Nonlinearity and Self-similarity

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Abstract

We propose a generalized Lagrangian capable of describing in an unifying mode different nonlinear dynamical systems and wavelets; Korteweg-de Vries (KdV) solitons, K(2,2) compactons and Morlet continuous wavelets. A procedure to associate for any discrete wavelet, that is for any multi-scale finite-difference equation, the corresponding Lagrangian and algebraic structure, is given. We focus upon introduction of a nonlinear wavelet-like basis made by localized analytical nonlinear solutions. A dimentional wavelet analysis that provide a qualitative relation between the amplitude, width and velocity of the traveling solutions of certain nonlinear equations is introduced.

1 Introduction

Nonlinear science, though relatively new and far less understood is an important frontier for probing fundamentals of Nature. Intense research has developed world-wide in the mathematics and physics of nonlinear dynamical systems [1] occuring in different fields like fluid dynamics, plasma physics and quantum optics through string theory. Among other, nonlinear physics presents a variety of patterns [2] and particle-like traveling waves [3]. Notable examples include dynamics of solitons or breathers [3], nonlinear molecular and solid state physics and nonlinear optics, [1, 4-7].

Traditional nonlinear models base on specific nonlinear partial differential equations (NPDE) and, contrary to the linear systems exhibiting smooth regular motions, NPDE show strong interaction between initial conditions and dynamics or couplings between different parts of the system. Also, nonlinear interactions involve many scales [8], and produce self-similar or fractal patterns [2]. The NPDE solutions of physical interest are usually localized and have good stability in time and through scattering each other (fulfilling local conservation laws). Their shapes are related to the velocity, thus making the nonlinear patterns distinctly observable from the linear cases. In the asymptotic range these solutions consist in izolated traveling pulses (solitons, breathers, compactons [9], rotons [10, 11]), free of interaction. Close to the scattering range, the nonlinear solutions obey nonlinear superposition principles.

The main point in nonlinear analysis is the construction of analytical, localized or finite supported solutions for a given NPDE. This is still an open question, especially because of this nonlinear superposition. Recent examples were found where the traditional nonlinear tools (inverse scattering, group symmetry, functional transform) are inapplicable, [9]. On the other hand, from the experimental point of view, one knows that observed patterns in Nature, either stationary, growing or propagating, are generaly of finite space-time extension and have amultiscale structure. Since the soliton and soliton-like solutions, even localized, are basically of infinite extent, one needs adequate nonlinear and eventualy self-similar bases in order to invetigate such structures.

One good support for this challenge is given by the multiresolution analysis [12]. Traditional tools, like the Fourier integrals or linear harmonic analysis are inefficient for nonlinear systems compared to wavelet systems. Wavelets are analysing functions (bases) having the scale space dependent. That is they are characterized by non-uniform scales, having thus the possibility of better local analysis at any scale, [12]. They have applications in signal processing, singular potentials in quantum mechanics, q-algebras and nuclear physics, [13], etc. The wavelet analysis of NPDE brings a new line since they can see very steep variation, singularities together with very smooth ones..

There are many physical premise favoring wavelets in the construction of nonlinear bases. For example the processes of breaking up of fluid drops was shown to be self-similar, the specific singularities (the neck) look identical at any scale, [14]. In the limit of separation this singularity can still be expressed within the same basis.

Another example of interconnection between the nonlinear features and the self-similar behavior is provided by the welknown cnoidal solution of the KdV equation. This solution can describe some of the nonlinear oscillation and rotation of a liquid drop, [10, 11], fluid shell or bubble modes, neutron star dynamics, spots in atmosphere, etc. We plot in the farthest half of the box, in Fig. 1, this cnoidal solution. In the range $s \in (0, 1)$ in the figure, the cnoidal wave (denoted

$cn(x,s)$) depends on the space coordinate x and the parameter s, running from 0 to 1. For $s=0$ the cnoidal wave approaches a linear wave (cos function) and in the limit $s=1$ the cnoidal function approaches the soliton (hyperbolic cos function). In the closest half of the box (negative s) we plot a continuous dilation of the cnoidal wave in its cos limit, that is we keep $s = 0 = constant$ and just apply a dilation to this function. That can be writen analyticaly as

$$u(x,s)\Big|_{|x|\leq 10} = \begin{cases} cn^2[x,s] & s\in[0,1] \\ cn^2[\alpha x, 0] & \alpha = (1-s)^{1/6}, \\ & s \in [0,-1], \end{cases}$$

where, for the sake of simplicity, we denoted by s the dilation parameter, too. This picture shows how the cnoidal nonlinear solution can be aproximately constructed by similarity, in this case by dilation, from its linear limit.

In this paper we introduce new wavlet inspired approaches in invetigation of NPDE localized solution. For that, we first construct a generelized Lagrangian which yields a class of NPDE allowing both compact supported solution (and in some limit approaching the KdV, MKdV solitons) and continuous wavelet solutions. We extend this approach in order to associate, for any discrete wavelet, and for any multiscale finite-difference equation, the corresponding Lagrangian. Also some nonlinear algebraic structure. A dimensional wavelet analysis, that provide a qualitative relation between the amplitude, width and velocity of the traveling solutions of certain nonlinear equations is introduced, too. Finaly, we introduce a nonlinear wavelet-like basis made by localized analytical nonlinear solutions.

The paper is organized as follows. In the second section we introduce a Lagrangian which generalize the KdV (MKdV) Lagrangian and the compacton Lagrangian. We obtain a generalized KdV equation which is later reduced to a new equation with compact supported solutions, that is K(n,m). In the third section we construct the Hamiltonian system associated with any discrete wavelet. In section four we use a reduction of the NPDE to a simple algebraic realtion among the parameters of the localized solution. We also introduce a continuous wavelet basis based on the compacton and dilated-compacton traveling waves. In the fifth section we review the structure of wavelet systems and we give a general method to construct wavelets with given symmetry.

2 The generalized Lagrangian

Observed patterns in nature - stationary or propagating are usually of finite extent. In this section we introduce a generalized Lagrangian from which we derive different NPDE producing traveling localised solutions. Namely, compact supported solitons, conventional non-compact solitons, and wavelets.

By reducing the $\{x,t\}$ variables of the solutions to one traveling coordinate $(x-Vt)$, any NPDE transforms into a nonlinear ordinary differential equation (NODE). This one can be further reduce into a first-order nonlinear differential system.

