# The Efficiency of Pyramidal Algorithms for Image Representation using Morphological Filters

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*Abstract:* - Pyramidal multiresolution image representation is an ordered sequence of decreasing spatial resolution images in which each image is filtered version of its predecessor. Pyramidal representation of an image is applied in various fields of image processing: progressive image transmission used in interactive image communication over low-bandwidth channels, for image compression used in transmission and storage of visual information, in computer vision for image filtering and analysis at multiple scales. The pyramid approach is attractive due to low computational complexity and simple parallel implementation.
A morphological hierarchical representation reflects the natural decomposition of shapes in the image. Mathematical morphology operates directly with shapes without affecting the remaining image structure and therefore provides a better basis than linear techniques for constructing pyramids. The morphological pyramidal coder has several attractive features: 8 bit integer operations only, a perfect reconstruction mode, progressive transmission and progressive computation property.
Pyramidal image representation and compression techniques using morphological filters are studied through experimental analysis based on a collection of five images with a broad spectrum of structural detail. Comparisons of results for three types of morphological pyramidal representation techniques, basic and modified expansive and non-expansive, are made based on different structuring elements, different morphological antialiasing and interpolation filters. The focus in the research is put on morphological filters. Pyramidal algorithms for image representation are implemented with different antialiasing morphological filters: dilate, erode, close, open, open(close), close(open), with two types of structuring elements: square and cross-type. Morphological interpolation filters dilate and close with square type structuring element is implemented. As a result of such analysis, modified expansive pyramidal algorithm for image representation is more efficient than basic expansive algorithm. The superiority in terms of lossless compression performance and computational simplicity is obtained using non-expansive algorithm with morphological filters.

*Key-Words:* - pyramidal image representation and compression, mathematical morphology CSCC'99 Proc.pp.1351-1356

## 1 Introduction

The best known image compression techniques are based on image transforms (Fourier, Hadamard, cosine, Karhunen-Loeve), statistical modelling (predictive compression), linear filtering (subband compression), wavelet image decomposition, visual pattern image coding, pyramidal image decomposition (Gaussian and Laplacian pyramids) [1]. Many of these techniques are inherently linear and are limited when effective compression of high-resolution visual information (e.g. edge information) is of interest. Furthermore, linear techniques explore the algebraic structure of images and therefore are not efficient in cases where compression of geometric visual information is required. Therefore, it is desirable to compress an image by means of a nonlinear technique, which effectively exploits the geometric, rather than the algebraic structure in images. Mathematical morphology provides such a technique, primarily because of its potential for simplifying complex image data by removing redundancy in a non-linear fashion, while preserving important shape/size information. Morphological decomposition algorithm is an efficient scheme for multidimensional band-pass representation of an image [3].

The purpose of the paper is to study the effectiveness and efficiency of pyramidal decomposition techniques, expansive and non-expansive, with various morphological filters for lossless image compression.

## 2 Morphological Filters

The main advantages of morphological filters are their ability to preserve geometric structure of the image and their simplicity and efficiency in implementation [5]. The main idea of mathematical morphology is to examine the geometric structure of an image by probing it with small patterns, called "structuring elements" at various locations [1]. Useful shape information can be extracted from the image by varying the size and shape of the structuring elements. In the following, short review of morphological filters, which are implemented in the algorithms for pyramidal image representation, is presented.

For the image *X* and structuring element *B*, basic morphological operations dilation (*D*) and erosion (*E*) are defined as follows [6], [7]:

$$D:\ X \oplus B = \max_{b\in B}[X(x-b)+B(b)] \quad (1)$$

$$E:\ X \ominus B = \min_{b\in B}[X(x+b)-B(b)] \quad (2)$$

The simplest morphological filters are open (*O*) and close (*C*) [6], [7]:

$$O:\ X \circ B = (X \ominus B) \oplus B \quad (3)$$

$$C:\ X \bullet B = (X \oplus B) \ominus B \quad (4)$$

Alternating sequential filters, *OC* and *CO* are made through an iterative application of opening and closing, with the same structuring element [1]:

$$CO:\ (X \circ B) \bullet B \quad (5)$$

$$OC:\ (X \bullet B) \circ B \quad (6)$$

These alternating sequential filters are denoted as morphological low-pass filters.

