Novel quantification for chaotic dynamical systems with large state attractors

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Abstract: In this paper a novel quantification method for large state space attractors is proposed. The suggested approach is briefly described and tested on several dynamical systems with three degrees of freedom. Generalization of the method is for higher dimensional deterministic dynamical systems is also presented. The preliminary results shows that the method can be used for rough recognition of attractor nature and geometry. The significant contribution of proposed approach lies in speed-up the calculation process due to the reduction of one manifold.

Key–Words: Sphere, n-sphere, spherical quantification, chaos, Gotthans-Petrzela system, time-series.

1 Introduction

The recent discovery of cyclically symmetrical system composed of the differential equations containing signum functions [1], [2], [3] and [4] defined by

$$\begin{aligned}\dot{x} &= -a_x x \pm \operatorname{sign}[\sin(b_y y)]\\ \dot{y} &= -a_y y \pm \operatorname{sign}[\sin(b_z z)]\\ \dot{z} &= -a_z z \pm \operatorname{sign}[\sin(b_x x)],\end{aligned} \tag{1}$$

where $a_{x,y,z}$ and $b_{x,y,z}$ are constants and dots denote the first derivatives of the state variables. The system is so-called Gotthans-Petrzela (GP) oscillator. It turns out that there are serious problems during its analysis and using well known mathemaical tools. The difficulties are obvious especially in the case of Lyapunov exponents (LE) estimation [5], [6] and [7]. This is caused by extreme numerical values of the derivatives substitued into Jacobi matrix in the main loop of the calculation routine. Moreover there are associated troubles with precision while analytical formulas can not be utilized otherwise the procedure indicates completely incorect results.

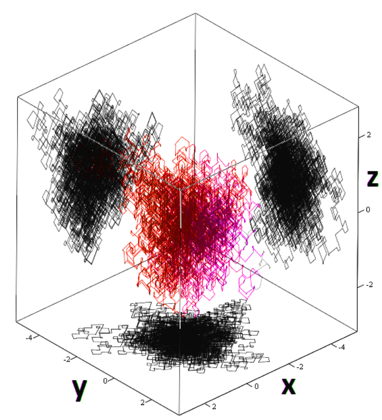

Figure 1: Solution of dynamical system obtained by integration of ODE with Monge's projections, GP system with uniform parameters $a_x = a_y = 0.1$ and $a_z = 0$.

2 2-Spherical quantification

The lack of the suitable methods for quantifying the behavior familiar to the conservative systems with large strange attractors leads the authors to develop different ones. The proposed approach is based on the fundamental nature of the strange attractors, namely ergodicity and mixing property. To reduce the computation time in R^3 demanded by standard box-counting methos it should be replaced by the mathematical operations on the plane in R^2, in detail on the surface of a sphere. The principle of calculation is analyzed in the transformation from Cartesian coordinates of the attractor into the spherical coordinates, respectively on the surface of 2-sphere. The surface of 2-sphere has to be normalized to obtain a general quantifier. Using spherical coordinates, the unit sphere can be parameterized by

$$\begin{aligned}&\vec{r}(\theta,\varphi) = (\cos\varphi\sin\varphi, \sin\theta\sin\varphi, \cos\varphi),\\ &0 \le \theta \le 2\pi, 0 \le \varphi \le \pi.\end{aligned} \tag{2}$$

Than the surface S of sphere $Sphere$ is set to be equal to 1 (normalization) can be expressed as

$$S(Sphere) = \int_{Sphere} |\vec{r}_u \times \vec{r}_v| \mathrm{dudv} = 1. \quad (3)$$

Thus for the square radius r^2 of R^3 sphere stands

$$r^2 = \frac{1}{4\pi}. \quad (4)$$

Radius of this ball is chosen $r = \frac{\sqrt{\pi}}{2\pi}$ such that the sum of all surface pieces (SP) is unity. This globe splits into elemental SP depending on the $\Delta\varphi$, $\Delta\theta$ angle steps

$$\theta = \arctan\left(\frac{y}{x}\right), \quad (5)$$

$$\varphi = \arccos\left(\frac{z}{r}\right). \quad (6)$$

Assuming that attractor will fill the entire space of R^3 where integral step limit approaches zero, meaning the likehood has a continuous uniform distribution $f(x)$ is described as

$$f(x) = \begin{cases} \frac{1}{b-a} & for\ a \leq x \leq b \\ 0 & for\ x < a\ or\ x > b \end{cases} \quad (7)$$

Than the surface of whole sphere can be described as

$$S_{Sphere} = \int_0^{\pi} \int_0^{2\pi} \frac{1}{4\pi} \sin\theta d\varphi d\theta = 1. \quad (8)$$

