The Optimum Radius of the Robust Stability of Schur Polynomials

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Abstract : The problem of the Robust Stability of Schur Polynomials is investigated. Recently, a new approach based on Rouche's theorem of the classical complex analysis has been adopted for the solution of this problem. In this paper, an improvement of this solution is presented. This is the optimum solution of the Robust Stability problem of Schur Polynomials and is obtained by solving a minimization problem. and it is better than all the other methods in the robust stability literature.

I. INTRODUCTION

In linear, time invariant (LTI), discrete time systems we must be interested not only in whether the polynomial (system) is stable but also whether the polynomial (system) will remain stable in the presence of system parameter deviations. This is the problem of the *robust stability*. Over the last two decades, this problem has attracted much attention by scientists and engineers working in the area of analysis, synthesis, simulation, reduction, modelling of physical and artificial systems.

One of the basic robustness problems was to determine the robust stability of a given family of characteristic polynomials. The most notable result is Kharitonov's theorem and its various genaralizations [1]÷[7]. According to Kharitonov's theorem, a whole class of polynomials is stable if and only if *four* special, well-defined polynomials are stable. In other words, the problem of robust stability of the polynomial is reduced to the problem of the stability of a whole family of polynomials which is further examined by considering the stability of four "corner" polynomials of the family. So, the problem is solved by examing only four polynomials with respect the stabilty. Many recent elegant results of the literature concerning the robustness of LTI, (discrete time or continuous time) systems can be found in the book of Barmish [13].

In [8], [9], [12] and [13], a different robust stability problem is stated and discussed. This problem could be called as the inverse Kharitonov's problem since here the starting point is only the particular given polynomial instead of a family of many polynomials.

A simple formulation of this problem in the case of LTI, discrete time systems is as follows, [12].

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Suppose that the polynomial $f(z)=\sum_{i=0}^{N}\alpha_i\cdot z^i$ with $\alpha_i\in\boldsymbol{R},\ z\in\boldsymbol{C}$ is Schur stable that is $f(z)\neq 0$ for $|z|\leq 1$. *Find* the supremum R (R>0) such that all the polynomials $g(z)=\sum_{i=0}^{N}\beta_i\cdot z^i$ with $\beta_i\in(\alpha_i-R,\alpha_i+R),\quad i=0,1,\ldots,n$ to be Schur stable.

In this paper, we find a better result than those in the literature, [1]÷[9], [12], [13]. To this end we use Rouche's theorem, [10].

Rouche's theorem: If the two functions $f(z)$ and $g(z)$ are analytic in a region G and if

$$|g(z)-f(z)|<|f(z)| \tag{1}$$

holds at every point z of the boundary ∂G of G, then the two functions $f(z)$ and $g(z)$ have the same numbers of zeros in G.

We will consider as $f(z)$ the original given polynomial and $\Delta f(z)=g(z)-f(z)$ the perturbation under which the new polynomial $g(z)$ is obtained from $f(z)$. Here, the main result is derived in Section II where we achieve a better (bigger) radius of stability than in [12] as well as a more convenient result than the relevant results of [13]. Furthermore, in Section III, three illustrative examples support the obtained result.

II. ANALYSIS

Consider the Schur stable polynomial $f(z)$

$$f(z)=\sum_{i=0}^{N}\alpha_i\cdot z^i \qquad \text{with } \alpha_i\in\boldsymbol{R},\ z\in\boldsymbol{C} \tag{2}$$

the robust stability of which is examined under variation of the (real) coefficients a_i. The polynomial

$$\Delta f(z)=\sum_{i=0}^{N}\Delta a_i\cdot z^i \qquad \text{with } \Delta a_i\in\mathbf{R},\ z\in\boldsymbol{C} \tag{3}$$

represents the "perturbated" $f(z)$. The problem in question is to determine necessary conditions under which $g(z)$ (where $g(z)=f(z)+\Delta f(z)$) is also Schur stable. For this reason, (1) is rewritten as $|\Delta f(z)|<|f(z)|$ or

$$|\Delta f(z)|^2<|f(z)|^2 \tag{4}$$

At this point, one could notice that this method can be also applied for LTI, continuous time systems too since, it is known one can assume that ∂G is the unit circle since this is always possible via an appropriate conformal mapping (Riemann's theorem, [10]). In the case of the transition of the continuous time to the discrete time system this is achieved via a bilinear transformation.

