3 – D Systems: Transfer Function Computation [1]

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Abstract — **A theoretically attractive and computationally fast algorithm is presented for the determination of the coefficients of the determinantal polynomial and the coefficients of the adjoint polynomial matrix of a given 2–D state space model of Fornasini–Marchesini type. The algorithm uses the discrete Fourier transform (DFT) and can be easily implemented on a digital computer.** [1]

INTRODUCTION

During the past two decades there has been extensive research in multidimensional systems. This is due to the extensive range of applications, especially in signal and image processing, geophysics etc., [1]–[3]. State space techniques play a very important role in the analysis and synthesis of 3–D systems. An interesting theoretical problem is to determine the coefficients of a transfer function from its state space representation and vice versa. In going from the transfer function to state space model a various number algorithms have been proposed. In the case where a state space model is available the Leverrier, Vanderlmode matrices or the DFT algorithms can be used [4]–[7]. The DFT has been used for the evaluation of the transfer functions for multidimensional systems of the Roesser type [8].

In this paper a computer implementable algorithm is proposed, using the DFT, for the computation of the 3–D transfer function for the Fornasini–Marchesini 3–D state space models [9]. The proposed algorithm determines the coefficients of the determinantal polynomial and the coefficients of the adjoint polynomial matrix, using the DFT algorithm. The computational speed of the method could be improved using the Fast Fourier Transform.

Three–dimensional (3–D) state space models of the Fornasini – Marchesini type have the following structures [9]:

First F–M model

$$\begin{aligned} x(i_1+1,i_2+1,i_3+1) &= \mathbf{A}_1 x(i_1+1,i_2,i_3) \\ &+ \mathbf{A}_2 x(i_1,i_2+1,i_3) \\ &+ \mathbf{A}_3 x(i_1,i_2,i_3+1) \\ &+ \mathbf{b}u(i_1,i_2,i_3) \qquad (1) \\ y(i_1,i_2,i_3) &= \mathbf{c}' x(i_1,i_2,i_3) \end{aligned}$$

Second F–M model

$$\begin{aligned} x(i_1+1,i_2+1,i_3+1) &= \mathbf{A}_1 x(i_1+1,i_2,i_3) \\ &+ \mathbf{A}_2 x(i_1,i_2+1,i_3) \\ &+ \mathbf{A}_3 x(i_1,i_2,i_3+3) \\ &+ \mathbf{b}_1 u(i_1+1,i_2,i_3) \qquad (2) \\ &+ \mathbf{b}_2 u(i_1,i_2+1,i_3) \\ &+ \mathbf{b}_3 u(i_1,i_2,i_3+1) \\ y(i_1,i_2,i_3) &= \mathbf{c}' x(i_1,i_2,i_3) \end{aligned}$$

where, $x(i_1,i_2,i_3) \in \mathcal{R}^n$, $u(i_1,i_2,i_3) \in \mathcal{R}^m$, $y(i_1,i_2,i_3) \in \mathcal{R}^p, \mathbf{A}_k, \mathbf{B}_k$ for $k = 1,2,3$ and $\mathbf{c}$, are real matrices of appropriate dimensions.

Applying the 3–D z_i, $(\forall\, i = 1,2,3)$ transform to the systems (1) and (2), with zero initial conditions, their transfer functions respectively become:

$$T_1(z_1,z_2,z_3) = \mathbf{c}' \left[\mathbf{I}z_1z_2z_3 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2 - \mathbf{A}_3 z_3\right]^{-1} \mathbf{b} \qquad (3)$$

and

$$\begin{aligned} T_2(z_1,z_2,z_3) &= \mathbf{c}' \left[\mathbf{I}z_1z_2z_3 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2 - \mathbf{A}_3 z_3\right] \\ &\times \ (b_1 z_1 + b_2 z_2 + b_3 z_3) \qquad (4) \end{aligned}$$

In the following section an interpolative approach is developed for determining the transfer function $T(z_1,z_2,z_3)$, given the matrices $\mathbf{A}_k, \mathbf{B}_k$ for $k = 1,2,3$ and $\mathbf{c}$, using the DFT. For the sake of completness a brief description of the DFT follows.

