ON THE ABSENCE OF THE STATE VARIABLES DISCONTINUITIES OF LTI SYSTEMS IN THE PRESENCE OF SINGULAR INPUTS

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Abstract

This paper investigates the problem of the absence of the state variables discontinuities of a linear time-invariant (LTI) system in the presence of singular inputs. Necessary and sufficient conditions are given for this absence of state variables discontinuities. These conditions may be useful for many practical problems. An attempt is also made to find a general solution to the above problem. Numerical examples illustrate the theoretical results.

1. Introduction

Finite linear combinations of Dirac unit impulses and their generalized derivatives are often called singular functions. They are not functions in the ordinary sense, but they can be manipulated (e.g., shifted, differentiated, integrated) in a way similar to that in which other time signals are manipulated. The operation of adding two singular signals is also well defined, whereas in general the product is not.

In systems theory, a system is defined as an algorithm (or equivalently, a signal transducer) that transforms an input function (signal) into an output function (signal).

In this paper, we are dealing with input signals that may be represented in the form of a sum of a regular function (i.e., ordinary functions that have right and left limits at each point) and a singular function. Our objective is to investigate the following problem: when do singular inputs of an LTI system not cause discontinuities in state variables? This problem was stated for the first time by Novak [1], in whose work find necessary and sufficients conditions involving eigenvalues and eigenvectors of the companion matrix. These conditions are of practical interest, as we will see in Section 2. In Section 3 we attempt to find the general solution of the above problem.

With these ends in view, we consider the linear time-invariant (LTI) system given by the following vector equation:

$$\dot{x} = A\cdot x + b(t)\cdot u(t) + \beta_0\cdot\delta(t) + \beta_1\cdot\delta^{(1)}(t) + \ldots + \beta_s\cdot\delta^{(s)}(t) \quad (1)$$

with:

$$x(0^-) = x_0 \quad (2)$$

where x is a state n-vector, x_0 is a constant n-vector, A is a constant n x n matrix, $b(t)$ is a continuous n-vector function, $\beta_o, \beta_1, \ldots, \beta_s$ are constant n-vectors, and $u(t)$, $\delta(t)$ and $\delta^{(k)}(t)$ are the Heaviside unit step function, the Dirac unit impulse (delta) function, and the $k-$th derivative of $\delta(t)$ respectively. Henceforward, we call the matrix A the companion matrix of the LTI system described by (1).

It has been shown [1] that:

$$x(0^+) = x(0^-) + A^s\beta_s + \ldots + A\beta_1 + I\beta_0 \quad (3)$$

Therefore, the necessary and sufficient condition for the equality $x(0^+) = x(0^-)$ to hold in the nontrivial case (i.e., when at least one of the constants β_0, ..., β_s is different from the zero vector) is:

$$A^s\beta_s + \ldots + A\beta_1 + I\beta_0 = 0 \quad (4)$$

where I is the n x n Identity matrix. In the next section, sufficient and necessary conditions for equation (4) to hold are presented.

2. Main Results

We start by re-writing (4) as follows:

$$b_s D^s + \ldots + b_1 D + b_0 I = 0 \quad (5)$$

where $b_s = \beta_s^T$, ..., $b_0 = \beta_0^T$ and $D = A^T$ where the superscript T denotes the transpose matrix. We then prove the following theorem.

Theorem 1. D is a solution of (5) if and only if all the simple (usual) eigenvectors x of D (corresponding to a simple or a multiple eigenvalue λ of (D), fulfil the relation:

$$(b_s\lambda^s + \ldots + b_1\lambda + b_0)\cdot x = 0 \quad (6a)$$

as well as the generalized eigenvectors $x_1, \ldots, x_p$ of D, corresponding to a multiple eigenvalue λ (of D), fulfil the relation:

$$\sum_{j=0}^{k} \frac{1}{j!} \frac{d^j (b_s \lambda_1^s + \ldots + b_1 \lambda_1 + b_0)}{d\lambda_1^j} |_{\lambda_1=\lambda} \cdot x_{k-j} = 0$$

with $k = 1, \ldots, p$. In (7), x_0 is defined to be the simple eigenvector of D from which the generalized eigenvectors $x_1, \ldots, x_p$ have been produced via the relation $A \cdot x_k = \lambda \cdot x_k + x_{k-1} (k = 1, \ldots p)$.

