A Flat Output Approach for the Steering of the Deflection of a Piezoelectric Bender

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Abstract: Motion planning, open–loop (steering) and closed loop control synthesis for a cantilever beam with a uniformly distributed piezoelectric actuator on the whole surface, called a piezoelectric bender is considered. With a flatness based control design method an exact formula is derived for the control voltage which must be applied in order to achieve a prescribed deflection in a specified finite time. This control is given as an infinite power series the convergence of which is ensured by an appropriate choice of the desired trajectory of the "flat" output. Experimental results are presented for the application of the open loop control law and the closed loop control law.

Keywords: motion planning, flatness approach, vibration control, piezoelectric actuator, infinite dimensional system.

1 Introduction

This work is concerned with the motion planning of a cantilever beam with a uniformly distributed piezoelectric actuator on the whole surface. Such structures are called piezoelectric benders (see Fig. 1), too. The control task is to apply a voltage V in such a way that the deflection achieves a prescribed deflection in a specified finite time. The effect of the control voltage V in the lateral direction can be summarized as a torque at the free end of the beam. The resulting open loop control problem thus consists in steering a distributed parameter system via boundary control. The derivation of the open loop control law is based on the possibility to parametrize the system trajectories via a time depending function. Accordingly, this variable is called flat or basic output. Therefore, choosing the trajectory of this flat output, determines the evolution of the whole distributed parameter system. Via few Laplace transform-like symbolic computations one can calculate a formulae for the system trajectories, including the control input, in form of power series in the space coordinate. These series are parametrized by the flat output together with an infinite number of its time derivatives. Their con-

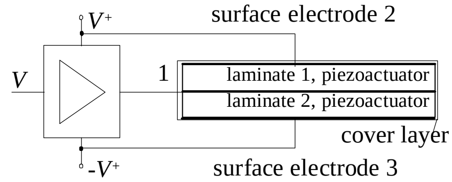

Figure 1: The piezoelectric bender.

vergence can be guaranteed by a proper choice of the flat output trajectory [3]. This trajectory will be chosen in such a way that there results a lateral deflection of the beam from rest to rest, and in a finite time.

The paper is structured as follows. In the next section, the problem is described in detail and the mathematical model of the bender is introduced. In Section 3 a flat output is determined and the exact series formulae of the system trajectories and of the open loop control law are computed. Section 4 contains a short description of the experimental setup. Experimental results are presented in section 5 for the application of the open loop control law. It will be shown that the measurement results coincide in a good way with the computed trajectories. Nevertheless, an open loop control approach has some inherent problems. For this reason we have extended our method with a feedback law. Section 6 presents the measurement results related to this improved approach.

2 Mathematical model

The structure under consideration is depicted in Fig. 1. It is composed of two piezoelectric laminates with thickness d, each comprising n thin layers, and the electrodes are covering the whole surface. The piezoelectric bender (of length l, width b, thickness $h = 2d$) is clamped at one end, free at the other one. The piezolayers are electrically in parallel in order to enable working with a reduced actuator voltage $-V^+ \leq V \leq V^+$, which is applied by a power amplifier. The voltage V leads to a bending of the structure in the (x_1, x_3)–plane. The relative displacement in the lateral x_3-direction with respect to the undeformed beam is denoted as u_3 (see Fig. 2). The modeling leads to the Euler-Bernoulli partial differential equation [6]

$$\rho A \frac{\partial^2 u_3}{\partial t^2} + EI \frac{\partial^4 u_3}{\partial x_1^4} = 0 \ ,$$

with the boundary conditions

$$u_3(0,t) = 0, \qquad \frac{\partial u_3}{\partial x_1}(0,t) = 0,$$
$$EI \frac{\partial^2 u_3}{\partial x_1^2}(l,t) = -k_V V, \qquad \frac{\partial^3 u_3}{\partial x_1^3}(l,t) = 0 \ ,$$

where E is the Young's modulus of the material, I is the moment of inertia of the beam cross section about the x_2–axes, $A = bh$ denotes the cross section area and ρ is the mass density. The parameters k_V and I are determined by

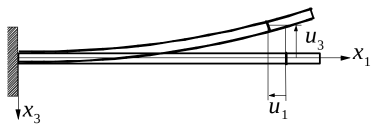

Figure 2: The deflection of the bender.

$$k_V = n d_{31} EA/2, \quad I = bh^3/12$$

with the piezoelectric stress constant d_{31}.