The generalization which we will study in this section is described by the Lagrangian [15]

$$L = \int_R dx \Big[\frac{1}{2}\varphi_x\varphi_t + \frac{\alpha}{(p+1)(p+2)}(\varphi_x)^{p+2}$$

$$-\beta(\varphi_x)^m(\varphi_{xx})^n + \frac{\gamma}{2}(\varphi_x)^l(\varphi_{xx})^q(\varphi_{xxx})^r\Big], \quad (1)$$

which includes extra terms with higher order derivatives compared to the traditional non-relativistic field theory Lagrangians [16]. Here p,m,n,l,p,q,r and α,β,γ are free parameters. After writing the Hamilton equation, we make the functional substitution $\eta = \varphi_x(x,t)$. The first term in eq.(1)

gives the dynamical structure of the resulting equation, that is the time dependence. The second term is the typical term, most involved in nonlinear field equations [1,3-5, 7,9,10,15,17]. This term generates the nonlinear couplings in the corresponding NPDE. The next two terms in eq.(1) are responsible for the dispersion. The balance between the second term and the third (and higher) terms controls the existence of localized solutions.

We begin our investigation by studying the simpler version of eq.(1), that is a Lagrangian with only three terms, $\gamma = 0$. The parameter n is the most important one and controls the different cases. For $n = 0, 1$, the third term in eq.(1) reduces to the second one or vanishes, respectively. The corresponding equation does not contain dispersion so it is not interesting for our purposes. Its solutions are unstable in time, decaying or blowing up. For $n = 2$, $L[p, m, n, l]$ leads to a generalized sequence of KdV-like equation of the form

$$\eta_t + \alpha\eta^p\eta_x + \beta[2\eta^m\eta_{xxx} + 4m\eta^{m-1}\eta_x\eta_{xx}$$

$$+m(m-1)\eta^{m-2}(\eta_x)^3], \qquad (2)$$

called the K(p+2,m) nonlinear equation. These equations admit compact supported traveling solutions, known as compactons [9,15,17]. The compactons are trigonometric functions defined on a half-period, and zero in the rest. In general, they have the form $Acos^a d(x - ct)$.

These compact waves have the remarkable property that after colliding with other compactons they reemerge with the same coherent shape. If the initial data of the K(2,2) equation are compact but not a compacton (a dilated or compressed version of it) the solution decomposes into a number of compactons, Fig. 3. Hence, as it was earlier suggested in [9], these solutions can provide a nonlinear basis.

In the special case when $0 < m < 2$ and $p = m$, the solutions are compactons for which the width is independent of the amplitude. This is the fact which provides the connection with wavelet bases. They are characterize by a unique scale, and it is this feature which makes later possible the introduction of a nonlinear basis starting from this "mother" function. For $m = p = 1$ the resulting equation

$$\eta_t + \alpha 2(\eta^2)_x + 2\beta(\eta^2)_{xxx} - 8\beta\eta_x\eta_{xx} = 0, \quad (3)$$

is a generalized KdV equation with nonlinear dispersion (the third term in eq.(3)) and a suplimentar nonlinear term. A compacton solution of eq.(3) is

$$\eta = \eta_0 cos^2\left[\sqrt{\alpha 24\beta}\left(x - \alpha(\eta_0 + 2\delta)3t\right)\right] + \delta, \quad (4)$$

if $|x - Vt| < \pi\sqrt{3\beta/\alpha^2}$ and zero in the rest. Here δ and η_0 are free parameters and the velocity is a function of the amplitude. We notice that the width $L = \sqrt{24\beta/\alpha}$ of the wave is independent of the amplitude. This equation has the same terms and the same solution as the K(2,2) considered in [9]. Contrary to that one (which has four independent invariants), eq.(3) has only one integral invariant, the area of the solution. The quadratic dispersion term is characteristic for the nonlinear coupling in the chain.

The general solution of eq.(3) is not only the cos function but a dilation of lenght l of the peak value, Fig. 4 Although the second derivative of this generalized compacton is discontinuous at its edges and is still a strong solution of eq.(3) because the third derivative acts on u^2 which is of class C_3.

For $m = 0$ eq.(2) reduces to the class of modified KdV equations, MKdV.

$$\eta_t + \alpha\eta^p\eta_x + 2\beta\eta_{xxx} = 0 \qquad (5)$$

which are integrable for $p = 1, 2$ with soliton traveling solutions, respectively

$$\varphi_x(x, t) \equiv$$

$$\eta(x-Vt)=\begin{cases}\eta_0 sech^2\left[\sqrt{24\beta\alpha\eta_0}\left(x-\alpha\eta_0 3t\right)\right],\\ \eta_0 sech\left[\sqrt{12\beta\alpha\eta_0^2}\left(x-\alpha\eta_0^2 6t\right)\right],.\end{cases} \quad (6)$$

For $p=1,2$, respectively. For $n>2$ the equations become too complex and their investigation beyond the aim of the present paper. Moreover, we have no knowledge of physical systems described by the third power of the derivative terms.

In the case when the last term in eq.(1) is not any more neglected ($\gamma \neq 0$) it was shown that compacton solutions occur only if $p=m=l+q$ and $r=2$ [15,17]. Another interesting case occurs if $l=q=2$, $r=1$, $m=1$ and $r=4$. In this case eq.(1) has the form

$$L_{wavelet}[\varphi]=\int dx\left[\frac{1}{2}\varphi_x\varphi_t-\beta\varphi_x\varphi_{xx}^4+\frac{\alpha}{2}\varphi_x^2\right]. \quad (7)$$

After the substitution $u(x,t)=\varphi_x(x-ct)$ the corresponding NPDE becomes

$$-(v+2\alpha)u-\gamma u_x^4+2\partial_x(2\alpha u u_x^3+\beta u^2 u_x u_{xx})$$

$$-2\beta u u_x^2 u_{xx}-\partial_{xx}(\beta u^2 {u_x}^2=0, \quad (8)$$

where $\varphi_x(x,t)$ is a travelling wave $u(x-vt)$. One specific solution of eq.(8) is a modulated (or complex shifted) Gaussian

$$\varphi_x(x,t)=u(x-vt)=\frac{1}{\sqrt[4]{\pi}}e^{-ik(x-vt)}e^{-\frac{(x-vt)^2}{2}}. \quad (9)$$

This is a traveling wave with fixed velocity and amplitude and arbitrary half-width. Also, for $t=0$ this function becomes the continuous Morlet wavelet [12,18] and can be extended to a basis of localized solutions [12,19]. This is a complex-valued function, depending on a real (in general integer) parameter k. The set $\{\Phi(x;k,d)\}_{k,d\in Z}$ is a quasi-orthogonal basis in $L^2_{[R]}$, where $\Phi(x;k,d)=\Phi(x-d,k)$ is the translated version of the Morlet wavelet. The coefficient k controls the scale or the number of zeros of each function of the basis. When the coefficient k which labels the scale is small the basis functions describe breather solutions of the KdV and MKdV equations. When $k\rightarrow 0$, the complex exponential can be expanded in Taylor series and the basis function approach the Hermit polynomials. That is consistent with the harmonic oscillator limit. Ususally there is no sense to consider small the coefficient of a complex exponential. In the case the Morlet basis functions, having narrow support, the variable x has a limited range.

