has nonnegative and bounded solution. If in addition the system is detectable then the solution is unique and the optimal close loop system is stable. In the time invariant case this results generalise the existing results for discrete JLQ on infinite time interval.

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