# 3 Pyramidal Image Representation

Three types of pyramidal algorithms for image representation are reviewed: basic and modified expansive and non-expansive.

## 3.1 Basic Expansive Algorithm

Pyramidal image decomposition approach has been first proposed by Burt and Adelson [4]. They used linear filters for generation of Gaussian and Laplacian pyramid. As a result, important image features such as edges are smoothed. This results in an overall blurring of the image. In the following, a multiresolution basic expansive representation similar to Burt's Gaussian pyramid will be discussed. Multiresolution schemes decompose a signal $x$ into a sequence $\{x_0 = x, x_1, x_2, ...\}$ of lower resolution signals [3]. Clearly, loss of information represented by error signals $y_j$ occurs when moving from $x_j$ to $x_{j+1}$, Fig. 1*a*. Decomposition scheme contains antialiasing *AF* filter for removing high frequency components from the signal in order to reduce aliasing in the subsampling process. Filtered image is than subsampled by removing every other pixel in both dimension and reduced image $x_{j+1}$ is obtained from input image $x_j$. To obtain difference image $y_j$, reduced image $x_{j+1}$ should be interpolated. Through interpolation, reduced-resolution approximation $x_{j+1}$ is expanded to the image $\hat{x}_j$, which is the same size as the input image $x_j$.

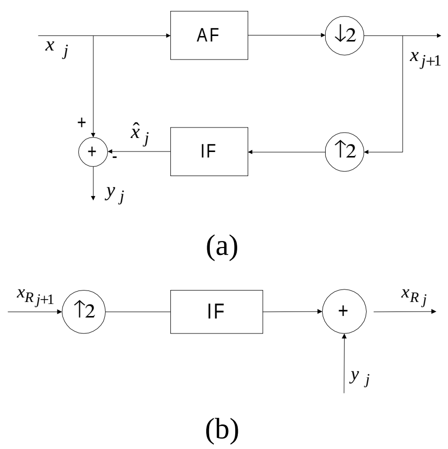



**Fig. 1** One stage in (a) generating image of approximation pyramid $x_{j+1}$ and detail pyramid image $y_j$ from input image $x_j$; (b) image reconstruction: $x_{R\,j}$ reconstructed image in pyramids level *j*; AF antialiasing and IF interpolation filter

Morphological error pyramid is a complete representation of the original image. Reversing the morphological decomposition steps, the original image can be recovered completely. One step in the image reconstruction generating reconstruction image $x_{R\,j}$ is shown on Fig. 1*b*. Reconstruction process starts from the image at the top of detail pyramid $y_N$. This image is interpolated and added to the lower level (higher resolution) detail image $y_{N-1}$ to generate image $x_{R\,N-1}$. This higher resolution image is then interpolated and the entire process is repeated multiple times until the original image $x_{R1}$ is completely reconstructed. The goal of interpolation filtering is to provide the smoothest, visually most pleasing reconstruction of the original image at each pyramid stage [5].

In order to measure the compression performance of pyramidal image decomposition algorithms, entropy of pyramidal image representation which estimates bit rate for pyramid transmission is calculated. The entropy of image provides the minimum number of bits per pixel required for coding. Since error images at different levels have different sizes, their entropy is normalised depending on the size of the image to produce the total entropy of the error pyramid. If the original image is decomposed on *K* pyramid images, total entropy of basic expansive pyramidal image representation, $H_{P_B}$, is [1]:

$$H_{P_B} = \sum_{k=1}^{K} \left( \frac{1}{2^{k-1}} \right)^2 H_k \tag{7}$$

The entropy of pyramid image at level $k$ is:

$$H_k = \sum_i (-p_i) ld(p_i) \; [bpp] \tag{8}$$

where $p_i$ denotes the frequency of grey level $i$ occurrence in the image on pyramid level $k$.