Another interpretation for solution in eq.(9) is given in the frame of the covariant phase-space representation for light [20] since it gives a localized probability for photons as localized waves. The Hamiltonian associated with eq.(9) has the form:

$$H=\int dx[\varphi_t\frac{\partial L}{\partial\varphi_t}-L]=\int dx[-\frac{1}{16}u^4+\frac{1}{2}u^2u_x^2$$

$$-\frac{1}{16}u^2u_{xx}^2+\frac{5}{16}uu_x^2u_{xx}]. \quad (10)$$

We notice that there are lower order NPDE which have the Morlet wavelet, eq.(10), for solution

$$uu_{xx}-u_x^2+u^2=0. \quad (11)$$

Like in the case of the Lagrangian found in [15], $L_{wavelet}[\varphi]$ leads to the three laws of conservation: mass, energy and momentum. Also, $H_{wavelet}$ is conserved by Noether's theorem under the transformations: $\varphi\rightarrow\varphi+c_1, x\rightarrow x+c_2$ and t$\rightarrow$ $t+c_3$, where c_1,c_2,c_3 are constants. By introducing the Morlet solution in the Hamiltonian, eq.(10), we can analyse $H_{wavelet}(k)$ as a function of k. The dependence of the real part of $H_{wavelet}$ on k, given in Fig. 2, shows that the spectrum is bounded. For small k the system has one local minimum, since in this approximation the system is a harmonic oscillator. For large k's the Hamiltonian shows a self-similar behavior. The Morlet functions have features between similarity and nonlinearity

$$\Phi_M^2(x;k) = \Phi_M(\sqrt{2}x, k'), \qquad (12)$$

with $k' = \sqrt{2}k$. The nonlinear combinations of Ψ_M are transformed into dilations in the $k-$space and dilations in the coordinate x-space, both with a factor of $\sqrt{2}$. Loosely speaking, the square of function is equivalent to a dilation plus a translation in the basis, $k \to \sqrt{2}k$. The Morlet basis fulfils a special multiplicative algebra

$$\Phi(x;k)\Psi(x;k') = \Phi\left(\frac{x}{\sqrt{2}}, \sqrt{2}(k+k')\right). \qquad (13)$$

3 Discrete wavelets. Hamiltonian densities for the dilation equation

In this section we show that any finite-difference equation is related to a certain type of Lagrange or Hamiltonian problem. These types of equations mainly occur in wavelet theory. A wavelet basis for $L^2_{[\mathbf{R}]}$ consists in a set of dilated and translated copies $\Psi_{n,k}(x) = \Psi(2^n x - k)$, $k, n \in \mathbf{Z}$, of an initial wavelet $\Psi(x)$ defined itself as a dilation of a linear combination of translations of a scaling function $\Phi(x)$

$$\Psi(x) = \sum_{k=-N+1}^{1} (-1)^k C_{1-k}\Phi(2x-k). \qquad (14)$$

The scaling function $\Phi(x)$ fulfills the so called two-scale equation (or dilation equation)

$$\Phi(x) = \sum_{k=0}^{N} C_k \Phi(2x-k) = \sum_{k=0}^{N} C_k \Phi_{1,k}, \qquad (15)$$

with coefficients C_k restricted by the conditions $\sum_{k=0}^{N} C_k = 2 and \quad \sum_{k=0}^{N} C_k C_{k+2l} = 2\delta_{0l}$, [12,129]. The operations involved in Eq. (14,15) are handled by the translation operator $T^k f(x) = f(x+k)$ and the dilation operator $D^n f(x) = f(2^n x)$. Eq. 15 can also be written as $\Phi = Dj(T)\Phi$ where $j(T)$ is a N-th order polynomial in T^{-1}. If we introduce the differential realizations $T^k = e^{k\partial_x}$, $D^n = e^{nx \ln 2 \partial_x}$, valid only when acting on $C^\infty_{(\mathbf{R})}$ functions then the two operators can be expressed in terms of infinite series of derivatives. Equations (14, 15) describe the selfsimilarity of the basis $\{\Psi_{n,k}(x)\}_{n,k\in\mathbf{Z}}$ and $L^2(R) = \bigoplus_{n\in Z} V_n$ where each V_n is generated by all k-translations of $D^n\Psi$ for any positive integer n [19]. First natural attempt to relate wavelets to algebraic structures was their description in terms of the affine group symmetry [21]. Recently, a more complete algebraic description was introduced [13] by proving that for any discrete wavelet system a nonlinear algebra can be constructed in terms of T and D. In the following we generalize this structure and give an algorithm to construct wavelets with given symmetry.

The procedure works for functions having a certain degree of smoothness such that both the translation T and dilation D operators can be expanded in formal operator Taylor series. Under such circumstances any q-difference (two or many scales) or finite-difference equation, containing T and D operators, is equivalent with an infinite-order PDE equation. To show that this PDE emerges from a stationary Hamilton equation, we first focus on the translation operator which will result in a finite-difference equation. We introduce a class of scalar field Hamiltonian $\mathcal{H}_T$ in the form

$$\mathcal{H}_T = \int_{-\infty}^{\infty} H_T[x, u, \partial_x u, ..., \partial_x^n u, ...]dx,$$

where the Hamiltonian density H_T depends on the x-coordinate and on the field $u(x,t)$ together with its derivatives. The corresponding Hamilton equation are

$$\frac{\partial u}{\partial t}=\partial_x\frac{\delta H_T}{\delta u}=\sum_{k\geq 0}(-1)^k\partial^{k+1}\frac{\partial^k H_T}{\partial u^{(k)}}, \qquad (16)$$

where $u^{(k)}=\partial^k u$. The simplest type of Hamiltonian density, which can provide a finite-difference equation (related to translations) has the form