In (4), we substitute $z = e^{j\theta}$, $0 \le \theta \le 2\pi$. Then after some algebraic manipulation one finds

$$f(z)\,f(z^{-1}) = \left[\sum_{i=o}^{n} a_i^2 \quad \middle| 2\sum_{i=0}^{n-1} a_i a_{i+1} \quad \middle| \cdots \middle| 2\sum_{i=0}^{n-k} a_i a_{i+k} \middle| \cdots \middle| 2\sum_{i=0}^{n-n} a_i a_{i+n} \quad \right] \cdot \boldsymbol{c} \tag{5.1}$$

as well as

$$\Delta f(z)\,\Delta f(z^{-1}) = \left[\sum_{i=o}^{n} \Delta a_i^{\,2} \quad \middle| 2\sum_{i=0}^{n-1} \Delta a_i \Delta a_{i+1} \quad \middle| \cdots \middle| 2\sum_{i=0}^{n-k} \Delta a_i \Delta a_{i+k} \middle| \cdots \middle| 2\,\Delta a_0 \Delta a_n \quad \right] \cdot \boldsymbol{c} \tag{5.2}$$

where

$$\boldsymbol{c} = \begin{bmatrix} 1 \\ cos\,\theta \\ \vdots \\ cos\,k\theta \\ \vdots \\ cos\,n\theta \end{bmatrix} \tag{5.3}$$

So, if one uses the usual trigonometrical relation

$$cos(j\theta) = \frac{1}{2}\left\{ (2\,cos\,\theta)^{j} - \frac{j}{1}(2\,cos\,\theta)^{j-2} + \frac{j}{2}\binom{j-3}{1}(2\,cos\,\theta)^{j-4} - \frac{j}{3}\binom{j-4}{2}(2\,cos\,\theta)^{j-6} + \ldots \right\}$$

$$j = 1,\ldots,N_1$$

then $\boldsymbol{c}$ can be written as a multiplication of one rectangular matrix $\boldsymbol{T}$ with the vector $\boldsymbol{x}$ as follows.

$$\boldsymbol{c} = \boldsymbol{T} \cdot \boldsymbol{x} \tag{6}$$

where $\boldsymbol{x}^t = \begin{bmatrix} 1 & x & x^2 & \cdots & x^{N_1} \end{bmatrix}$ with $x = cos\,\theta$. Matrix $\boldsymbol{T}$ depends on the "length" of $\boldsymbol{c}$. As an example, for $N_1 = 5$, (6) is written as

$$\begin{bmatrix} 1 \\ cos\,\theta \\ cos\,2\theta \\ cos\,3\theta \\ cos\,4\theta \\ cos\,5\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 4 & 0 & 0 \\ 1 & 0 & -8 & 0 & 8 & 0 \\ 0 & 5 & 0 & -20 & 0 & 16 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ cos\,\theta \\ cos^2\,\theta \\ cos^3\,\theta \\ cos^4\,\theta \\ cos^n\,\theta \end{bmatrix} \tag{7}$$

Substituting (5.1) and (5.2) into (4) and using (6) one finally obtains the relation

$$F(\Delta a_1,\dots,\Delta a_n,x) = |f(z)|^2 - |\Delta f(z)|^2 = \left[\sum_{i=o}^{n} a_i^2 \quad \middle| 2\sum_{i=0}^{n-1} a_i a_{i+1} \quad \middle| \cdots \middle| 2\sum_{i=0}^{n-k} a_i a_{i+k} \middle| \cdots \middle| 2\sum_{i=0}^{n-n} a_i a_{i+n} \quad \right] \cdot \boldsymbol{T} \cdot \boldsymbol{x}$$