3 Open loop control design

For the open loop control design, we normalize the model. The normalized equations are obtained by introducing the dimensionless space variable z via $lz = x_1$ and the dimensionless time τ by

$$\tau l^2 \sqrt{\rho A/E/I} = t \ . \tag{1}$$

In order to avoid any confusion between the lateral deflection u_3 and the dimensionless control variable u introduced below we also introduce w as the normalized deflection. With

$$\frac{\partial}{\partial x_1} = \frac{1}{l}\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial t} = \sqrt{\frac{EI}{\rho A}}\frac{1}{l^2}\frac{\partial}{\partial \tau} \ ,$$

this normalization yields the dimensionless model

$$\frac{\partial^4 w}{\partial z^4} + \frac{\partial^2 w}{\partial \tau^2} = 0 \ , \tag{2}$$

with the boundary conditions

$$w(0,\tau) = 0, \qquad \frac{\partial w}{\partial z}(0,\tau) = 0,$$
$$\frac{\partial^3 w}{\partial z^3}(1,\tau) = 0, \qquad \frac{\partial^2 w}{\partial z^2}(1,\tau) = -u(\tau) \ .$$

The initial conditions can be supposed to be zero. This last condition involves the new dimensionless control variable u defined as

$$u(\tau) = \frac{k_V l^2}{EI} V(\tau) \ .$$

These equations can easily be transformed into operational ones using either Laplace transforms or Mikusiński's operational calculus [10]. (see also Remark 3.1). Both leads to the same result: By introducing s as an operator corresponding to the differentiation with respect to τ, and distinguishing the transforms of w and u by hats, we obtain

$$s^2\hat{w} + \frac{\partial^4 \hat{w}}{\partial z^4} = 0$$

$$\hat{w}(0,s) = 0, \qquad \frac{\partial \hat{w}}{\partial z}(0,s) = 0,$$

$$\frac{\partial^2 \hat{w}}{\partial z^2}(1,s) = -\hat{u}, \qquad \frac{\partial^3 \hat{w}}{\partial z^3}(1,s) = 0 \ .$$

The solution of this system satisfies ($i = \sqrt{-1}$)

$$\hat{w} = A\cosh\left(\sqrt{is}z\right) + B\sinh\left(\sqrt{is}z\right) + C\cos\left(\sqrt{is}z\right) + D\sin\left(\sqrt{is}z\right)$$

with the parameters A, B, C, and D determined by the following linear system of equations, which results by evaluating the boundary conditions:

$$A + C = 0, \qquad B + D = 0$$

$$Ais\cosh\sqrt{is} + Bis\sinh\sqrt{is} - Cis\cos\sqrt{is} - Dis\sin\sqrt{is} = -\hat{u}$$

$$A\sinh\sqrt{is} + B\cosh\sqrt{is} + C\sin\sqrt{is} - D\cos\sqrt{is} = 0$$

Equivalently, by eliminating A and B, we get

$$\begin{aligned} is\left(\cosh\sqrt{is} + \cos\sqrt{is}\right)C + is\left(\sinh\sqrt{is} + \sin\sqrt{is}\right)D &= \hat{u} \\ \left(\sinh\sqrt{is} - \sin\sqrt{is}\right)C + \left(\cosh\sqrt{is} + \cos\sqrt{is}\right)D &= 0 \end{aligned} \tag{3}$$

The equations (3) are satisfied if we introduce the flat or basic output $\hat{y}$ via

$$C = \frac{i\left(\cosh\sqrt{is} + \cos\sqrt{is}\right)}{s}\hat{y}$$

$$D = -\frac{i\left(\sinh\sqrt{is} - \sin\sqrt{is}\right)}{s}\hat{y}$$

(see also Remark 3.1). Now equation (3) yields a formula for $\hat{u}$ in terms of $\hat{y}$:

$$\hat{u} = -\left[2 + \cosh\left(\sqrt{is}(1+i)\right) + \cosh\left(\sqrt{is}(1-i)\right)\right]\hat{y} \ . \tag{4}$$