$$H_T=\frac{1}{2}\sum_{j\geq 0}^{N}\sum_{k\geq 0}^{\infty}(-1)^k C_j\frac{a_j^{2k}}{(2k)!}(\partial^k u)^2, \qquad (17)$$

where N is arbitrary and gives the order of the highest translation and a_k are the steps of translations. Eq.(16) results in the Hamilton equation

$$\frac{\partial u}{\partial t}=\sum_{j\geq 0}^{N}C_j(T^{a_j}+T^{-a_j})u, \qquad (18)$$

and consequently, the stationary solutions ($\partial_t u=0$) fulfil a finite-difference equation. The Hamiltonian in eq.(17) introduces a limitation, since in the corresponding Hamilton equation all translation operators occur only in pairs, that is symmetric combinations T^a+T^{-a}. Since the coefficients C_j are arbitrary, the Hamiltonian in eq.(17) can generate *any* type of symmetric finite-difference equation. Conversely, for any such equation, it exists such a Hamiltonian, having the finite-difference equation, eq.(18), as its stationary Hamilton equation.

There is a geometrical interpretation of the translation Hamiltonian, so far. If expressed in a compact form, the Hamiltonian in eq.(17) becomes

$$H=\sum_k \mathcal{C}_k\int(uT^k u+\mathcal{I}uT^{-k}u)dx, \qquad (19)$$

where $\mathcal{C}_k$ are the corresponding constants and the operator $\mathcal{I}$ acts as inversion, $\mathcal{I}f(x)=f(-x)$. Actualy, this Hamiltonain is a linear combination of scalar products between the solution and its translated versions. The real solution minimizes the Hamiltonian and hence tryes to cancel as many such scalar products as possible. This is nothing but the tendency of cutting out (reducing) the support of solutions in order to make all translations orthogonal. This is the understanding of the fact that finite-difference equations usually provide finite supported solutions, like discrete wavelets.

Another example of constructing Hamiltonian systems for finite-difference equations is inspired by the 1-dimensional linear damped oscillator described by the formal Hamiltonian $H_{ldo}=e^{bt/m}(m\dot{x}^2+kx^2)/2$ where $x(t)$ is the law of motion of a particle of mass m in an elastic field k with friction coefficient b. In this case, from the Rund-Trautman identity $-H_{ldo}\tau+p\xi=$const., with $p=\dot{x}(t)$, $\xi\partial_x+\tau\partial_t$ being the infinitesimal generator of space-time dilatation symmetry of H_{ldo}, it results a conserved quantity (even the energy is not conserved) associated with a continuous scale invariance $C=e^{bt/m}(m\dot{x}^2+kx^2+bx\dot{x})$. We introduce the Hamiltonian density

$$H_T=\frac{e^x}{2}\sum_{k\geq 0}^{\infty}a_k(\partial_x^k u)^2, \qquad (20)$$

with a_k arbitrary coefficients. The corresponding Hamilton equations for the stationary solution ($\partial_t u=0$) reads

$$\qquad (21)$$

If we choose in eq.(20) the coefficients a_k such that $\mathcal{A}_l=b^l/l!$, eq.(21) becomes simply $T^b u(x,0)=0$, for any translation of a shift b. Further more, in order to obtain the most general N^{th}-order finite-difference equation, $\sum_{j=0}^N C_j T^{jb}u(x,0)=0$, we have to identify the coefficients of the same order of derivative of u in eq.(16) with those resulting from the expansion $T^{jb}=\sum_{k\geq 0}(jb)^k/k!\partial^k$ where $j=0,1,...$ acting on u. This results in a system of linear equations

$$A_0=C_0+C_1+...+C_N,$$

$$A_k = \frac{b^k}{k!}\left(C_1 + 2^k C_2 + 3^k C_3 + ... + N^k C_N\right). \tag{22}$$

By coupling eqs.(20-21) we have

$$\tag{23}$$

which is the final equation to fix the a_j's as functions of C_k's. Here $j = [l/2 + 1/2]$ means integer part. In other words, for any finite-difference equation described by the coefficients C_k, step b and order N, we can find the corresponding coefficients a_j which provide the Hamiltonian eq.(20) for this equation. Conversely, to any such Hamiltonian it corresponds a specific finite-difference equation.

In order to construct the dilation part of the two-scale equation from a Hamiltonian formalism, we introduce the density

$$H_D = \frac{1}{2}\sum_{k\geq 0}^{\infty} C_k x^{2k} (\partial_x^k u)^2. \tag{24}$$

For a stationary field, the corresponding Hamilton equation is

$$\sum_{j\geq 0}^{\infty}\sum_{k=0}^{j}(-1)^j C_j x^{2j-k} \mathcal{D}_{jk} \partial_x^{2j-k} u(x,0) = 0, \tag{25}$$

where the coefficients $\mathcal{D}_{jk}$ are zero if $k \geq 2j+1$ or $k = 0$ and in the rest are given by

$$\mathcal{D}_{jk} = \frac{j!(2j)!}{k!(j-k)!(2j-k)!}. \tag{26}$$

Eq.(25) can still be written in a more compact form

$$\sum_{n\geq 0}^{\infty} A_n x^n \partial_x^n u(x,0) = 0, \tag{27}$$

where

$$A_n = \sum_{[(n+1)/2]}^{N} (-1)^j C_j \mathcal{D}_{j,2j-n}, \tag{28}$$

Eq.(27) belongs to the typical form of dilation equation with D operator expanded in Taylor series. For example, if we choose $A_n = 1/n!$ in eq.(28), eq.(27) becomes $u(2x,0) = 0$. The procedure can be extended to any discrete dilation equations (q-difference) involving any number of scales.