Considering the discrete time series, the step $\Delta\theta$ and $\Delta\varphi$ needs to be set, otherwise certain SP needs to be deleted to have particular list of SP for measuring the complexity of the analyzed attractor. Following the flow $\Phi(x, y, z)$ of attractor in R^3 having N elements, each SP is indexed by the natural numbers i and j. In the main calculation routine the individual SP occupied by a state trajectory are summarized. The surface of occupied area on the R^3 sphere can be calculated using the following discrete formula

$$S_{\Phi} = \sum_{i=1}^{N} \sum_{j=1}^{N} |\frac{1}{4\pi} \sin(i\frac{\pi}{N}) 2ij \frac{\pi^2}{N^2}|. \quad (9)$$

The graphical interpretation of this novel motion quantifier is demonstrated in Fig. 2, Fig. 3, Fig. 4, Fig. 5, Fig. 6 and Fig. 7 for some interesting situations. There is one serious drawback of this procedure leading to the indispensable numerical errors. The shape and orientation of the state attractor can be right-lined as it is visible in the first two examples. If so, a huge amount of the information about attractor

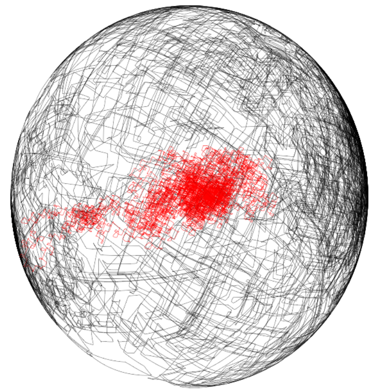

Figure 2: The dynamical motion quantification, analyzed attractor (red) and its projection on sphere (black), GP system with uniform parameters $a_x = 0, a_y = 0.1, a_z = 0.1$ and $b_x = b_y = b_z = 10$.

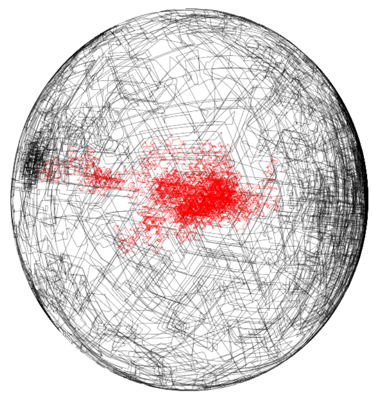

Figure 3: The rotated view on the dynamical motion quantification, analyzed attractor (red) and its projection on sphere (black), GP system with uniform parameters $a_x = 0, a_y = 0.1, a_z = 0.1$ and $b_x = b_y = b_z = 10$.

geometric structure is lost. To improve this disadvantage the linear change of the coordinates in order to spread the studied attractor should be performed before transformation on the sphere.

It is known, that the chaotic systems exhibit special sort of trajectories in state space. Thus the proposed method can be used as quantifier but should not be misinterpreted as metric dimension. There are also certain disadvantages, some of them revealed by the authors. Consider the rotation of attractor in Cartesian space. Certain rotation of attractor (very rare cases) can cause problems with transformation. This can be removed using linear coordinate transformation before applying the algorithm.

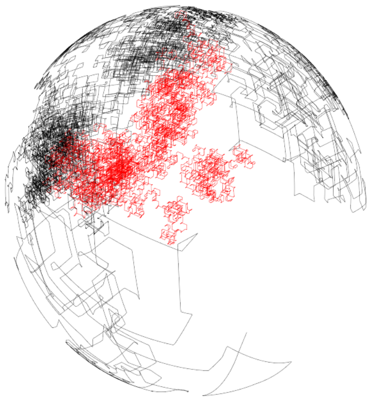

Figure 4: The dynamical motion quantification, analyzed attractor (red) and its projection on sphere (black), GP system with uniform parameters $a_x = 0, a_y = 0.1, a_z = 0$ and $b_x = b_y = b_z = 10$.

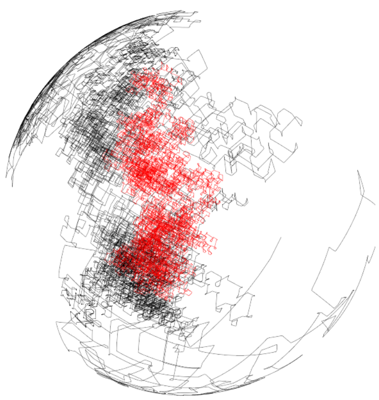

Figure 5: The rotated view on the dynamical motion quantification, analyzed attractor (red) and its projection on sphere (black), GP system with uniform parameters $a_x = 0, a_y = 0.1, a_z = 0$ and $b_x = b_y = b_z = 10$.