Pyramidal image representation obtained using basic expansive algorithm is overcomplete image representation with number of pixels greater than the number of pixels in the original image [3]. For the image of size *N*x*N* pixels, expansive pyramidal representation contains *NP* pixels:

$$NP = \sum_{k=1}^{K} \left( \frac{N}{2^{k-1}} \right)^2 \tag{9}$$

$$K \to \infty, NP \to \frac{4}{3} N^2 \tag{10}$$

### 3.2 Modified Expansive Algorithm

Using basic expansive pyramidal decomposition algorithm, image representation with greater number of pixels than in the original image is generated. Increased number of pixels in pyramidal representation influences negatively to the efficiency of decomposition. Applying Critical morphological sampling theorem proposed by Florencio and Schafer [11], more efficient pyramidal representation can be obtained reducing the number of pixels in image representation. According to that theorem, after morphological filtering of the image using structuring element of size 3x3, subsampling by 3 (instead of 2) in each direction is allowed, keeping the same reconstruction error. That is applied in the modified expansive pyramid representation scheme, where instead of decimation and interpolation by 2, decimation and interpolation by 3 is performed.

In this way, total number of pixels in the pyramidal image representation is reduced, and therefore, total entropy of pyramid which estimates bit rate for image transmission is lower. The entropy of the modified expansive pyramidal image representation is:

$$H_{P_B} = \sum_{k=1}^{K} \left( \frac{1}{3^{k-1}} \right)^2 H_k \tag{11}$$

For the image of size *N*x*N* pixels, pyramidal representation obtained using modified expansive algorithm contains *NP* pixels:

$$NP = \sum_{k=1}^{K} \left( \frac{N}{3^{k-1}} \right)^2 \tag{12}$$

$$K \to \infty, \quad NP \to \frac{9}{8} N^2 \tag{13}$$

### 3.3 Non-expansive Algorithm

Further improvement of image decomposition efficiency is obtained using non-expansive pyramidal algorithm proposed by Florencio and Schafer [8]. In the non-expansive pyramid scheme, antialiasing filters are completely removed. In this way, one out of every fourth sample in each difference image $y_j$ is zero and there is no need to be transmitted. Therefore, total number of pixels in non-expansive pyramid is not greater than the number of pixels in the original image. While some non-expansive pyramids, e.g. Subband/Wavelet tend to produce ringing, or blocking, e.g. DCT, non-expansive pyramid with morphological filters doesn't suffer neither blocking nor ringing effect.

The entropy of pyramidal image representation generated by non-expansive algorithm in $K$ levels is:

$$H_{P_B} = \frac{3}{4} \sum_{k=1}^{K-1} \left( \frac{1}{2^{k-1}} \right)^2 H_k + \left( \frac{1}{2^{K-1}} \right)^2 H_K \tag{14}$$

Pyramidal representation obtained using non-expansive algorithm is a complete image representation with number of pixels not greater than in the original image. For the image of size *N*x*N* non-expansive pyramidal representation contains *NP* pixels:

$$NP = \sum_{k=1}^{K-1} \frac{3}{4} \left( \frac{N}{2^{k-1}} \right)^2 + \left( \frac{N}{2^{K-1}} \right)^2 \tag{15}$$

$$K \to \infty, \quad NP \to N^2 \tag{16}$$

## 4 Experimental Results

Compression performance of pyramidal image decomposition algorithms with various morphological filters for lossless image transmission is calculated. Three types of pyramidal image decomposition algorithms described in Section 3, basic and modified expansive and non-expansive, with morphological filters are implemented. The efficiency of an image compression technique is evaluated for each of three types of pyramidal algorithms calculating the entropy of pyramidal image representation using eqns. 7, 11, 14. The smaller the entropy, the smaller bit rates for lossless image transmission, the better efficiency of decomposition is achieved.

