A STUDY OF THE EFFECT OF AN INPUT ON THE STATE VARIABLES OF DISCRETE-TIME LTI SYSTEMS

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ABSTRACT. The paper examines the following problem: "When does the effect of an input on the state variables of a discrete-time LTI systems cease to exist at the moment when the input ceases to exist too?". An analogous problem has originally been studied for the case of continuous-time systems by Novak [7]. Three examples are included to support the theoretical results.

1. INTRODUCTION

In this paper, the problem regarding the absece of the effect of an input on the state variables of a discrete-time LTI system at the moment when this input ceases to exist is considered and examined.

The analogous problem in continuous time refers to the absence of state variables' discontinuities in the presence of singular inputs [7].

The present paper is organized as follows" In Section 2, the problem is stated. In Section 3, the results are presented. The discrete-time necessary and sufficient condition derived is similar to the continuous-time one but not the same. In Section 4, three examples are given that illustrate how the results are applied.

2. STATEMENT OF THE PROBLEM

Consider a discrete linear time-invariant (LTI) system describred by the following state vector equation

$$x(n+1) = A \cdot x(n) + b(n) \cdot u(n) + \beta_0 \delta(n) + \beta_1 \delta(n-1) + \ldots + \beta_s \delta(n-s) \qquad (1)$$

with $x(0)=x_0$

where x is a state p-vector, x_0 is a constant p-vector, A is a constant pxp matrix, $b(n)$ is a p-vector, $\beta_0, \beta_1, \ldots, \beta_s$ are constant p-vectors, $u(n)$ is an arbitrary input and $\tilde{u}(n) \equiv \beta_0 \delta(n) + \beta_1 \delta(n-1) + \ldots + \beta_s \delta(n-s)$ is a finite time input, i.e. $\tilde{u}(n) = 0 \quad for \quad n > s$

Here

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}$$

Eqn. (1) may be rewritten as

$$x(n+1) + Ax(n) + b(n)u(n) + \tilde{u}(n) \quad (3)$$

Our aim here is to find a necessary and sufficient condition under which $x(n)$ does not depent on when $\tilde{u}(n) n>s$.

3. THE PRESENT RESULTS

The response of system (1) or (3) is given by the sum of the responses of the following three systems:

a)

$$x_1(n+1) + A \cdot x_1(n) \quad (4)$$

with

$$x_1(0) = x(0) \quad (5)$$

b)

$$x_2(n+1) = A \cdot x_2(n) + b(n) \cdot u(n) \quad (6)$$

with

$$x_2(0) = 0 \quad (7)$$

c)

$$x_3(n+1) = A \cdot x_3(n) + \beta_0 \delta(n) + \beta_1 \delta(n-1) + \ldots + \beta_s \delta(n-s) \quad (8)$$

with

$$x_3(0) = 0$$

There is

$$x(n) = x_1(n) + x_2(n) + x_3(n) \quad (9)$$

The condition "$x(n)$ does not depend on $\tilde{u}(n) = \beta_0 \delta(n) + \beta_1 \delta(n-1) + \ldots + \beta_s \delta(n-s)$, when $n>s$" is equivalent with the condition "$x_3(n)=0$, when $n>s$".

The response $x_3(n)$ is given by:

$$x_3(n) = \sum_{i=0}^{n-1} A^{n-i} \tilde{u}(i) \quad (10)$$

Since $\tilde{u}(i) = 0$ for $i>s$, eqn. (10) is equivalent to:

$$x_3(n) = \sum_{i=0}^{s} A^{n-1} \tilde{u}(i) \quad (11)$$

or

$$x_3(n) = A^{n-s}\sum_{i=o}^{s} A^{s-i}\tilde{u}(i) \qquad (12)$$

Now, since it can not be[1]: $A^{n-s} = 0 \forall n > s$,eqn. (12) yields

$$\sum_{i=0}^{s} A^{s-i}\tilde{u}(i) = 0 \qquad (13)$$

or

$$A^s\beta_0 + A^{s-1}\beta_1 + \ldots + I \cdot \beta_s = 0 \qquad (14)$$

Eqn. (14) is the necessary and sufficient condition in order for $x(n)$ not to depend on $\tilde{u}_n(n)$, when n>s.
Consider the special case

$$\beta_k = \alpha_k\beta \qquad k = 1,2,\ldots,s \qquad (15a)$$

where α_k are constant scalars and β is a constant vector. Then, eqn. (1) reduces to