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3–D DFT

Consider the finite sequences $X(k_1, k_2, k_3)$ and $\tilde{X}(r_1, r_2, r_3)$, $\kappa_i, \lambda_i = 0, \cdots, M_i, \quad \forall\ i = 1, 2, 3$. In order for the sequences $X(k_1, k_2, k_3)$ and $\tilde{X}(r_1, r_2), r_3)$, to constitute a 3–D DFT pair the following relations should hold [10]:

$$\begin{aligned} \tilde{X}(r_1, r_2, r_3) &= \sum_{k_1=0}^{M_1} \sum_{k_2=0}^{M_2} \sum_{k_3=0}^{M_3} X(k_1, k_2, k_3) \\ &\times \quad W_1^{-k_1 r_1} W_2^{-k_2 r_2} W_3^{-k_3 r_3} \end{aligned} \qquad (5)$$

$$\begin{aligned} X(k_1, k_2, k_3) &= \frac{1}{R} \sum_{r_1=0}^{M_1} \sum_{r_2=0}^{M_2} \sum_{r_3=0}^{M_3} \tilde{X}(r_1, r_2, r_3) \\ &\times \quad W_1^{k_1 r_1} W_2^{k_2 r_2} W_3^{k_3 r_3} \end{aligned} \qquad (6)$$

where,

$$R = \prod_{i=1}^{3} (M_i + 1) \qquad (7)$$

$$W_i = e^{(2\pi j)/(M_i+1)}, \quad \forall\ i = 1, 2, 3 \qquad (8)$$

X, $\tilde{X}$ are discrete argument matrix valued functions, with dimensions

$$\begin{aligned} X &= [x_{i_1, i_2, i_3}], \quad i = 1, \ldots, p \\ \tilde{X} &= [\tilde{x}_{i_1, i_2, i_3}], \quad j = 1, \ldots, m \end{aligned}$$

FIRST F–M: ALGORITHM

The transfer function of the first Fornasini – Marchesini 3–D state space model (1) has the structure,

$$T(z_1, z_2, z_3) = \frac{N(z_1, z_2, z_3)}{d(z_1, z_2, z_3)} \qquad (9)$$

where,

$$\begin{aligned} \mathbf{N}(z_1, z_2, z_3) &= \mathbf{c}' \, \mathtt{adj}\, [\mathbf{I} z_1, z_2, z_3 \\ &\quad -\mathbf{A}_1 z_1 - \mathbf{A}_2 z_2 - \mathbf{A}_3 z_3] \mathbf{b} \quad (10) \\ d(z_1, z_2, z_3) &= \mathtt{det}\, [\mathbf{I} z_1, z_2, z_3 \\ &\quad -\mathbf{A}_1 z_1 - \mathbf{A}_2 z_2 - \mathbf{A}_3 z_3] \quad (11) \end{aligned}$$

Taking into consideration that

$$\begin{aligned} n &= \mathtt{deg}_{z_1}[N(z_1, z_2, z_3)] = \mathtt{deg}_{z_2}[N(z_1, z_2, z_3)] \\ &= \mathtt{deg}_{z_3}[N(z_1, z_2, z_3)] \end{aligned}$$

and

$$\begin{aligned} n &= \mathtt{deg}_{z_1}[d(z_1, z_2, z_3)] = \mathtt{deg}_{z_2}[d(z_1, z_2, z_3)] \\ &= \mathtt{deg}_{z_3}[d(z_1, z_2, z_3)] \end{aligned}$$

where, $\mathtt{deg}_{z_1}[\ ]$, $\mathtt{deg}_{z_2}[\ ]$, $\mathtt{deg}_{z_3}[\ ]$ denote the degrees with respect to z_1, z_2, *and* z_3, respectively. Equations (10) and (11) can be written in polynomial form as follows:

$$\mathbf{N}(z_1, z_2, z_3) = \sum_{\kappa=0}^{n} \sum_{\lambda=0}^{n} \sum_{\mu=0}^{n} \mathbf{P}_{\kappa,\lambda,\mu} z_1^{\kappa} z_2^{\lambda} z_3^{\mu} \qquad (12)$$

$$d(z_1, z_2, z_3) = \sum_{\kappa=0}^{n} \sum_{\lambda=0}^{n} \sum_{\mu=0}^{n} q_{\kappa,\lambda,\mu} z_1^{\kappa} z_2^{\lambda} z_3^{\mu} \qquad (13)$$

where, $P_{\kappa,\lambda,\mu}$ are matrices with dimensions $(p \times m)$, while $q_{\kappa,\lambda,\mu}$ are scalars.