All experiments are made using test images Lena of size 512x512 pixels, Goldhill of size 640x512 pixels, Boats of size 640x512 pixels, Cameraman of size 256x256 pixels, Baboon of size 512x512 pixels, with 256 grey levels, shown on Fig. 2. This collection of images contains images with a variety of structural details.

In the experiments, image of size 512x512 pixels is decomposed in five levels until the image of size

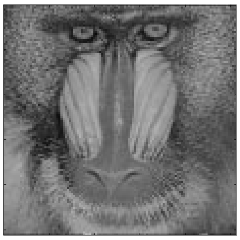
32x32 is generated, because it is experimentally proved as the most efficient decomposition.

Lena

Boats

Cameraman

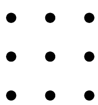
Goldhill



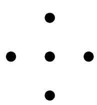
Baboon

**Fig. 2** Test images

Crucial components to pyramidal system performance using expansive pyramidal algorithms are smoothing *AF* filters used before decimation in decomposition and interpolation *IF* filters used at the transmitter and receiver. Experiments are performed varying these elements to find out their affection to the system performance. As antialiasing filters, morphological filters erode *E*, dilate *D*, open *O*, close *C*, open(close) *OC*, close(open) *CO* are implemented. Two types of structuring element, square and cross-type, Fig. 3, are used for antialiasing filtering in pyramidal decomposition schemes. According to Morphological sampling theorem [9], [10], morphological filters dilate and close with square structuring element from Fig. 3*a*, are implemented as interpolation *IF* filters in pyramidal algorithms for image representation.



(a)



(b)

**Fig. 3** Structuring elements used for antialiasing filtering (a) square (b) cross-type

Some combinations of morphological [*AF*, *IF*] filters in basic expansive pyramidal algorithm for image decomposition are used in literature: [*O*, *D*] filters used Sun and Maragos [12], [*E*, *D*] used Heijmans and Toet [10], AF filter *CO* used Toet [1]. Also, some comparison in the performances of basic expansive pyramidal algorithm for image representation and compression with several combination of [*AF*, *IF*] filters: [*CO*, *D*], [*CO*, *C*], [*O*, *D*], [*O*, *C*], [*E*, *D*] made Kong and Goutsias [1]. For completeness, the efficiency of pyramidal representation obtained using both expansive algorithms, basic and more efficient modified, with various combinations of morphological [*AF*, *IF*] filters: [*O*, *D*], [*O*, *C*], [*E*, *D*], [*D*, *C*], [*C*, *D*], [*C*, *C*], [*CO*, *D*], [*CO*, *C*], [*OC*, *D*], [*OC*, *C*] and two types of structuring elements, square and cross-type for *AF* filtering, for test images is summarised in Table 1.

Using basic expansive pyramidal algorithm for image decomposition, the most efficient decomposition is performed using [*E*, *D*] filters and square structuring element of size 3x3 for antialiasing filter for all images except for the image Boats, where the most efficient decomposition is performed using filters [*D*, *C*] also with square structuring element of size 3x3 for antialiasing filter. Using [*D*, *C*] filters the efficiency of pyramidal image decomposition is just behind the efficiency obtained using [*E*, *D*] filters for all test images except for the image Boats. The error images obtained using [*E*, *D*] and [*D*, *C*] filters require a total number of grey levels smaller than that of the other filters. This means that the [*E*, *D*] and [*D*, *C*] pyramids can be coded with fewer bits per pixel and thus a higher compression ratio may be achieved.

For all combination of [*AF*, *IF*] filters in expansive algorithms applied on test images, the efficiency obtained using modified algorithm is greater than the efficiency obtained using basic algorithm, except for the filters [*D*, *C*]. Applying modified expansive pyramidal algorithm, the efficiency of decomposition increases to 11 % comparing to the efficiency obtained using basic expansive algorithm. The combination of morphological filters [*AF*, *IF*] for the most efficient decomposition using modified expansive pyramidal algorithm are: [*E*, *D*] for images Lena and Baboon with square structuring element of size 3x3 for antialiasing filter for the image Lena and cross-type structuring element for the image Baboon, [*OC*, *D*] with cross-type structuring element for *OC* filter for the image Boats, [*CO*, *D*] for the image Cameraman with cross-type structuring element for *CO* filter, [*O*, *D*] for the image Goldhill with square structuring element for *O* filter.