In the following, we introduce a dynamical system described by a pair of functions $u(\pm x - Vt)$, representing two traveling profiles of shapes $u(\xi_{1,2})$ ($\xi_{1,2} = \pm x - Vt$) running with fixed velocity V in the same direction on the real axis x. We introduce an infinite-dimensional Hamiltonian for $u(\xi_{1,2})$, associated with the two-scale equation

$$H_{DT}[u] = \int_{-\infty}^{\infty}\sum_{n\geq 0}\Big[t_n \partial_x^n\Big(u(\xi_1)u(\xi_2)\Big)$$

$$+ d_n x^{2n} \partial_x^n u(\xi_1)\cdot \partial_x^n u(\xi_2)\Big]\frac{dx}{2}, \tag{29}$$

depending on the arbitrary constants t_n and d_n. By taking the functional derivative, the Hamilton equations associated with eq.(29), $\dot{u}(\pm x - Vt) = \partial_x \delta H_{DT}/\delta u(\mp x - Vt)$ [24] for $u(\pm x - Vt)$ become

$$\sum_{n\geq 0}\sum_{k=0}^{N}(-1)^n d_n \mathcal{D}_{nk} x^{2n-k} u^{(2n-k)}(\xi_{1,2}) + V u(\xi_{1,2})$$

$$\pm \sum_{n\geq 0}(-1)^n t_n u^{(n)}(\xi_{1,2}) = 0. \tag{30}$$

Eq.(30) can be related with any dilation equation for an appropriate choice of the coefficients t_n and d_n. By using the differential realization for T and D operators and by identifying in eq.(15) and eq.(30) the same order of derivative for the function (i.e. Φ in eq.(15) and u in eq.(30)) one obtains an algebraic system of linear equations.

For the coefficients related with translation we have

$$t_n = \frac{1}{n!}\sum_{k=1}^{N} k^n C_k - V\delta_{n0}. \qquad (31)$$

For the coefficients related to the dilation part we have

$$\frac{(2^\beta - 1)^n}{n!} = \sum_{k=[(n+1)/2]}^{n} (-1)^k d_k \mathcal{D}_{nk}. \qquad (32)$$

The system eqs.(31-32) can be solved in both directions. For any Hamiltonian in the form eq.(29) with given coefficients t_k and d_k, we can solve eqs.(31-32) with respect to the coefficients C_k and β. Conversely, for a given two-scale eq.(15), starting from its coefficients C_k and dilation factor β, we can solve eqs.(31-32) and obtain the coefficients in the Hamiltonian in eq.(30). The series in eqs.(31-32) are convergent for a large class of functions since $|t_n| \simeq \frac{N^{n+1}}{n!} \to 0$ when $n \to \infty$. If $u(\pm x - Vt) = \pm\Phi(x,t)$ in H_{DT}, eq.(30) is recovered but contains only even (odd) powers of $T^{\pm 1}$, respectively. This duality suggests, as a further study, the introduction of a pair (Φ, Ψ) for $u(\pm x - Vt)$.

In the case of the dilation equation of the Haar wavelet, the solution is a discontinuous distribution, $\Phi_{0,0} = H(x) - H(x-1)$. In order to calculate the formal coefficients for the Hamiltonian in eq.(29) (with $\beta = -1$) we can use the dilation expansion. In the second order, the corresponding Haar Hamiltonian density is

$$h = \Phi(x)\Phi(1-x) + \Phi(x-1)\Phi(-x)2$$

$$+\left[\Phi^2 + \frac{x^2}{4}\left(d\Phi dx\right)^2 + \frac{x^4}{32}\left(d^2\Phi dx^2\right)^2 + ...\right]_x \qquad (33)$$

One can ether calculate the derivatives in the above expression, or the derivatives in eqs.(29-30). To avoid this problem we have to use a functional sequence of differentiable functions Δ_n, generated by a test function Δ_0, weakly converging towards $\Phi_{0,0}$. Hence, for each n we can use the corresponding for the differential terms, the corresponding term in the functional sequence. Details of calculations can be found in [22].

Compacton-wavelet bases]Qualitative wavelet analysis of nonlinear equations.

Compacton-wavelet bases

In this chapter we present qualitative analysis of any NPDE in terms of their localized traveling solutions by using wavelet methods. In general the localized solutions (that is soliton-like pulses) are characterized by three parameters. Two geometrical, the amplitude (A) and the width (L) and one dynamic, the group velocity of the pulse (V). This analysis provides simple realtions between these parameters, without actualy solving the NPDE. These usefull relation describe the general behavior of the solutions, their degree of compactness, the shape-dependence of the velocity, etc.

For a given equation the procedure consists in the substitution of all the terms of the NPDE according to the following rule: $u_t \to \pm V u_x$; $u \to \pm A$; $u_x \to \pm A/L$; $u_{xx} \to \pm A/L^2$ and so on. Consequently, the NPDE is maped into a nonlinear algebraic equation in A, L and V. In Table 1 we present several examples of application of the method in specific NPDE used in physics or other applications.

The validity of this substitution can be proved by writing the NPDE on a very small interval of the x-axis, where the behavior of the localized function is enough close to a modulated Gaussian. Then, apply a Morlet wavelet transform to the equation. Details of the method are given in Appendix 2. In Table 1 we have choosen nine typical NPDE, as presented in the first column by name and equation. In the second column we presented the analytical traveling localized solution, if is known. This solution provides special nonlinear relations between the amplitude, width and velocity, given in the third column. Finaly, in the last column is presented the result of the wavelet method. This consists again in the relation betwen the three parameters (A, L, V) obtain through the above substitution, without actualy solving the equation. The valability of the procedure is provided by the comparison betwen column four and five, for any case, where analytical solutions are available.

The first row present a linear case of the wave equation, just for comparison. The wavelet method provides the exact dispersion relation between V and no information about the amplitude which is consistent with its arbitrariness in the general solution.

The case of the KdV equation is described in the second row. The method provide a general expression for $L(A, V)$. If we need L to be proportional with A, we have to put V proportional with A. In this case we re-obtain the expression among the parameters provided by the exact solution, in column three. A first prediction of the method is if we allow V to depend on a power of the amplitude. This means solutions a higher nonlinear coupling between shape and kinematics. The side effect would be a lower limit for A. Smaller solitons than this limit can move vith velocity proportional to the amplitude, only. The same result is obtained happens for the MKDV equation (third row), except that V is proportional with the square of the amplitude in order to have L a function of A only, like in the exact solution case. Also, in the fifth row, we introduce the Nonlinear Schrödinger equation (NLS) of order three with soliton solution, [3]. This equation occurs in nonlinear optics or in the polaron model in solid state physics [4].

In the general case of a NLS of order n, when no analytical solution is known, we can predict the $L(A, V)$ dependence.

Similar situation occurs nn the case of sine-Gordon equation, if we ask velocity to be proportional with L^2 (in the sixth column) , obtaining a transcendent equation in A which is exactly the case of its soliton solution (third column).