The numerator polynomial matrix $\mathbf{N}(z_1, z_2, z_3)$ and the denominator polynomial $d(z_1, z_2, z_3)$ can be numerically computed at $R = \prod_{i=1}^{3}(M_i + 1)$, points, equally spaced on the unit $3 - D$ space. The R points are chosen as

$$W_i = W = e^{(2\pi j)/(M_i+1)}, \quad \forall\ i = 1, 2, 3 \qquad (14)$$

The values of the transfer function (9) at the R points are its corresponding 3–D DFT coefficients.

Moreover, we define

$$v_1(r) = v_2(r) = v_3(r) = W^r, \quad \forall\ \ r = 0, \ldots, n \qquad (15)$$

Denominator polynomial

To evaluate the denominator coefficients $(q_{\kappa,\lambda,\mu})$, define,

$$\begin{aligned} a_{i_1, i_2, i_3} &= \mathtt{det}\, [\mathbf{I} v_1(i_1) v_2(i_2) v_3(i_3) \qquad (16) \\ &\quad -\mathbf{A}_1 v_1(i_1) - \mathbf{A}_2 v_2(i_2) - \mathbf{A}_3 v_3(i_3)] \end{aligned}$$

Therefore using equations (13), (16) yield

$$a_{i_1, i_2, i_3} = d[v_1(i_1), v_2(i_2), v_3(i_3)] \qquad (17)$$

Provided that at least one of $a_{i_1, i_2, i_3} \neq 0$.

Equations (13), (15) and (17) yield

$$a_{i_1, i_2, i_3} = \sum_{\kappa=0}^{n} \sum_{\lambda=0}^{n} \sum_{\mu=0}^{n} q_{\kappa,\lambda,\mu} W^{-(i_1 \kappa + i_2 \lambda + i_3 \mu)} \qquad (18)$$

Using equations (18) and (5) it is obvious that, $[a_{i_1, i_2, i_3}]$, $[q_{\kappa,\lambda,\mu}]$ form a DFT pair. Therefore the coefficients $q_{\kappa,\lambda,\mu}$ can be computed, using the inverse 3–D DFT, as follows:

$$q_{\kappa,\lambda,\mu} = \frac{1}{R} \sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \sum_{i_3=0}^{n} a_{i_1, i_2, i_3} W^{(i_1 \kappa + i_2 \lambda + i_3 \mu)} \qquad (19)$$

where, $\kappa, \lambda, \mu = 0, \ldots, n$

Numerator Polynomial

To evaluate the numerator matrix polynomial $\mathbf{P}_{\kappa,\lambda,\mu}$, define

$$\begin{aligned}\mathbf{F}_{i_1,i_2,i_3} &= \mathbf{c}' \,\mathtt{adj}\, [\mathbf{I}v_1(i_1)v_2(i_2)v_3(i_3) \\ &\quad -\mathbf{A}_1 v_1(i_1) - \mathbf{A}_2 v_2(i_2) - \mathbf{A}_3 v_3(i_3)]\mathbf{b}\end{aligned} \tag{20}$$

Provided that at least one of $\mathbf{F}_{i_1,i_2,i_3} \neq 0$.

Therefore using equations (10),(20), yields

$$\mathbf{F}_{i_1,i_2,i_3} = \mathbf{N}[v_1(i_1), v_2(i_2), v_3(i_3)] \tag{21}$$

Equations (12), (15) and (21) yield

$$\mathbf{F}_{i_1,i_2,i_3} = \sum_{k=0}^{n}\sum_{r=0}^{n}\sum_{r=0}^{n} \mathbf{P}_{\kappa,\lambda,\mu} W^{-(i_1\kappa+i_2\lambda+i_3\mu)} \tag{22}$$

Using equations (19) and (5) it is obvious that, $[F_{i_1,i_2,i_3}]$, $[P_{\kappa,\lambda,\mu}]$ form a DFT pair. Therefore the coefficients $P_{\kappa,\lambda,\mu}$ can be computed, using the inverse 3–D DFT, as follows:

$$\mathbf{P}_{\kappa,\lambda,\mu} = \frac{1}{R}\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\sum_{i_3=0}^{n} \mathbf{F}_{i_1,i_2,i_3} W^{(i_1\kappa+i_2\lambda+i_3\mu)} \tag{23}$$

where, $\kappa, \lambda, \mu = 0, \ldots, n$.