$$-\left[\sum_{i=o}^{n} \Delta a_i^{\,2} \quad \middle| 2\sum_{i=0}^{n-1} \Delta a_i \Delta a_{i+1} \quad \middle| \cdots \middle| 2\sum_{i=0}^{n-k} \Delta a_i \Delta a_{i+k} \middle| \cdots \middle| 2\,\Delta a_0 \Delta a_n \quad \right] \cdot \boldsymbol{T} \cdot \boldsymbol{x}$$

$$= \left[\sum_{i=o}^{n} a_i^2 \quad -\sum_{i=o}^{n} \Delta a_i^{\,2} \quad \middle| 2\sum_{i=0}^{n-1} a_i a_{i+1} - 2\sum_{i=0}^{n-1} \Delta a_i \Delta a_{i+1} \quad \middle| \cdots \middle| 2\sum_{i=0}^{n-k} a_i a_{i+k} - 2\sum_{i=0}^{n-k} \Delta a_i \Delta a_{i+k} \middle| \cdots \middle| 2\sum_{i=0}^{n-n} a_i a_{i+n} \; - 2\,\Delta a_0 \Delta a_n \right] \cdot \mathbf{T} \cdot \mathbf{x} > 0 \tag{8}$$

where one verifies that the difference $F(\Delta a_1,\dots,\Delta a_n,x)$ is a polynomial in $n+1$ variables $\Delta a_1,\dots,\Delta a_n,x$. Considering now that

$$|\Delta a_i| < R \qquad\qquad i = 0,1,\dots,n \tag{9}$$

the problem in question is to find the optimum R for which (8) holds and consequently the polynomial in (2) remains stable under the considered coefficients' deviation. Now, (8) yields

$$\left[\sum_{i=o}^{n} a_i^2 \quad \middle| 2\sum_{i=0}^{n-1} a_i a_{i+1} \quad \middle| \cdots \middle| 2\sum_{i=0}^{n-k} a_i a_{i+k} \middle| \cdots \middle| 2\sum_{i=0}^{n-n} a_i a_{i+n} \quad \right] \boldsymbol{T} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix} >$$

$$R^2 \begin{bmatrix} n+1 & 2n & 2(n-1) & \dots & 2 \end{bmatrix} \boldsymbol{T} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix} \tag{10}$$

For R=0 , (10) holds for any x with $-1 \le x \le 1$ since $f(z)$ is Schur stable. The *supremum* R^2 for which (10) holds for every x $(-1 \le x \le 1)$ is the minimum value of the fraction:

$$a(x) = \frac{\begin{bmatrix} \sum_{i=o}^{n} a_i^2 & \left| 2\sum_{i=0}^{n-1} a_i a_{i+1} \right| & \cdots \left| 2\sum_{i=0}^{n-k} a_i a_{i+k} \right| & \cdots \left| 2\sum_{i=0}^{n-n} a_i a_{i+n} \right| \end{bmatrix} \boldsymbol{T} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix}}{\begin{bmatrix} n+1 & 2n & 2(n-1) & \ldots & 2 \end{bmatrix} \boldsymbol{T} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix}} \tag{11}$$

Therefore the supremum of R^2, and consequently the supremum of R, is found by the following one-variable minimization problem

$$\begin{aligned} &minimize \qquad a(x) \\ &over\ x \\ &(-1 \le x \le 1) \end{aligned} \tag{12}$$

If we denote the solution of the above minimization problem as a^*, then the supremum of R which will be denoted as R^* will be equal to $\sqrt{a^*}$.

In (11), one should notice that the numerator of $a(x)$ is positive for $-1 \le x \le 1$, since it corresponds to $f(z)$ which is a Schur stable polynomial. In additional, the denominator is positive or 0 since it corresponds to the polynomial $1+z+\ldots z^n$ which has n roots on the unit circle. Therefore $a(x)$ is positive or $+\infty$ for $-1 \le x \le 1$. Therefore the minimum of $a(x)$ over the closed interval [-1,1] is meaningful.