Moreover, we get a formula for the operational function $\hat{w}$ representing the deflection:

$$\begin{aligned} \hat{w} = &\left[-\cosh\left(\sqrt{is}(1-z)\right) + \cosh\left(\sqrt{is}i(1-z)\right) + \right. \\ &\tfrac{1+i}{2}\left(\cosh\left(\sqrt{is}(1+iz)\right) - \cosh\left(\sqrt{is}(i-z)\right)\right) + \\ &\tfrac{1-i}{2}\cosh\left(\sqrt{is}(1-iz)\right) - \\ &\left.\tfrac{1-i}{2}\cosh\left(\sqrt{is}(i+z)\right)\right]\frac{i\hat{y}}{s} \ . \end{aligned} \tag{5}$$

We thus have a first set of formulae expressing the system variables $\hat{w}$ and $\hat{u}$ in terms of the flat output $\hat{y}$. Expanding the formula (5) in a power series in z yields:

$$\begin{aligned} \hat{w} = &\sum_{k=0}^{\infty}\frac{1}{(4k+2)!}(-1)^k\left[2(1-z)^{4k+2} + \right. \\ &\left.(1+i)(i-z)^{4k+2} + (1-i)(i+z)^{4k+2}\right]s^{2k}\hat{y} \end{aligned} \tag{6}$$

Analogously, a formula for $\hat{u}$ is obtained either by derivation w.r.t. z and evaluation at $z = 1$ or directly by expanding equation (4)

$$\hat{u} = -2\left[1 + \sum_{k=0}^{\infty}\frac{1}{(4k)!}(2s)^{2k}\right]\hat{y} \ . \tag{7}$$

The trajectories in the (dimensionless) time domain are now obtained by interpreting the operator s as differentiation w.r.t. τ. The dimensionless deflection of the beam results as

$$\begin{aligned} w = &\sum_{k=0}^{\infty}\frac{1}{(4k+2)!}(-1)^k\left[2(1-z)^{4k+2} + \right. \\ &\Re\left\{(i+z)^{4k+2}\right\} + \Im\left\{(i+z)^{4k+2}\right\} + \\ &\left.\Re\left\{(i-z)^{4k+2}\right\} - \Im\left\{(i-z)^{4k+2}\right\}\right]y^{(2k)} \end{aligned} \tag{8}$$

and the explicit formula of the control is

$$u = -2\left[1 + \sum_{k=0}^{\infty}\frac{1}{(4k)!}2^{2k}\right]y^{(2k)} \ . \tag{9}$$

We thus have a formal parametrization of all system variables in terms of the trajectory of the flat output.

Remark 3.1 In [3] and [5] a mathematical framework for linear boundary controlled distributed parameter systems is proposed which extends Fliess' module theoretic approach to linear systems beyond the class of time delay systems. In this algebraic approach a system is intrinsically described by a module over an appropriate ring of operational functions in the sense of Mikusiński. From a computational point of view, which is the one of the present paper, these operational functions behave very much like Laplace transforms. The advantage lies in the ability to perform formal algebraic operations and to postpone questions of convergence to the choice of desired trajectories. The flat output is defined as a basis of a free module in this context. □

In order to actually get a parametrization of the solution, the trajectory of y must be chosen in such a way that the series (8) converges (and with this also series (7)). In order to meet the zero initial conditions, we need a solution which starts at rest, namely at $w(z,0) = 0$ and reaches another equilibrium after a finite time T. This is equivalent to $y^{(k)}(0) = 0, k \geq 0$ and $y^{(k)}(T) = 0, k > 0$ and thus a non-analytic function is required. A possible choice for y is:

$$\begin{cases} 0 & \text{if } t < 0 \\ \dfrac{\int_0^{\tau/T} \exp\left(-\dfrac{1}{(q(1-q))^\sigma}\right) dq}{\int_0^1 \exp\left(-\dfrac{1}{(q(1-q))^\sigma}\right) dq} & \text{for } \tau \in [0,T] \\ 1 & \text{for } \tau > T, \end{cases} \tag{10}$$

with $\sigma > 0$.