$$x(n+1) = Ax(n) + b(n)u(n) + \left[\sum_{k=0}^{s} \alpha_k\delta(n-k)\right] \cdot \beta \qquad (15b)$$

and eqn.(14) to

$$p_s(A) \cdot \beta \equiv 0 \qquad (16)$$

where $p_s(\lambda) = \alpha_0\lambda^s + \alpha_1\lambda^{s-1} + \ldots + \alpha_s$ is a polynomial with scalar coefficients.
Let p_c be the characteristic polynomial of the matrix A, i.e. let $p_c(\lambda) = det(\lambda I - A)$
Then, [5]

$$p_c(A) \equiv 0 \qquad (17)$$

and so: eqn. (16) is trivially satisfied.

The minimal order polynomial that satisfies eqn. (17) is called the minimal annulating polynomial of the matrix A. This polynomial divides the characteristic polynomials [2]. The minimal annulating polynomial is determined as the quotient of the characteristic polynomial and the greatest common divisor of the elements of the matrix $adj(\lambda I-A)$. Now

[1] Here it is assumed that we have not the case: $\det|A - \lambda I| = \lambda^p$ for which $A^p=0$ (deadbeat control).

consider the minimal polynomial p^* that satisfies the identity eqn. (16). This polynomial is equal to the quotient of the characteristic polynomial of the matrix A and the greatest common divisor of the elements of the vector $[adj(\lambda I-A)]\beta$. According to [6], a polynomial p, satisfies eqn. (16) if and only if p^* divides p_S. On the basis of the above, the following proposition is formulated.

Proposition

For the system (15b) with the associated polynomials p_S and p^*, $x(n)$ does not depend on $\bar{u}(n)$ when $n>s$, if and only if p^* divides p_S.

Owing to the fact that p^* always divides p_C, one can state the following:

Corollary I: If, for the system (15b) with the associated polynomials p_C and p_S, p_C divides p_S, $x(n)$ does not depend on $\bar{u}(n)$ when n>s.

4. EXAMPLES

Example 1

Consider the second-order system

$$x_1(n+1) = 2x_1(n) - x_2(n) + b_1(n)u(n) - \delta(n) + \delta(n-1) + 5\delta(n-1)$$

$$x_2(n+1) = x_1(n) + b_2(n)u(n) + 2\delta(n-1) + 3\delta(n-3)$$

with the initial conditions

$$x_1(0) = x_{10} \qquad x_2(0) = x_{20}$$

Thus, the system matrices are:

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\beta_3 = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \quad \beta_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \beta_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \beta_0 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Here,

$$A^3\beta_0 + A^2\beta_1 + A\beta_2 + I\beta_3 =$$

$$= \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 1 & -0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} =$$

$$= \begin{bmatrix} -4 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, according to eqn. (14) $x_1(n)$ and $x_2(n)$ does not depend on

$$\tilde{u}(n) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}\delta(n) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\delta(n-1) + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\delta(n-2) + \begin{bmatrix} 5 \\ 3 \end{bmatrix}\delta(n-3),$$

when $n \geq s = 4$.

Example 2

Consider an LTI system, as in eqn. (15a) and (15b), for which

$$A = \begin{bmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{bmatrix} \quad b(n) = \begin{bmatrix} 1 \\ n \\ -1 \end{bmatrix} \quad \beta = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

and

$$p_2(\lambda) = \lambda^2 - \lambda - 2$$

The state equations of this system are

$$x_1(n+1) = 3x_1(n) - 3x_2(n) + 2x_3(n) + u(n) - 2\delta(n) + 2\delta(n-1) + 4\delta(n-2)$$

$$x_2(n+1) = -x_1(n) + 5x_2(n) - 2x_3(n) + nu(n)$$

$$x_3(n+1) = -x_1(n) + 3x_2(n) - u(u) + \delta(n) + \delta(n-1) - 2\delta(n-2)$$

The matrix A has the following characteristic polynomial

$$p_c(\lambda) = det(\lambda I - A) = \lambda^3 - 8\lambda^2 + 20\lambda - 16$$

The minimal annulating polynomial of the matrix A is

$$p_0(\lambda) = \lambda^2 - 6\lambda + 8$$

and the minimal polynomial satisfying eqn.(16) is

$$p^*(\lambda) = \lambda - 2$$

Clearly, $p_2(\lambda) = (\lambda + 1)p^*(\lambda)$, i.e. p^* divides p_2. Therefore p_2 satisfies eqn.(16) (see proposition 1).