Proof: Before proceeding to the proof of this theorem, note that the eigenvalues of a matrix D are found by the equation $\det(D - \lambda I) = 0$. The simple roots of the polynomial $\det(D - \lambda I)$ are called simple eigenvalues of D, whereas the multiple roots of the same polynomial are called multiple eigenvalues of D. For a simple eigenvalue λ, one finds a simple eigenvector of D via the relation: $D \cdot x = \lambda \cdot x$, whereas for a multiple eigenvalue λ (with m multiplicity) one may find m_1 simple eigenvectors via the relation: $D \cdot x = \lambda \cdot x$, (where $m_1 \leq m$) and $p = m - m_1$ generalized eigenvectors via the recursive relation: $D \cdot x_k = \lambda \cdot x_k + x_{k-1}$ where x_0 is one of the m_1 simple eigenvectors.

Now we are ready to prove the theorem.

Necessary: Suppose that (5) holds. Then, if x is a simple eigenvector of D corresponding to the eigenvalue λ, we have $D^i x = \lambda^i x$ $(i = 0, \ldots, s)$. Multiplying both sides of (5) by x, one obtains (6). A generalized eigenvector of order k corresponding to an m-tuple eigenvalue is defined by the relations:

$$Dx_0 = \lambda x_0$$

$$\vdots$$

$$Dx_k = \lambda x_k + x_{k-1}$$

where $k = 1, \ldots, p$. From the above equations, one can find:

$$D^i x_k = \sum_{j=0}^{k} \frac{1}{j!} \frac{d^j \lambda_1^i}{d\lambda_1^j} |_{\lambda_1=\lambda} \cdot x_{k-j} \quad (8)$$

For example: $D^5 x_2 = \lambda^5 x_2 + 5\lambda^4 x_1 + 10\lambda^3 x_0$. Multiplying both sides of (5) by x_k one obtains:

$$b_s \cdot \sum_{j=0}^{k} \frac{1}{j!} \frac{d^j \lambda_1^s}{d\lambda_1^j} \bigg|_{\lambda_1=\lambda} \cdot x_{k-j} + \ldots + b_1 \cdot \sum_{j=0}^{k} \frac{1}{j!} \frac{d^j \lambda_1^1}{d\lambda_1^j} |_{\lambda_1=\lambda} \cdot x_{k-j} +$$

$$+ b_0 \cdot \sum_{j=0}^{k} \frac{1}{j!} \frac{d^j \lambda_1^0}{d\lambda_1^j} |_{\lambda_1=\lambda} \cdot x_{k-j} = 0 \quad (9)$$

from which equation (7) follows directly.

Sufficient: Suppose that (6) and (7) hold. Then:

$$(b_s D^s + \ldots + b_1 D + b_0 I) \cdot x = 0 \quad (10)$$

where x is a simple or a generalized eigenvector of D. Because we have n independent eigenvectors x (simple or generalized), it follows that:

$$b_s D^s + \ldots + b_1 D + b_0 I = 0 \quad (11)$$

QED

For some practical problems, theorem 1 may be very useful. Consider, for example, an LTI system of the form (1) for which (4) does not hold. For such a system, the particular singular input cause discontinuities in the state variables.