Remark 3.2 (See [3] for additional references.) A function f of the real variable t is said to be *Gevrey of class* $\gamma \in \mathbb{R}$ on $[0,T]$ if for all $k \geq 0$

$$| f^{(k)} | \leq C_T^{k+1}(k!)^\gamma, \qquad t \in [0,T]$$

where C_T is a real constant depending on T and f. The analytic functions are thus Gevrey of class 1 and the function y in (10) is Gevrey of class $\gamma = (1+\sigma)/\sigma$. This implies that for $\sigma < 2$ the series (8) converges. □

4 Experimental setup

The experimental setup, as shown in Fig. 3 consists of the piezoelectric cantilever beam, a laser measurement unit for measuring the tip deflection, a digital signal processor (DSP) unit with additional A/D- and D/A-converters, the real time workshop of MATLAB/SIMULINK, and a voltage amplifier. The piezoelectric bender is suitable

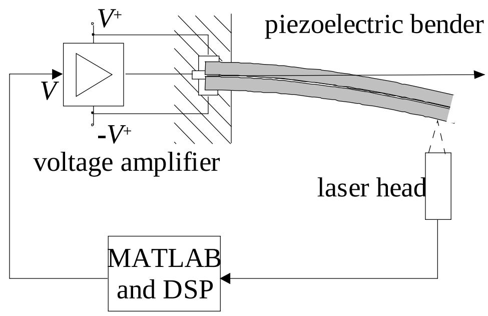

Figure 3: The experimental setup.

for low power electromechanical applications. The characteristics of the material and the geometric dimensions are as follows:

$$\begin{array}{l} \text{piezoelectric stress } d_{31} = -220\,10^{-12}\,[\text{m/V}] \\ \text{Young's modulus } E = 0.55\,10^{11}\,[\text{m}^2/\text{N}] \\ \text{mass density } \rho = 7.9\ 10^3\,[\text{kg/m}^3] \\ \text{length} \times \text{width} \times \text{height } 35 \times 12.5 \times 0.7\,[\text{mm}] \\ \text{number of layers/laminate } n = 7 \\ \text{maximal tip deflection } u_3(l) = 300\,[\mu\text{m}] \\ \text{breaking tip force } F \sim 1\,[\text{N}]\ . \end{array} \tag{11}$$

The tip deflection is measured with a laser head offering a measurement accuracy of $3\,[\mu m]$ sampled every $80\,[\mu s]$. It should be noted, that the laser measurement unit produces significant noise depending on the sample rate.

5 Experimental Results, Open Loop Control

The open loop design requires a choice of the normalized time T and the parameter σ (see function (10)). Because of the time scaling with equation (1), the time T corresponds to the rise time $t_r = 2.3 \cdot 10^{-3} T$. Further the infinite series (7) and (8) must be truncated after a appropriate number N. In this work we have included the results of experiments which were obtained with $t_r = 11.5\,[\text{ms}]\,, \sigma = 1.1, N = 5$. Fig. 4 shows a

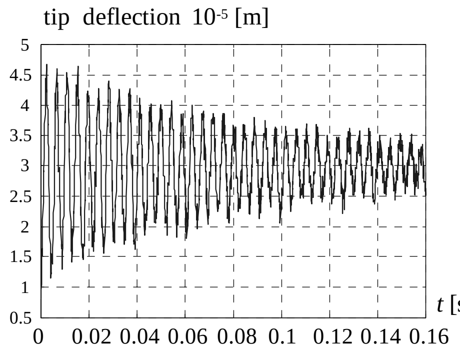

Figure 4: Step response of the piezoelectric bender.

step response (applied voltage $V = 3\,[\text{V}]$) of the piezoelectric bender. The first three of the infinite number of eigenfrequencies $\omega_{p,i}$ can be determined from the modeling with $\omega_{p,1} = 1530\,[\text{rad/s}]$, $\omega_{p,2} = 9590\,[\text{rad/s}]$, and $\omega_{p,3} = 26850\,[\text{rad/s}]$. They are verified with identification methods, too.

Remark 5.1 *For the modeling and the open loop control law design, the damping was not taken into account. This leads to a significant simplification in the design but has no essential influence on the results.*

Fig. 5 shows the results of the experiment with rise time $t_r = 11.5$ [ms] which shows a good correspondence between calculated and measured trajectories. There exist some physical effects not taken into account for the modeling which have essential influence on the experimental results. In our experience two effects play a mandatory rule: a) The hysteretic effect leads to steady state errors, and b) We have designed the open loop control law from a continuous time system point of view. But the experiment (sample time $T_a = 80\,[\mu\text{s}]$) requires a discrete time description. The related theory is not available at the moment.