The power of prediction of the method is best exemplified in the cases of the K(n,m) equations, when there is no known localized (soliton or compacton) solution [9,15,17]. The general relation provided among the parametrs approaches the correct relations for the exact solutions, in particular cases: $n = m$, $n = 2m = 2$, $n = m = 3$, $n = 3, m = 2$ and $n = 2, m = 3$, [15]. For example, in the last two columns of the Table 1 when $n = m$, there are the exact compacton solutions. The original form of this solution for K(n,n) was written with a small error in eq.(3) of the first article in [17]. Many other examples can be taken for different values of n, m provided with a correct prediciton for the behavior of the solutions. In the typical trigonometric case of a K(2,2) compacton (eq.(2) with $\alpha = 1$, $m = 1$ and $p = 0$) we obtain from the Table the exact relations $A = 4V/3$ and $L = 4$, [15].

Some new physical facts occur from this qualitative analysis, too. The traditional soliton (compacton, breather, roton, etc...) always moves with constant velocity on a rectilinear path (exception the roton case, soliton on the sphere produced in the absence of gravity and presence of surface pressure [10], when the trajectory is a circle, however the angular velocity being still constant) This picture can hardly provide a reliable model for compact physical systems in interaction, like particles, nuclei, molecules, etc. The soliton class knows only uniform (asymptotic) motion and elastic scattering. A better approach would be provided by solitons with variable speed.

For instance the case of a constantly decreasing velocity associated with a nonlinear dispersion doubled by dissipation. If so, the amplitude and the velocity of the soliton (compacton) will decrease untill the total extinction. Before the extinction, the velocity will be reversed and the soliton will move the opposite direction. This is the case when the velocity is still a linear function of amplitude but can change its sign, and it was introduced in [17] for a K(2,2)+KdV equation

$$u_t + (u^2)_x + (u^2)_{xxx} + \epsilon(u)_{xxx} = 0.$$

In this case one has the relation $V = C_1 A - C_2$ which produces a breaking of symmetry. Higher compactons ($A \geq C_2/C_1$) will travel to the right, smaller ones will travel to the left, while compactons with the critical amplitude (C_1/C_2) will stay at rest. We notice that the wavelet method gives the correct guess in the case of the above equation, too. In the case of the above equation, one obtains the dependence

$$L = \sqrt{(A+\epsilon)/(V \pm A)},$$

which still gives a constant width if $V = \mp A + \epsilon$.

Such variable speed -variable amplitude solutions are not necessarily a feature of linear plus nonlinear dispersion. If we introduce in the last column, last row expression a dependence of the type $V = 2A + C$, for the K(2,2) equation (having nonlinear dispersion only) the half width depends as a square root of A. Indeed, the shifted compacton

$$u(x,t) = Acos^2\left(x - Vt4\right) + \delta, \qquad (34)$$

with $V = 32\left(\delta + A2\right)$ is a solution of the typical K(2,2) equation, $u_t + (u^2)_x + (u^2)_{xxx} = 0$, with variable velocity induced by the amplitude. For $A = -2\delta$ the solution is stationary. An example is presented in Fig. 5 for a time dependent amplitude (oscillating) with a rate much slower than the motion of the solution (adiabatic decoupling). The wavelet method cannot predict this kind of solutions. They are either discontinuous (case in which the Proposition in Appendix 2 is not valid) or not-localized, case in whic the method itself does not work.

However, these examples of variable speed solitons are only kinetic pictures. There is so far no dynamic mechanism capable to provide the slow modification of the amplitude creating hence the change in the direction of the motion. This problem has been solved and it will be published soon. The key for the introduction of such a dynamics is the coupling between the traditional nonlinear picture (nonlinearity+ dispersion+diffusion) and the potential picture (Schrödinger additional terms). Moreover, if the amplitude oscillates around the critical value, the soliton will have an oscillating trajectory.

Another situation coverd by this method ocurs if the KdV equation has an additional term depending on the square of the curvature

$$u_t + uu_x + u_{xxx} + \epsilon(u_{xx}^2)_x = 0. \qquad (35)$$

This is the case of extremely sharp surfaces (fluid, surface oscillations of solids or vibrated granular materials) when the surface pressure cannot be linearized in terms of the curvature. The introduction of the last term predicts a new type of solitons having a superior bound for the amplitude, at a critical width. Solitons narrower than this critical width beging to decrease again to zero amplitude. This solitons could exist in pairs of the same amplitude at very different width. They may be related with the recent observed "oscillons" in granular materials, [22].

Besides the examples presented so far in the Table, the method provides a relaible criterium for finding of compact suported solutions. The reason for this simple estimation works in so many cases is given by the advantages of wavelet analysis on localized solutions. We stress that this method has little to do with the traditional similarity (dimensional) analysis, [15,17]. In that case one obtains relations only among powers of A, L and V, and not relations with numeric coefficients, like in our method.

The solution given in eq.(4) has a unique width, that is a unique scale. From the point of view of multi-resolution analysis the K(2,2) equations acts like a δ-band filter, allowing only this scale to come out of initial condition as a stable solution. However, this solution is not the unique. One can extend it to larger supported functions. The most general compact supported solution is a $C_2(\mathbf{R})$ combinations of piece-wise constant functions and piece-wise cos^2 functions. An example is shown in Fig. 6 In the case of the general K(2,2) equation, eq.(4), this piece-wise solution is a smoothly connected series of plateaus and squared cos functions arranged one on the top of each other in any combination. The amplitude η_0, the hight of the plateau δ are related to the velocity. The length of the plateau is given by λ. This solution is not stable in time since its different parts travel with different velocities. The higher the amplitude the faster travels, $V = 2\eta_0 + 4\delta$. A simple compacton can scatter such a smooth plateau. Because of the area conservation they combine together in a general solution, with the compacton traveling on the top of the plateau with a higher velocity. We still cannot say anything about the moments when the compacton interfere with the ends of the plateau. First comment is that such a family of solutions can be organized in a spline basis, of order two. The second comment is that the class of the general solution includes all scales up with respect to the simple compacton. Actually, the K(2,2) equation acts like a low-pass filter in terms of space-time scales of the solutions.

In order to prove that the most stable solution is the compacton, as it has been find out by many numerical experiments [9], we introduce in the Hamiltonian, eq.(1), different combinations of plateaus and compactons. For example series of disjunct compactons, or disjunct plateaus with amplitude and heigth given by a certain series. Also combinations of such series. In all these calculations, it appeared that, for constant area, the minimum energy is approached for combinations of simple compactons of different heights.

The existence of these general solutions give the opportunity to construct bases of functions, from the wavelet model. The robustness of the simple compactons and the inaplicability of the inverse scattering tools [9], that worked so well for the KdV, MKdV, NLS and Sine-Gordon NPDE, makes it clear that a new mechanism is underlying the dynamics of such solutions, like for instance the existence of a nonlinear basis in which compactons play the role of generic (mother) functions.