The problem in question is to find another linear system with the same poles but, in general, different zeros for which, the same singular input cannot cause state variables discontinuities, that is, the companion matrix A_1 of the new system satisfies (4). Equivalently, we seek for a matrix D_1 $(D_1 = A_1^T)$ with the same eigenvalues as D, but satisfying (5). To this end, we should use theorem 1. The problem is reduced to one of finding new eigenvectors (simple or generalized)—corresponding to the same eigenvalues—that satisfy (6) and (7). One can write:

$$D = S \cdot L \cdot S^{-1} \quad (12)$$

where L is the diagonal or Jordan form of D $(D = A^T)$ and S is the matrix of the eigenvectors of D. Denoting by S_1 the matrix of the n new (desirable) eigenvectors that satisfy (6) and (7), the matrix D_1 results as follows:

$$D_1 = S_1 \cdot L \cdot S_1^{-1} \quad (13)$$

Combining (12) and (13) yields:

$$A_1 = H^{-1} \cdot A \cdot H \quad (14)$$

where: $H = (S^T)^{-1} \cdot S_1^T$. Therefore, the new system that has the same poles as the original but different zeros, and satisfies the condition (4) (i.e., it does not have state variables discontinuities if the input is the particular singular function) described by:

$$\dot{x} = A_1 x + b(t) u(t) + \beta_0 \delta(t) + \beta_1 \delta^{(1)}(t) + \ldots + \beta_s \delta^{(s)}(t) \quad (15)$$

In control theory, it is well known that we can change the zeros of an LTI system, without changing the poles, via an appropriate controller.

Example 1. Consider an LTI system with:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \beta_0 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \beta_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\beta_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \beta_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

One easily verifies that A does not satisfy condition (4). For this system, the particular singular input causes discontinuities in the state variables. The problem of finding another linear system with the same poles but different zeros, for which the condition (4) is satisfied, is dealt with as follows:

First, we obtain that the matrix $D = A^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ has a double eigenvalue: $\lambda = 1$ with one simple eigenvector $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and one generalized eigenvector $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. It is desired to find other x_0 and x_1 eigenvectors (say, $\hat{x}_0$ and $\hat{x}_1$) for which (6) and (7) hold, that is:

$$(b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0) \cdot \hat{x}_0 = 0$$

and

$$(b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0) \cdot \hat{x}_1 + (3b_3\lambda^2 + 2b_2\lambda + b_1) \cdot \hat{x}_0 = 0$$

where $b_3 = \beta_3^T, \ldots, b_0 = \beta_0^T$. One solution of the above two equations is: $\hat{x}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and: $\hat{x}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Hence:

$$D_1 = S_1 \cdot L \cdot S_1^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

and:

$$A_1 = D_1^T = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

It is now easy to verify that the condition (4) is satisfied with A_1 instead of A. Therefore, the new system that satisfies the condition (4) is:

$$\dot{x} = A_1 \cdot x + b(t) \cdot u(t) + \beta_0 \cdot \delta(t) + \beta_1 \cdot \delta^{(1)}(t) + \beta_2 \cdot \delta^{(2)}(t) + \beta_3 \cdot \delta^{(3)}(t) \quad (16)$$

For this system the singular input cannot cause discontinuities in the state variables.

3. The General Solution of Equation(4)

In this section, the following interesting general problem is examined: *given* a generalized input-function, *find* the systems for which this input does not cause discontinuities in their state variables. The algebraic formulation of this problem is: *Find* the general solution of (4) with respect to the matrix A *when* $\beta_0, \beta_1, \ldots, \beta_s$ *are given.*

To this end, equation (4) is written as:

$$X^s \beta_s + \ldots + X\beta_1 + I\beta_o = 0 \quad (17)$$

To solve (17), we set:

$$X = T^{-1} \cdot Q \cdot T \quad (18)$$

where T is an unknown nonsingular matrix of order n and Q is an arbitrary $n \times n$ matrix. (In a very special case, Q can be considered as the diagonal or the Jordan form of X). Introducing (18) into (17), we obtain the following equation:

$$T^{-1} \cdot (Q^s \cdot T \cdot \beta_s + \ldots + Q \cdot T \cdot \beta_1 + T \cdot \beta_0) = 0 \quad (19)$$

As $\det(T^{-1}) \neq 0$, we have:

$$Q^s \cdot T \cdot \beta_s + \ldots + Q \cdot T \cdot \beta_1 + T \cdot \beta_0 = 0 \quad (20)$$