Finally, the transfer function sought is,

$$T(z_1, z_2, z_3) = \frac{N(z_1, z_2, z_3)}{d(z_1, z_2, z_3)} \tag{24}$$

where,

$$\mathbf{N}(z_1, z_2, z_3) = \sum_{\kappa=0}^{n}\sum_{\lambda=0}^{n}\sum_{\mu=0}^{n} \mathbf{P}_{\kappa,\lambda,\mu} z_1^{\kappa} z_2^{\lambda} z_3^{\mu} \tag{25}$$

$$d(z_1, z_2, z_3) = \sum_{\kappa=0}^{n}\sum_{\lambda=0}^{n}\sum_{\mu=0}^{n} q_{\kappa,\lambda,\mu} z_1^{\kappa} z_2^{\lambda} z_3^{\mu} \tag{26}$$

SECOND F-M: ALGORITHM

The transfer function of the second Fornasini–Marchesini 3–D state space model has the structure,

$$T(z_1, z_2, z_3) = \frac{N(z_1, z_2, z_3)}{d(z_1, z_2, z_3)} \tag{27}$$

where,

$$\begin{aligned}\mathbf{N}(z_1, z_2, z_3) &= \mathbf{c}' \,\mathtt{adj}\, [\mathbf{I}z_1, z_2, z_3 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2 \\ &\quad -\mathbf{A}_3 z_3] \\ &\quad \times \; (\mathbf{b}_1 z_1 + \mathbf{b}_2 z_2 + \mathbf{b}_3 z_3)\end{aligned} \tag{28}$$

$$\begin{aligned}d(z_1, z_2, z_3) &= \mathtt{det}\, [\mathbf{I}z_1, z_2, z_3 - \mathbf{A}_1 z_1 - \mathbf{A}_2 z_2 \\ &\quad -\mathbf{A}_3 z_3]\end{aligned} \tag{29}$$

Taking into consideration that

$$\begin{aligned} n &= \mathtt{deg}_{z_1}[N(z_1, z_2, z_3)] = \mathtt{deg}_{z_2}[N(z_1, z_2, z_3)] \\ &= \mathtt{deg}_{z_3}[N(z_1, z_2, z_3)]\end{aligned}$$

and

$$\begin{aligned} n &= \mathtt{deg}_{z_1}[d(z_1, z_2, z_3)] = \mathtt{deg}_{z_2}[d(z_1, z_2, z_3)] \\ &= \mathtt{deg}_{z_3}[d(z_1, z_2, z_3)]\end{aligned}$$

where, $\mathtt{deg}_{z_1}[\ ]$, $\mathtt{deg}_{z_2}[\ ]$, $\mathtt{deg}_{z_3}[\ ]$ denote the degrees with respect to $z_1, z_2,$ *and* z_3, respectively. Equations (28) and (29) can be written in polynomial form as follows:

$$\mathbf{N}(z_1, z_2, z_3) = \sum_{\kappa=0}^{n}\sum_{\lambda=0}^{n}\sum_{\mu=0}^{n} \mathbf{P}_{\kappa,\lambda,\mu} z_1^{\kappa} z_2^{\lambda} z_3^{\mu} \tag{30}$$

$$d(z_1, z_2, z_3) = \sum_{\kappa=0}^{n}\sum_{\lambda=0}^{n}\sum_{\mu=0}^{n} q_{\kappa,\lambda,\mu} z_1^{\kappa} z_2^{\lambda} z_3^{\mu} \tag{31}$$

where, $P_{\kappa,\lambda,\mu}$ are matrices with dimensions $(p \times m)$, while $q_{\kappa,\lambda,\mu}$ are scalars.

The numerator polynomial matrix $\mathbf{N}(z_1, z_2, z_3)$ and the denominator polynomial $d(z_1, z_2, z_3)$ can be numerically computed at $R = \prod_{i=0}^{3}(M_i + 1)$ points, equally spaced on the unit 3–D space. The R points are chosen as

$$W_i = W = e^{(2\pi j)/(M_i+1)}, \;\; \forall\, i = 1, 2, 3 \tag{32}$$

The values of the transfer function (27) at the R points are its corresponding 3–D DFT coefficients.

Moreover, we define

$$v_1(r) = v_2(r) = v_3(r) = W^r, \;\; \forall \;\, r = 0, \ldots, n \tag{33}$$

Denominator Polynomial

To evaluate the denominator coefficients $(q_{\kappa,\lambda,\mu})$, define,

$$\begin{aligned} a_{i_1,i_2,i_3} &= \mathtt{det}\, [\mathbf{I}v_1(i_1)v_2(i_2)v_3(i_3) - \mathbf{A}_1 v_1(i_1) \\ &\quad -\mathbf{A}_2 v_2(i_2) - \mathbf{A}_3 v_3(i_3)]\end{aligned} \tag{34}$$

Therefore using equations (5) and (34) yield

$$a_{i_1,i_2,i_3} = d[v_1(i_1), v_2(i_2), v_3(i_3)] \tag{35}$$

Provided that at least one of $a_{i_1,i_2,i_3} \neq 0$.