Further reduction of bit rate for lossless image transmission is obtained using non-expansive pyramidal algorithm for image representation with morphological filters *D* or *C*. Entropy of non-expansive pyramidal representations for test images are presented in Table 2.

| image | struct. el. | *AF* | *O* | *O* | *E* | *D* | *C* | *C* | *CO* | *CO* | *OC* | *OC* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | *IF* | *D* | *C* | *D* | *C* | *D* | *C* | *D* | *C* | *D* | *C* |
| Lena | square | basic | 6.09 | 5.79 | **5.49** | 5.56 | 5.87 | 6.1 | 6.15 | 5.95 | 6 | 6.15 |
| | | modif. | 5.46 | 5.62 | **5.35** | 5.68 | 5.49 | 5.76 | 5.48 | 5.64 | 5.49 | 5.78 |
| | cross-type | basic | 5.99 | 5.81 | 5.58 | 5.62 | 5.88 | 5.97 | 6.06 | 5.92 | 5.97 | 6.04 |
| | | modif. | 5.42 | 5.65 | 5.41 | 5.65 | 5.43 | 5.71 | 5.39 | 5.64 | 5.4 | 5.72 |
| Boats | square | basic | 5.96 | 5.84 | 5.49 | **5.39** | 5.75 | 5.96 | 6.03 | 5.93 | 5.87 | 6.02 |
| | | modif. | 5.43 | 5.68 | 5.39 | 5.56 | 5.39 | 5.62 | 5.43 | 5.62 | 5.41 | 5.67 |
| | cross-type | basic | 5.86 | 5.86 | 5.57 | 5.5 | 5.78 | 5.89 | 5.96 | 5.86 | 5.86 | 5.98 |
| | | modif. | 5.38 | 5.70 | 5.45 | 5.55 | 5.35 | 5.62 | 5.35 | 5.64 | **5.33** | 5.66 |
| Cameraman | square | basic | 6.16 | 5.95 | **5.59** | 5.76 | 6.1 | 6.13 | 6.22 | 6.06 | 6.11 | 6.21 |
| | | modif. | 5.50 | 5.73 | 5.43 | 5.83 | 5.55 | 5.73 | 5.53 | 5.71 | 5.55 | 5.78 |
| | cross-type | basic | 6.07 | 5.95 | 5.65 | 5.75 | 6.08 | 6.01 | 6.15 | 6.02 | 6.14 | 6.12 |
| | | modif. | 5.45 | 5.73 | 5.47 | 5.75 | 5.48 | 5.69 | **5.42** | 5.68 | 5.44 | 5.73 |
| Goldhill | square | basic | 6.51 | 6.19 | **5.76** | 5.86 | 6.26 | 6.44 | 6.51 | 6.34 | 6.38 | 6.50 |
| | | modif. | **5.43** | 5.93 | 5.62 | 5.99 | 5.8 | 5.97 | 5.79 | 5.92 | 5.81 | 6.02 |
| | cross-type | basic | 6.33 | 6.20 | 5.88 | 5.95 | 6.26 | 6.33 | 6.44 | 6.33 | 6.36 | 6.43 |
| | | modif. | 5.73 | 5.95 | 5.71 | 5.95 | 5.75 | 5.95 | 5.71 | 5.92 | 5.72 | 5.98 |
| Baboon | square | basic | 7.65 | 7.46 | **6.95** | 6.98 | 7.51 | 7.69 | 7.73 | 7.59 | 7.73 | 7.75 |
| | | modif. | 6.82 | 6.96 | **6.58** | 7.06 | 6.86 | 7.01 | 6.85 | 6.95 | 6.88 | 7.05 |
| | cross-type | basic | 7.56 | 7.46 | 7.05 | 7.09 | 7.5 | 7.58 | 7.68 | 7.58 | 7.68 | 7.68 |
| | | modif. | 6.76 | 6.99 | 6.70 | 7.02 | 6.79 | 6.98 | 6.75 | 6.95 | 6.76 | 7.01 |