Equation (20) is a generalized Lyapunov equation with respect to T. Furthermore, it is easy to verify that (20) is equivalent to:

$$(Q^s \otimes \beta_s^T + \ldots + Q \otimes \beta_1^T + I \otimes \beta_0^T) \cdot t = 0 \quad (21)$$

where: $t = [t_{11} \ \ldots \ t_{1n} \ t_{21} \ \ldots \ t_{2n} \ \ldots \ t_{n1} \ \ldots \ t_{nn}]^T$, t_{ij} is the ij–element of T, and $\otimes$ denotes the usual Kronecker matrix product.

Defining:

$$G = Q^s \otimes \beta_s^T + \ldots + Q \otimes \beta_1^T + I \otimes \beta_0^T \quad (22)$$

we write equation (21) as:

$$G \cdot t = 0 \quad (23)$$

where G is an $n \times n^2$ matrix and t is an $n^2 \times 1$ vector. T is fully determined by finding t.

Therefore, the (general) solution of (17) is: $X = T^{-1} \cdot Q \cdot T$ for every T resulting from (23) for which $\det T \neq 0$.

Furthermore, let us define the $n \times n$ matrices, $G_1, \ldots, G_n$ by splitting G as:

$$G = \left[G_1 \vdots \ldots \vdots G_n \right] \quad (24)$$

and introduce the notation:

$$t = \left[t_1^T \vdots \ldots \vdots t_n^T \right]^T \quad (25)$$

where: $t_1 = \begin{bmatrix} t_{11} \\ \vdots \\ t_{1n} \end{bmatrix}, \ldots, t_n = \begin{bmatrix} t_{n1} \\ \vdots \\ t_{nn} \end{bmatrix}$. Using this notation, (23) is written as:

$$G_1 t_1 + \ldots + G_n t_n = 0 \quad (26)$$

The following two theorems provide necessary and sufficient conditions for $\det T = 0$, for every T that is a solution of (20).

Theorem 2. For the matrix G given in (24), suppose that $\det G_i \neq 0$, for at least one i, $i = 1, \ldots, n$. If the $n \times n$ matrices $G_1, \ldots, G_n$ are dependent on each other, that is,

$$\alpha_1 G_i = \alpha_i G_1 \ , \ \ldots \ , \ \alpha_{i-1} G_i = \alpha_i G_{i-1} \ ,$$

$$\alpha_{i+1} G_i = \alpha_i G_{i+1} \ , \ \ldots \ , \ \alpha_n G_i = \alpha_i G_n$$

then every solution T of (20) is a singular matrix.

Proof. Suppose that (27) holds. Then from (26), one obtains that: $\alpha_1 G_i t_1 + \ldots + \alpha_n G_i t_n = 0$ or $G_i(\alpha_1 t_1 + \ldots + \alpha_n t_n) = 0$. Because $\det G_i \neq 0$, one has: $\alpha_1 t_1 + \ldots + \alpha_n t_n = 0$. Therefore $t_1, \ldots, t_n$ are linearly dependent, and T is singular.

QED

Note that if the condition $\det G_i \neq 0$ for at least one i, $i = 1, \ldots, n$, then by virtue of (27), it follows that $\det G_i \neq 0$ for all i, $i = 1, \ldots, n$.

The inverse of the above theorem could be formulated as follows.

Theorem 3. For the matrix G given in (24), suppose that $\det G_i \neq 0$, for at least one i, $i = 1, \ldots, n$. If every solution T of (20) is a singular matrix, that is $\alpha_1 t_1 + \ldots + \alpha_n t_n = 0$ with $\alpha_i \neq 0$ then equation (27) holds, that is:

$$\alpha_1 G_i = \alpha_i G_1 \ , \ \ldots \ , \ \alpha_{i-1} G_i = \alpha_i G_{i-1} \ ,$$

$$\alpha_{i+1} G_i = \alpha_i G_{i+1} \ , \ \ldots \ , \ \alpha_n G_i = \alpha_i G_n$$