6 Experimental Results, Closed Loop Control

The results of the section 5 have shown that motion planning with the flat output approach leads

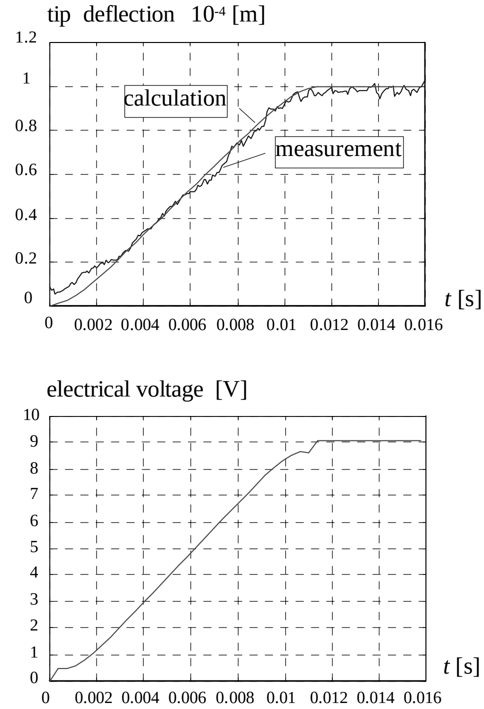

Figure 5: Open loop control, rise time $11.5\,[\text{ms}]$.

to fine results. Nevertheless any open loop design has inherent problems caused by modeling errors and unpredictable disturbances. These effects lead to steady state errors or oscillations which cannot be suppressed by an open loop control law. The application of closed loop control with a con-

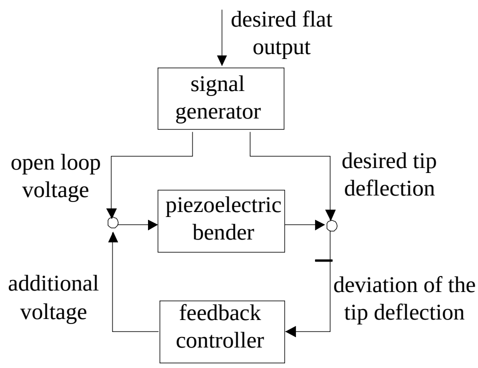

Figure 6: Closed loop control scheme.

trol scheme as shown in Fig. 6 offers a suitable way to avoid these problems. The signal generator calculates the open loop voltage V and the desired tip deflection $u_{\text{ref},3}\,(l)$. If the measured tip deflection $u_3\,(l)$ and the calculated tip deflection

$u_{\mathrm{ref},3}(l)$ differ for any reason, than the feedback controller will produce an additional voltage to suppress this deviation. Hence, the feedback part is designed from specifications like stability, disturbance rejection, and robustness. The controller design was made with a nonstandard H_2–design [7]. This method leads to a controller system of order six including an integral part. Fig. 7 shows the result of the closed loop experiment.

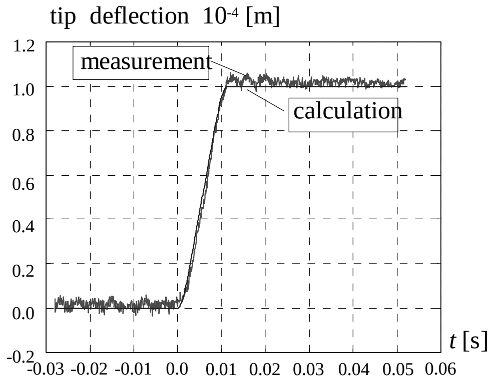

Figure 7: Closed loop control, rise time $t_r = 11.5\,[\mathrm{ms}]$.

7 Conclusion

We have presented a flat output approach for the motion planning of the deflection of a piezoelectric bender. The measurement results have shown that this method is an feasible way for the generation of trajectories.

This open loop control design method has so far been applied to two different flexible robot arm models [3], [1], (the latter reference includes experimental results), to the heat equation [9], and to tubular reactor models [4]. It is a generalization of the flatness based control of nonlinear systems [2], [9].

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