3 General n-spherical quantification

N-sphere can be considered as a generalization of the surface of an ordinary sphere to arbitrary dimension. For any natural number n, an n-sphere of radius r is defined as the set of points in $D = (n + 1)$ dimensional Cartesian space. The visualization of multidimensional objects is very difficult and provides only a fragment of entire picture. Thus it is left to the readers imagination and skipped in this paper. The distance r (radius) from a central point is any positive real number. Thus the n-sphere is defined by following term

$$S^n = \left\{ x \in R^{n+1} : ||x|| = r \right\}. \tag{10}$$

It is an n-dimensional manifold in Cartesian space. In particular, a 2-sphere is an ordinary sphere in three-

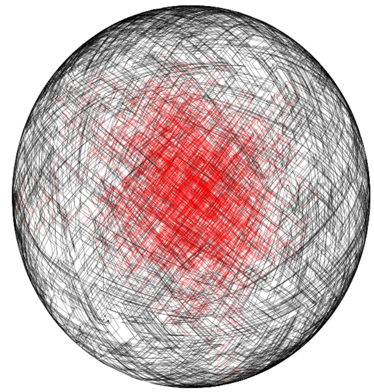

Figure 6: The dynamical motion quantification, analyzed attractor (red) and its projection on sphere (black), GP system with uniform parameters $a_x =_{ay}= a_z = 0.1$ and $b_x = b_y = b_z = 10$.

dimensional Cartesian space. Spheres of dimension $n > 2$ are called hyperspheres. The surface area of the n-sphere of radius r in $n + 1$ Cartesian space is defined as

$$S^n(r) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} r^n = 1, \tag{11}$$

where $\forall n \in Z \wedge n \geq 0$, Gamma function $\Gamma(l)$is an extension of the factorial function. Considering $\frac{n}{2} + 1) \geq 0$ is positive real number, the Gamma function reduces to

$$\Gamma(l) = (l - 1)!. \tag{12}$$

By defining a coordinate system in an D-dimensional spherical coordinate system from D-dimensional Cartesian space, in which the coordinates consist of a radial coordinate r, $n - 2$ angular coordinates $\Theta_1 \ldots \Theta_{n-2}$ with angle ranges $< 0, \pi >$ and another angular coordinate ϕ with range $< 0, 2\pi >$. Thus the radius of unity surface of n-sphere is defined as

$$r = \sqrt[n]{\left(\frac{S^n \Gamma(\frac{n+1}{2})}{2\pi^{\frac{n+1}{2}}} \right)}. \tag{13}$$

The other angular values can be calculated from Cartesian space consist of $x_1 \ldots x_D$

$$\begin{aligned}
\Theta_1 &= \text{arccotg} \frac{x_1}{\sqrt{x_D^2 + x_{D-1}^2 + \ldots x_2^2}} \\
\Theta_2 &= \text{arccotg} \frac{x_2}{\sqrt{x_D^2 + x_{D-1}^2 + \ldots x_2^3}} \\
&\vdots \\
\Theta_{D-2} &= \text{arccotg} \frac{x_{D-2}}{\sqrt{x_D^2 + x_{D-1}^2}} \\
\varphi &= 2 \cdot \text{arccotg} \frac{\sqrt{x_D^2 + x_{D-1}^2} + x_{D-1}}{x_D}
\end{aligned} \tag{14}$$

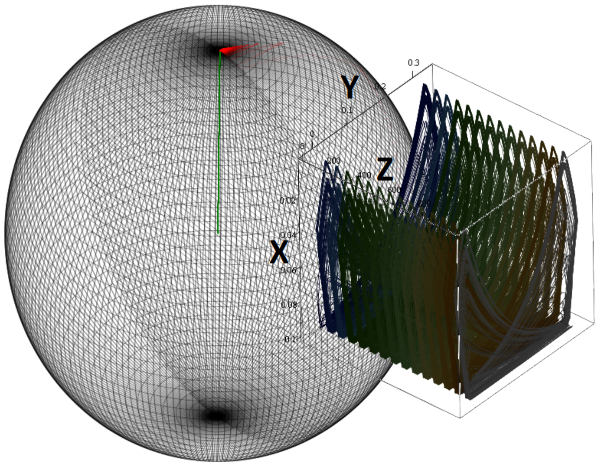

Figure 7: The dynamical motion quantification of periodical solution, analyzed attractor (green and in the extra plot) and its projection on sphere (black and red), GP system with uniform parameters $a_x = 10$ $a_y = a_z = 0.1$ and $b_x = b_y = b_z = 10$.

The single surface area of n-sphere can be calculated by

$$S^n_{Single} = r^n d\phi \prod_{m=1}^{D-2} (\sin \Theta_m)^m d\Theta_m, \tag{15}$$

where $D = (n+1)$. Than the whole filled surface, following the flow Ω of attractor consist of N discrete points is

$$S_\Omega = \sum_{i_1}^{N} \cdots \sum_{i_{D-2}}^{N} |r^n i_{D-2} \frac{2\pi}{N} \prod_{m=1}^{D-2} (\sin \Theta_m)^m \prod_{k=1}^{D-2} i_k \frac{\pi}{N}|. \tag{16}$$

The absolute value is used in equation 16. to meet the conditions of sum by changing the sign of negative angles.