Equations (31), (33) and (35) yield

$$a_{i_1,i_2,i_3} = \sum_{\kappa=0}^{n}\sum_{\lambda=0}^{n}\sum_{\mu=0}^{n} q_{\kappa,\lambda,\mu} W^{-(i_1\kappa+i_2\lambda+i_3\mu)} \tag{36}$$

Using equations (36) and (5) it is obvious that, $[a_{i_1,i_2,i_3}]$, $[q_{\kappa,\lambda,\mu}]$ form a DFT pair. Therefore the coefficients $q_{\kappa,\lambda,\mu}$ can be computed, using the inverse 3–D DFT, as follows:

$$q_{\kappa,\lambda,\mu} = \frac{1}{R} \sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \sum_{i_3=0}^{n} a_{i_1,i_2,i_3} W^{(i_1\kappa+i_2\lambda+i_3\mu)} \quad (37)$$

where, $\kappa, \lambda, \mu = 0, \ldots, n$

Numerator Polynomial

To evaluate the numerator matrix polynomial $\mathbf{P}_{\kappa,\lambda,\mu}$, define

$$\begin{aligned} \mathbf{F}_{i_1,i_2,i_3} &= \mathbf{c}' \, \texttt{adj} \, [\mathbf{I} v_1(i_1) v_2(i_2) v_3(i_3) - \mathbf{A}_1 v_1(i_1) \\ &\quad - \mathbf{A}_2 v_2(i_2) - \mathbf{A}_3 v_3(i_3)] \\ &\times \; (\mathbf{b}_1 v_1(i_1) + \mathbf{b}_2 v_2(i_2) + \mathbf{b}_3 v_3(i_3)) \end{aligned}$$

Provided that at least one of $\mathbf{F}_{i_1,i_2,i_3} \neq 0$.

Therefore using equations (5) and (38), yields

$$\mathbf{F}_{i_1,i_2,i_3} = \mathbf{N}[v_1(i_1), v_2(i_2), v_3(i_3)] \quad (38)$$

Equations (30), (33) and (39) yield

$$\mathbf{F}_{i_1,i_2,i_3} = \sum_{k=0}^{n} \sum_{r=0}^{n} \sum_{r=0}^{n} \mathbf{P}_{\kappa,\lambda,\mu} W^{-(i_1\kappa+i_2\lambda+i_3\mu)} \quad (39)$$

Using equations (40) and (5) it is obvious that, $[a_{i_1,i_2,i_3}]$, $[q_{\kappa,\lambda,\mu}]$ form a DFT pair. Therefore the coefficients $q_{\kappa,\lambda,\mu}$ can be computed, using the inverse 3–D DFT, as follows:

$$\mathbf{P}_{\kappa,\lambda,\mu} = \frac{1}{R} \sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \sum_{i_3=0}^{n} \mathbf{F}_{i_1,i_2,i_3} W^{(i_1\kappa+i_2\lambda+i_3\mu)} \quad (40)$$

where, $\kappa, \lambda, \mu = 0, \ldots, n$.

Finally, the transfer function sought is,

$$T(z_1, z_2, z_3) = \frac{N(z_1, z_2, z_3)}{d(z_1, z_2, z_3)} \quad (41)$$

where,

$$\begin{aligned} \mathbf{N}(z_1, z_2, z_3) &= \sum_{\kappa=0}^{n} \sum_{\lambda=0}^{n} \sum_{\mu=0}^{n} \mathbf{P}_{\kappa,\lambda,\mu} z_1^{\kappa} z_2^{\lambda} z_3^{\mu} \\ &\times \; (\mathbf{b}_1 z_1 + \mathbf{b}_2 z_2 + \mathbf{b}_3 z_3) \quad (42) \\ d(z_1, z_2, z_3) &= \sum_{\kappa=0}^{n} \sum_{\lambda=0}^{n} \sum_{\mu=0}^{n} q_{\kappa,\lambda,\mu} z_1^{\kappa} z_2^{\lambda} z_3^{\mu} \quad (43) \end{aligned}$$

CONCLUSION

In this paper the well known DFT algorithm has been used for determining the coefficients of a 3–D transfer function from its 3–D state space of Fornasini–Marchesini type. The algorithms are theoretically attractive and can be easily implemented. The results presented here may be extended tothe multidimensional case.

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