Proof, It is assumed that:

$$\alpha_1 t_1 + \ldots + \alpha_n t_n = 0, \ \ \alpha_i \neq 0 \tag{28}$$

for one i $(i = 1, \ldots, n)$. From (26) it follows that:

$$t_i = -G_i^{-1} G_1 t_1 - \ldots - G_i^{-1} G_{i-1} t_{i-1} - G_i^{-1} G_{i+1} t_{i+1} - G_i^{-1} G_n t_n \tag{29}$$

Combining (28) and (29) yields:

$$(\alpha_1 I - \alpha_i G_i^{-1} G_1) t_1 + \ldots + (\alpha_{i-1} I - \alpha_i G_i^{-1} G_{i-1}) t_{i-1} +$$

$$+ (\alpha_{i+1} I - \alpha_i G_i^{-1} G_{i+1}) t_{i+1} + \ldots + (\alpha_n I - \alpha_i G_i^{-1} G_n) t_n = 0$$

where I is the identity matrix. As $\alpha_i \neq 0$, one can sets $t_1, \ldots, t_{i-1}, t_{i+1}, t_n$, in (28), as arbitrary vectors (linearly independent). Therefore, (30) renders:

$$\alpha_1 I - \alpha_i G_i^{-1} G_1 = 0, \ \ \ldots, \ \ \alpha_{i-1} I - \alpha_i G_i^{-1} G_1 = 0,$$

$$\alpha_{i+1} I - \alpha_i G_i^{-1} G_1 = 0, \ \ \ldots, \ \ \alpha_n I - \alpha_i G_i^{-1} G_n = 0$$

Thus, we obtain (27) as required. QED

Example 2. Consider again the system of Example 1 and assume that A is not given. We seek an A satisfying the equation:

$$A^3 \beta_3 + A^2 \beta_2 + A \beta_1 + I \beta_0 = 0$$

or, using the previous notation:

$$X^3 \beta_3 + X^2 \beta_2 + X \beta_1 + I \beta_0 = 0$$

Introducing (18) and following the previous analysis, one finds:

$$[Q^3 \otimes b_3 + Q^2 \otimes b_2 + Q \otimes b_1 + I \otimes b_0] \cdot t = 0$$

Now, setting:

$$Q = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

one finds:

$$\left(\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \otimes [-1 \quad 0] + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \otimes [1 \quad 2] + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \otimes [0 \quad 0] + \right.$$

$$\left. + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes [5 \quad 3] \right) \cdot t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $\begin{bmatrix} 5 & 5 & -1 & 4 \\ 0 & 0 & 5 & 5 \end{bmatrix} \cdot \begin{bmatrix} t_{11} \\ t_{12} \\ t_{21} \\ t_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence: $G_1 = \begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix}$, $G_2 = \begin{bmatrix} -1 & 4 \\ 5 & 5 \end{bmatrix}$.

Clearly, the 2×2 matrices G_1 and G_2 are not dependent on each other (theorem 2). So, a solution for which $\det T \neq 0$ holds is the following:

$$t_{11} = 0, \ t_{12} = 1, \ t_{21} = 1, \ t_{22} = -1$$

Thus: $T = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ and $T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and:

$$A = T^{-1} \cdot Q \cdot T \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

Remark. One may notice that $A = A_1$ (of example 1). This result is expected, as Q is the Jordan form of A_1 of example 1 and $T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is the matrix of the eigenvectors of A_1. Starting with other matrices Q, one can find further solutions.

4. Conclusion

In this paper, the matrix equation derived in [1] was considered. Necessary and sufficient conditions for the validity of this matrix equation were presented and its general solution was provided. The results of the paper, which are supported by two examples, are useful in circuits and systems analysis, as well as in control systems design, where the problem of checking the absence of discontinuities in state variables despite the presence of singular inputs plays a dominant role. An interesting problem for future investigation is to design an adaptive feedback control that guarantees that the system state variables remain continuous even if the inputs are singular functions of time. Some other recent relevant studies can be found in [2–7].

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