Remark: Generalizing the above formulated problem one can give various weights to the coefficients variations in a way to have

$$|\Delta a_i| < \lambda_i R \qquad\qquad i = 0,1,\ldots,n \tag{13}$$

instead of (9). In this case, after the usual algebraic manipulation, we have to solve the same problem

$$
\begin{array}{ll}
\text{minimize} & a(x) \\
\text{over } x & \\
(-1 \le x \le 1) &
\end{array}
\tag{14}
$$

with

$$
a(x) = \frac{\begin{bmatrix} \sum_{i=o}^{n} a_i^2 & \left|2\sum_{i=0}^{n-1} a_i a_{i+1}\right| \cdots \left|2\sum_{i=0}^{n-k} a_i a_{i+k}\right| \cdots \left|2\sum_{i=0}^{n-n} a_i a_{i+n}\right| \end{bmatrix} \boldsymbol{T} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix}}{\begin{bmatrix} \sum_{i=o}^{n} \lambda_i^2 & \left|2\sum_{i=0}^{n-1} \lambda_i \lambda_{i+1}\right| \cdots \left|2\sum_{i=0}^{n-k} \lambda_i \lambda_{i+k}\right| \cdots \left|2\sum_{i=0}^{n-n} \lambda_i \lambda_{i+n}\right| \end{bmatrix} \boldsymbol{T} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix}}
$$

One should note that these results are better, more elegant and simpler with respect the computation than those in [13]. They are also an apparent improvement with respect to the results of [12].

III. NUMERICAL EXAMPLES

To compare the results with the results of [12], the same examples are examined. Also $\lambda_i = 1 \ \ \forall\, i = 0,1,\ldots,n$.

(i) Consider the polynomial $f(z) = (z-2)(z-3) = z^2 - 5z + 6$. We have to find the "radius" R such that all the polynomials $g(z)$ with coefficients in the intervals (1-*R*,1+*R*), (-5-*R*,-5+*R*), (6-*R*,6+*R*) to be Schur stable. Therefore, one has to solve the following one-variable minimization problem

$$
\begin{array}{ll}
\text{minimize} & a(x) \\
\text{over } x & \\
(-1 \le x \le 1) &
\end{array}
$$

We have that $\boldsymbol{T}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & 0 & 2\end{bmatrix}$ and $\boldsymbol{x}=\begin{bmatrix}1\\ x\\ x^2\end{bmatrix}$. So finally one finds $a(x)=\dfrac{50-70x+24x^2}{1+4x+4x^2}$. The above one-dimensional minimization problem is solved numerically. So, one finds its minimum at x=1 and the minimum value is $a^*=4/9$ which yields R*=2/3=0.666.

Therefore the coefficients of the polynomial $f(z)=z^2-5z+6$ i.e. 1,-5,6 can be varied in the intervals (1-0.666,1+0.666), (-5-0.666,-5+0.666), (6-0.666,6+0.666) and the resultant polynomial to be Schur stable. So, the supremum R* coincides with that in [12].

(ii) $f(z)=(z-2)(z-3)(z+1+j)(z+1-j)=z^4-3z^3-2z^2+2z+12$. Here, one finds $a(x)=\dfrac{238+250x-224x^2-272x^3+96x^4}{1-4x-4x^2+16x^3+16x^4}$. The minimization of this fraction yields $a^*=88/25$ and R*=1.8762. This is a "better" supremum R than that of [12] where one had R*=1.513

(iii) $f(z)=(z-3j)(z+3j)(z+3)=z^3+3z^2+9z+27$. Here, we obtain that

$a(x)=\dfrac{80+48x+45x^2+27x^3}{x^2+x^3}$. Similarly one finds R*=10 which is a better evaluation of R* than that of [12] where R*= 5.

IV. CONCLUSION

In a recent publication, [12], a result concerning the problem of Schur polynomial stability has been obtained. In this paper, an improvement of this result i.e. of the radius of the robust stability is achieved. Since, in all steps of the procedure we actually have necessary and sufficient conditions, it is obvious that the present evaluation of the radius R is the *optimum* that can be achieved. As the one-variable minimization problem of an easily determined rational function is quite easy to be solved (even by plotting it over the interval [-1,1]), our approach is better, more elegant and simpler with respect the computation than those in [13]. Also, it is better than that of [12].

The extension of this result in the case of characteristic polynomial of continuous systems is possible via the well known Möbious (bilinear) transformation. In the robust control theory, the above result may be a useful contribution.

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