For the sake of simplicity we will renormalize the coefficients of the K(2,2) equation such that the support of the simple compacton is of length one. That is we take $\eta_c(x,t) = \eta(\pi x, \eta_0, 0, 0)$ on $[-1/2, 1/2]$.

We construct a multiresolution approximation of $L^2(\mathbf{R})$, that is an increasing sequence of closed subspaces V_j, $j \in \mathbf{Z}$ of $L^2(\mathbf{R})$ with the following properties, [19]

1. The V_j subspaces are all disjoint each other and their reunion is dense in $L^2(\mathbf{R})$.

2. For any function $f \in L^2(\mathbf{R})$ and for any integer j, we have $f(x) \in V_j$ if and only if $D^{-1}f(x) \in V_{j-1}$ where D^{-1} is an operator which will be defined later.

3. For any function $f \in L^2(\mathbf{R})$ and for any integer k, we have $f(x) \in V_0$ is equivalent with $f(x-k) \in V_0$.

4. There is a function, $g(x) \in V_0$, such that the sequence $g(x-k)$ with $k \in \mathbf{Z}$ is a Riesz basis of V_0, [19].

In the case of the length one support solutions of K(2,2) we chosse for the space V_0 to be generated by all translation of η_c with any integer k. The subspaces V_j for $j \geq 0$ are generated by all integer translations of the compressed version of this function, namely by $\eta(2^j \pi x, 2^{-j}\eta_0, 0, 0)$. The subspaces V_j for $j \leq 0$ are generated by all integer translations of the general solution in eq.(36) constructed by a half compacton (increasing from zero to η_0, a plateau of length $2^j - 1$ and the corresponding decreasing half-compacton. For example V_{-1} is generated by $\eta(\pi x, 2^j \eta_0, 1, 0)$. The spaces $V_j, j \geq 0$ are all solutions of K(2,2), the other are not any more. The function $g(x)$ is take to be $\eta(\pi x, \eta_0, 0, 0)$. It is easy to prove that these definitions fulfils restrictions one, three and four. As for the second criterium, we define the action of the operator $D^{-1} f(x) = f(2x)$ if $f(x) \in V_j$ with positive integer j, and $D^{-1}\eta(\pi 2^j x, 2^j \eta_0, 2^{-j} - 1, 0) = \eta(\pi 2^j x, 2^{-j+1}\eta_0, 2^{-j+1} - 1, 0)$, for negative j. In conclusion we constructed a basis of function made of contractions of the simple compacton solution (which are not any more solutions) and of a sequence of general solutions with non-dilated compacton ends and dilated plateau. These functions are still solutions of the equation. We can write the corresponding two scale equation which connects the subspaces (the equivalent of eq.(15))

$$\eta(\pi x, \eta_0, 1, 0) = (\pi x, \eta_0, 0, 0) + (\pi(x-1), \eta_0, 0, 0). \tag{36}$$

We will denote generically by $\eta_{k,j}$ the elemnets of this basis, that is $\eta_{k,j}(x) = \eta(\pi(x-k), 2^j \eta_0, 2^j - 1, 0)$.

In the following, we can expand any initial data for the K(2,2) equation in this basis.

$$u_0(x) = \sum_k \sum_j C_{k,j} \eta_{k,j}(x). \tag{37}$$

We notice that the following equality holds for $j' ¿ j$

$$\eta_{k,j}\eta_{k',j'} = \begin{cases} \neq 0 & k' = k \cdot 2^{j'-j}, ..., (k+1) \cdot 2^{j'-j} - 1 \\ = 0 & \text{in the rest.} \end{cases} \tag{38}$$

After some elaborate algebraic calculations, by using eq.(39), we show that the square of this function (necessary since the equations is nonlinear of order two) will be given by

$$u^2(x) = \sum_{k,j} \sum_{j' \geq j} \sum_{k' \in I} C_{k,j} C_{k',j'}$$

$$\times \left(\sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \cdots \sum_{i_{j'-j}=0}^{1} \eta_{\sigma(i_1, i_2, ..., i_{j'-j}), j'} \right) \eta_{k',j'}, \tag{39}$$

where I is the range of k' described in the first line of eq. (39), and

$$\sigma(i_1, i_2, ..., i_{j'-j}) = \sum_{l=1}^{j'-j} i_l 2^{j'-j'l+(j'-j)(j'-j+1)-l(l+1)2}$$

$$+k 2^{(j'-j)j+(j'-j)(j'-j+1)2}.$$

In eq.(40) the unique nonzero terms are, according with eq.(39), those for which $\sigma(i_1, i_2, ..., i_{j'-j}) = k'$, with $k' \in I$. This result express the following simple fact. The initial data is expanded in different scales, and different translations. The translations are mutual orthogonal so they donnot give any contribution when we calculate the square. When, in the expression of the square, we have to multiply two different scales, we reduce the wider scale in terms of linear combination of the narrower one, by using the two-scale equation, eq.(15). This is roughly what eq.(40) expresses. Out of all the terms in such a product only about $(2^{-j} - 1)/(2^{-j'} - 1) \simeq 2^{j'-j}$ of them give non-zero contribution. In other words this number is given by the number of solutions of equation $\sigma(i_1, i_2, ..., i_{j'-j}) = k'$, with $k' \in I$. This is the main advantage of treating nonlinear problems with a basis constructed with scale criteria.

Another advantage is that all the function in the basis are actually contractions or dilations, and translations of only two basic ones.

4 Comments and conclusions

Before concluding we would like to stress that the most common feature of NPDE and finite difference equations is the existence of compact supported solutions. Compactons and discrete wavelets are typical examples. In the following we introduce a criterium for the existence of compact supported solutions. We restrict to one-dimensional models described by a NPDE dynamical equation

$$\partial_t u = \mathcal{O}(x, \partial_x) u, \qquad (40)$$

where $\mathcal{O}$ is a nonlinear differential operator. By taking into account *only* traveling solutions, this NPDE reduces to a NODE in the coordinate $\xi = x - Vt$ for an arbitrary velocity V. Suppose $u(\xi)$ is a compact supported solutions, it results that this solution is not unique under given initial data. Indeed, if one fixes initial zero values for the solution and its derivatives up to the order of the eq.(42), at a certain point ξ_0 of the ξ axis, these conditions are fulfilled by the compact solution or a linear combination of disjoint translated versions of it, placed everywhere on the axis, except on ξ_0. Consequently, for such initial data the solution is not unique and hence compacticity implies non-uniqueness.