**Table 1**. Entropy of pyramidal image representation generated by basic and modified expansive algorithms with various morphological filters applied on test images

| | Lena | | Boats | | Cameraman | | Goldhill | | Baboon | |
|---|---|---|---|---|---|---|---|---|---|---|
| *IF* | *C* | *D* | *C* | *D* | *C* | *D* | *C* | *D* | *C* | *D* |
| *Hp* | 4.32 | 4.37 | 4.33 | 4.28 | 4.88 | 4.48 | 5.05 | 4.6 | 6.06 | 5.47 |

**Table 2**. Entropy of pyramidal image representation generated by non-expansive algorithm with morphological filters *D* and *C* applied on test images

Almost for all test images, greater efficiency of image decomposition for lossless image transmission is obtained using non-expansive pyramidal algorithm with dilate morphological filters. Only for the image Lena, better efficiency is obtained using close filters. Using non-expansive pyramidal algorithm for image representation, the efficiency of image decomposition is greater to 21.5 % than the efficiency obtained using basic expansive pyramidal algorithm.

Compression ratio *CR* as a measure of saving the number of bits for lossless image transmission using pyramid image decomposition is calculated using formula:

$$CR = \frac{H}{H_{P_B}}, \tag{17}$$

where $H$ is the image entropy and $H_{P_B}$ pyramid entropy.

The performance of three types of pyramidal algorithms with the most efficient morphological filters for each applied to test images is presented in Table 3. The most efficient algorithm for pyramidal image representation using morphological filters for lossless transmission is non-expansive morphological pyramidal algorithm. For lossless transmission of test images using non-expansive pyramidal algorithm with morphological filters for image decomposition, compression ratio 1.31-1.73 is achieved.

## 5 Conclusion

Morphological image decomposition technique is a natural choice for pyramidal image compression. Pyramidal algorithms are ideally suited for compression in progressive image transmission. Because of the inherent congruence between the morphological pyramidal decomposition scheme and human visual perception, the method appears well suited to eventual integration into artificial intelligent computer vision systems.

The compression performance of three types of pyramidal image representation algorithms, basic and modified expansive and non-expansive, with various morphological filters are compared. The best performance, e.g. the lowest bit rate for lossless image transmission is obtained using non-expansive pyramidal algorithm with morphological filters. Test images are perfectly reconstructed from their non-expansive pyramidal representations with compression ratio 1.73-1.31.