Thus for the system at hand $x_1(n)$, $x_2(n)$, $x_3(n)$ do not depend on the signal

$$\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \delta(n) + \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \delta(n-1) + \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} \cdot \delta(n-3) \qquad when \qquad n > 3$$

Example 3

Let the second order LTI discrete system in Fig. 1.

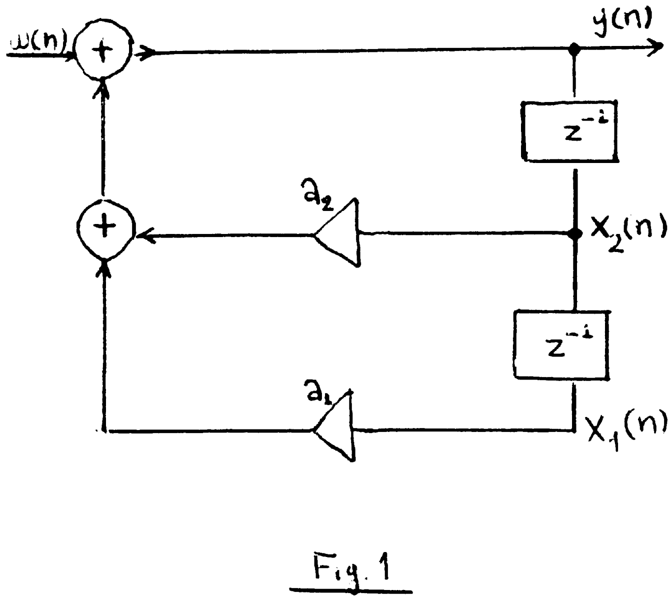

Figure 1

where z^{-1} represents a unit delay element, represents an analog amplifier, and + represents an adder. Here a_1 and a_2 are given real numbers.

The system state-space model can be found by inspection and is:

$$x_1(n+1) = x_2(n)$$

$$x_2(n+1) = -a_1 x_1(n) - a_2 x_2(n) + w(n)$$

$$y(n) = -a_1 x_1(n) - a_2 x_2(n) + w(n)$$

Now, suppose that

$$w(n) = b(n)u(n) + \beta_0 \delta(n) + \beta_1 \delta(n-1) + \beta_2 \delta(n-2)$$

Thus

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}$$

$$\beta_o = \begin{bmatrix} 0 \\ \beta_0 \end{bmatrix} \quad \beta_0 = \begin{bmatrix} 0 \\ \beta_1 \end{bmatrix} \quad \beta_2 = \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix}$$

and

$$A^2 \beta_0 + A\beta_1 + I\beta_2 = \begin{bmatrix} -a_1 & -a_2 \\ a_2 a_1 & -a_1 + a_2^2 \end{bmatrix} \begin{bmatrix} 0 \\ \beta_0 \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} 0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} -a_2 \beta_0 + \beta_1 \\ (-a_1 + a_2^2)\beta_0 - a_2 \beta_1 + \beta_2 \end{bmatrix}$$

Clearly, eqn.(14) is fulfilled if and only if $\beta_0 = \beta_1 / a_2$ and $\beta_2 = (a_1 - a_2^2)\beta_0 - a_2 \beta_1$ for arbitrary $\beta_1 \in \Re$.

Therefore, for the above selection of the parameters $\beta_0, \beta_1, \beta_2$, the state-variables $x_1(n)$, $x_2(n)$ (as well as the output $y(n)$ do not depend on the input signal $\beta_0 \delta(n) + \beta_1 \delta(n-1) + \beta_2 \delta(n-2)$ when $n > 2$.

Remark: The following typing errors were observed in [6]: (i) $\beta_0 I + \beta_1 A + \ldots + \beta_s A^s$ in eqn. (11) must be replaced by $A^s \beta_s + A\beta_1 + I\beta_0$, (ii) eqn. (19c) must be replaced by $x_3 = -x_1 + 3x_2 - h(t) + \delta^{(2)} - \delta^{(1)} - 2\delta$, and (iii) the matrix A of sec. 2 is of dimensionality nxn.

5. CONCLUSION

In this paper a proposition is formulated which provides a necessary and sufficient condition under which the effect of an input on the state variables of a discrete-time LTI system ceases to exist at the moment where the input ceases to exist too. This result which is the analogous of a previous result concerning continuous-time systems is very useful in some practical cases and is illustrated here by three simple but nontrivial examples.

6. REFERENCES

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