Since we can further transform the NODE into a nonlinear differential system of order one

$$d\vec{U} dx = \vec{F}(\xi, \vec{U}), \quad \vec{U} = (u, \partial_x u, ...), \qquad (41)$$

we can apply the fundamental theorem of existence and uniqueness to solutions of eq.(43), for given initial data $\vec{U}(\xi_0) = \vec{U}_0$. If the function $\vec{F}$ in eq.(43) fulfills the Lipschitz condition (its relative variation is bounded) than, for any initial condition, the solution is unique, [24].

For example, since any linear function is analytic and hence Lipschitz, we conclude that only nonlinear functions $\vec{F}$, coming out from NPDE, allow the existence of compact supported solutions. O compacton-like solution (in general compact supported) implies non-uniqueness in the underlying NPDE, which implies non-Lipschitzian structure of the NPDE and hence the existence of nonlinear terms.

In this paper we introduce some physical interpretation for wavelets, as being related to localized physical nonlinear solutions. Self-similarity, cluster expansion, bifurcations and universality are all intrinsically related with nonlinearities. Linear differential equations can be solved analytically and therefore their solutions are organized as linear spaces. On the contrary, in the case of NLPDE, the analytical solution, if it exists, is unique. In the present paper we present a new approach for NLPDE, towards a reparation of this feature. For a first time, we proved that starting from any unique soliton-like solution of a nonlinear partial differential equation, we can construct a whole basis which generates a special Hilbert space of solutions. This basis is a wavelet system, hence providing similarity properties.

We found a general Lagrangian which can generate NPDE and wavelet generic equations. This formalism provide the possibility of constructing nonlinear basis for NPDE. We show that frames of self-similar functions are related with nonlinear problems: nonlinear algebraic structures, nonlinear Hamiltonian systems, and especially with solitons with compact support, compactons. To any scaling function we can associate a nonlinear finite generated algebra. Also, for any two-scale equation we can construct a special infinite-dimensional Hamiltonian system such that the corresponding scaling function is one of its extrema. This analysis represents only a first step towards understanding the relations between systems described by NLPDE (self-similar solutions with or without blow-up behavior) and wavelets theory. In addition, we stress the evidence that compactons are objects which fulfil both characteristics of solitons and wavelets, suggesting possible new applications.

Such unifying direction between nonlinearity and selfsimilarity, could bring new applications of wavelets in cluster formation at any scale from supernovae through fluid dynamics to atomic and nuclear systems, droplet bubbles and shell physics, stable traveling patterns, fragmentation, cold fission, the dynamics of the pellet surface in inertial fusion, stellar models, etc.

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Figure Captions

Figure 1: A plot of the cnoidal solution of the KdV equation (the Jacobi elliptic $cn(x,s)^2$, versus $|x| \leq 10$ and $s \in [0,1]$, in the second half-box. In the first half-box, $s \in (-1,0)$ it is ploted a continuous dilation of the same function $cn[\alpha x, 0]$, with the dilation coefficient $\alpha = (1+s)^{1/6}$ a function of s. The cnoidal solution at any value of the parameter s is similar with a dilation of the cos function.

Figure 2: The Hamiltonian of the Morlet wavelet, function of its k-scale parameter. The degenerated ground state ($k = 0$) is related to Hermite polynomial (the valleys can be approximated with a harmonic oscillators). A $k = 5$ and a $k = 20$ asymptotic states are ploted.

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Figure 3: Breaking up of an initial profile into several compactons, and their separation in time. The four lines are separated by the same time interval.

Figure 4: The most general $C_2(\mathbf{R})$ localized solution of the K(2,2) equation. It begins with a constant plateau to the left, continued by a half-compacton increasing part and then another compacton, etc. They go on decreasing in the same smooth way. A typical compacton is shown to the left for comparison.

Figure 5: The solution of the mixed equation for linear and nonlinear dispersion, K(2,2)+KdV in the case of an oscillating velocity and amplitude.

Figure 6: Example of two compactons and a dilated-compacton structure before the breakingup process of a new compacton.

Figure 7: The eigenvalues a_2, a_4 and a_8 of the operator j_0 in the Haar wavelet algebra parametrically ploted against the first one, a_0. Wer present the spectrum for 5 tipical values of a_0 and in some places the continuous dependence is also shown. This spectrum strongly depends on this initial eigenvalue and its smooth variations is self-similar.

Table 1: Nonlinear equations, exact solutions and wavelet analysis.

Equation	**Solution**	**Relations**	**W**
Linear wave $u_{xx} - (1/c^2)u_{tt} = 0$	$\sum C_k e^{i(kx \pm \omega t)}$	$V = c$ A, L *arbitrary*	k^2
KdV $u_t + 6uu_x + u_{xxx} = 0$	$A\ sech^2 x - VtL$	$L = \sqrt{2A}$ $V = 2A$	$L = 1$
MKdV $u_t + 6u^2 u_x + u_{xxx} = 0$	$A\ sechx - VtL$	$L = 1/A$ $A = \sqrt{V}$	$L = 1\sqrt{}$
sine-Gordon $u_{xt} - \sin u = 0$	$A\ \tan^{-1} \gamma\ e^{x - VtL}$	$A = 4$ $V = L^2$	$\pm V$
NLS(3) $\Psi_t + \Psi_{xx} + \Psi^3 = 0$	$Ae^{i(\omega t + kx)} sechx - VtL$	$L = 1A$ $A \simeq V$	$L = \pm V \pm$
NLS(n) $\Psi_t + \Psi_{xx} + \Psi^n = 0$	unknown		$L = \pm V \pm$
K(n,m) $u_t + (u^n)_x + (u^m)_{xxx} = 0$	unknown for $n \neq m$		$L = \sqrt{n(n^2 +}$
K(n,n) $u_t + (u^n)_x + (u^n)_{xxx} = 0$	$\left[Acos^2\left(x - VtL\right)\right]^{1n-1}$ for $\lvert x - Vt\rvert \leq 2n\pi n - 1$	$A = 2Vnn + 1$ $L = 4n/(n-1)$	$L = \sqrt{n}$ if V
K(2,2) $u_t + (u^2)_x + (u^2)_{xxx} = 0$	$A \cos^2 x - VtL$ for $\lvert(x - Vt)/L\rvert \leq \pi/2$	$L = 4$ $V = 3A/4$	$L = \sqrt{8A}$ $V =$