| image | Lena | Goldhill | Cameraman | Boats | Baboon |
|---|---|---|---|---|---|
| *H* | 7.455 | 7.525 | 7.038 | 7.082 | 7.139 |
| basic expansive algorithm | | | | | |
| *AF* | *E* | *E* | *E* | *D* | *E* |
| *IF* | *D* | *D* | *D* | *C* | *D* |
| *SE* | square | square | square | square | square |
| *H1* | 4.138 | 4.41 | 4.31 | 4.12 | 5.42 |
| *H2* | 3.979 | 4.07 | 3.85 | 3.76 | 4.65 |
| *H3* | 4.226 | 4.025 | 3.77 | 3.86 | 4.34 |
| *H4* | 4.542 | 3.99 | 5.92 | 4.15 | 4.2 |
| *H5* | 6.488 | 6.14 | | 6.41 | 6.54 |
| $H_{P_B}$ | 5.493 | 5.76 | 5.6 | 5.39 | 6.948 |
| *CR* | 1.357 | 1.306 | 1.25 | 1.312 | 1.027 |
| modified expansive algorithm | | | | | |
| *AF* | *E* | *O* | *CO* | *OC* | *E* |
| *IF* | *D* | *D* | *D* | *D* | *D* |
| *SE* | square | square | cross-type | cross-type | square |
| *H1* | 4.69 | 4.75 | 4.83 | 4.7 | 5.87 |
| *H2* | 5.22 | 5.32 | 4.74 | 4.966 | 5.69 |
| *H3* | 6.07 | 6.02 | 5.02 | 5.57 | 5.76 |
| *H4* | 5.73 | 5.89 | 5.5 | 5.82 | 5.47 |
| *H5* | 4.23 | 4.975 | | 4.93 | 5.46 |
| $H_{P_B}$ | 5.355 | 5.427 | 5.424 | 5.33 | 6.581 |
| *CR* | 1.392 | 1.387 | 1.298 | 1.329 | 1.085 |
| non-expansive algorithm | | | | | |
| *IF* | *C* | *D* | *D* | *D* | *D* |
| *H1* | 4.17 | 4.45 | 4.36 | 4.147 | 5.37 |
| *H2* | 4.6 | 4.89 | 4.74 | 4.52 | 5.71 |
| *H3* | 5.06 | 5.23 | 4.95 | 4.92 | 5.83 |
| *H4* | 5.5 | 5.53 | 6.9 | 5.2 | 5.96 |
| *H5* | 7.27 | 7.34 | | 6.89 | 7.09 |
| $H_{P_B}$ | 4.32 | 4.6 | 4.47 | 4.28 | 5.47 |
| *CR* | 1.73 | 1.64 | 1.57 | 1.66 | 1.31 |

**Table 3**. Efficiency of pyramidal algorithms for image decomposition with morphological filters: entropy of the test images *H*, *H1-H5* entropy of pyramid's images, pyramid entropy $H_{P_B}$, compression ratio *CR* for lossless image transmission

*References*


[1] Kong X., Goutsias J., 'A study of pyramidal techniques for image representation and compression', *Journal of Visual Communication and Image Representation*, Vol.5, No.2, 1994, pp. 190-203

[2] Toet A., 'A morphological pyramidal image decomposition', *Pattern Recognition Letters*, Vol.9, No.4, 1989, pp.255-261

[3] Goutsias J., Heijmans H., 'An axiomatic approach to multiresolution signal decomposition', *International Conference on Image Processing, ICIP'98*, October 1998, Chicago, Illinois

[4] Burt P., Adelson E., 'The Laplacian pyramid as a compact image code', *IEEE Trans. on Communication*, Vol.31, No.4, 1983, pp. 532-540

[5] Overturf L., Comer M., Delp E., 'Color Image Coding Using Morphological Pyramid Decomposition', *IEEE Trans. on Image Processing*, Vol.40, No.2, 1995, pp. 177-185

[6] Haralick R., Sternberg S., Zhuang X., 'Image analysis using mathematical morphology', *IEEE Trans. PAMI*, Vol.9, No.4, 1987, pp. 532-550

[7] Maragos, P., Schafer R., 'Morphological filters-Part I: Their Set-Theoretic Analysis and Relations to Linear Shift-Invariant Filters', *IEEE Trans. on ASSP*, Vol.35, No.8, 1987, pp. 1153-1184

[8] Florencio D., Schafer R., 'A non-expansive pyramidal morphological image coder', *Proceedings of the International Conference on Image Processing ICIP'94*, November 1994, Austin, USA, vol. II, pp. 331-335

[9] Haralick R., Zhuang X., Lin C., Lee J., 'The Digital Morphological Sampling Theorem', *IEEE Trans. on ASSP*, Vol.37, No.12, 1989, pp. 2067-2090

[10] Heijmans H., Toet A., 'Morphological sampling', *CVGIP: Image understanding*, Vol.54, No.3, 1991, pp. 384-400

[11] Florencio D., Schafer R., 'Critical morphological sampling and its applications to image coding', *International Symposium on mathematical morphology ISMM'94*, September 1994, Fontainebleu, France

[12] Sun F., Maragos P., 'Experiments on image compression using morphological pyramids', *Proceedings of the SPIE Visual Communication and Image processing Conference*, Vol. IV, November 1989, pp. 1303-1312