4 Experimental Results

The proposed new algorithm is designed to recognize chaotic and non-chaotic solution. Hyperchaotic behavior can be also indicated since the volume element defined by neighborhood trajectories in the state space expands in two or more directions. To be more specific this quantifier responds to the orbit density in state space, higher value results into number closer to unity. Roughly speaking the state space attractor fills the Cartesian space and is extracted by the algorithm on the surface of sphere. If the solutions tend to be periodical, diverging or is represented by point, etc... , the result (or covered surface on the sphere) tends

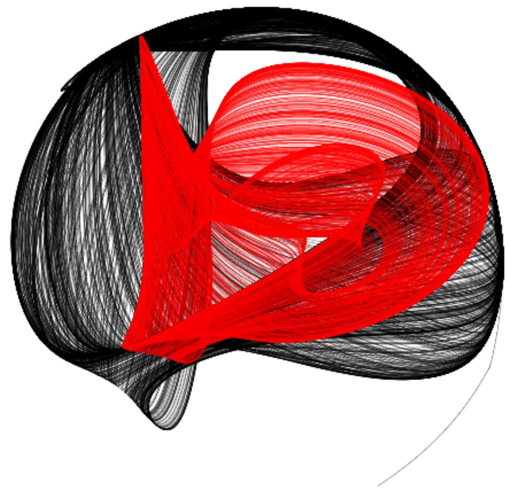

Figure 8: The dynamical motion quantification of periodical solution, analyzed attractor (green and in the extra plot) and its projection on sphere (black and red), The Halvorsen system with parameters $a = 1.3$ and $b = 4$.

to be minimal, sometimes it can be considered as almost zero. To identify the solution certain threshold can be set. The transient needs to be considered when analyzing the time series. Long transient can cause significant errors in setting the scale for the spherical transformations.

Table 1: Table of parameters and results for under test taken systems.

Eq.	Equation parameters	S_Ω	Lyapunov exp. λ_i
A	$a_x = 10, a_y = 0, a_z = 0, b_{x,y,z} = 10$	0.022	0.0238, -0.428, -9.595
	$a_x = 0.1, a_y = 0.1, a_z = 0.1, b_{x,y,z} = 10$	0.959	3.572, 0.005, -3.870
	$a_x = 2, a_y = 1, a_z = 1, b_{x,y,z} = 10$	0.225	1.186, -0.009, -5.177
	$a_x = 2, a_y = 2, a_z = 1, b_{x,y,z} = 10$	0.123	0.479, -0.005, -5.474
	$a_x = 2, a_y = 2, a_z = 1, b_{x,y,z} = 1$	0.003	-1.247, -1.256, -2.497
B	$a = 2, b = 2$	0.138	0.086, 0.007, -2.096
	$a = 3, b = 2$	0.061	0.005, -0.061, -1.944
C	$a = 1.1, b = 0.2$	0.135	0.101, 0.023, -1.113
	$a = 1.1, b = 0$	0	-0.004, -0.007, -0.992
	$a = 3.2, b = 0.1$	0.297	0.020, 0.015, -1.034
D	$p = 16, r = 45.92, b = 4$	0.143	1.444, -0.017, -22.335
	$p = 2, r = 45.92, b = 1$	0.103	0.534, -0.007, -4.526
	$p = 2, r = 80, b = 1$	0.056	0.048, -0.6730, -3.372
E	$a = 1.3, b = 4$	0.276	0.553, 0.052, -4.507
	$a = 1.3, b = 2$	0.054	0.016, -0.049, -3.866

Systems, more in [11] and [12], were taken under test. The systems are denoted by letters. System proposed in equation (1) is denoted in Table 1. as A. In this system the method of Fourier transform was used

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to obtain LE. The next tested system B is defined by

$$\begin{aligned}\dot{x} &= az\\ \dot{y} &= -by + z\\ \dot{z} &= -x + y + y^2.\end{aligned} \tag{17}$$

The next system denoted as C is defined as

$$\begin{aligned}\dot{x} &= y + az\\ \dot{y} &= bx^2 - y\\ \dot{z} &= 1 - x.\end{aligned} \tag{18}$$

More informations about systems (17) and (18) can the reader find in [8]. The next tested dynamical system D is the Lorenz's [9] famous system defined by

$$\begin{aligned}\dot{x} &= p(y - x)\\ \dot{y} &= -xz + rx - y\\ \dot{z} &= xy - bz.\end{aligned} \tag{19}$$

And the last Halvorsen's system denoted as E, 3-D system of chaotic flows that are symmetric with respect to cyclic interchanges of x, y, and z

$$\begin{aligned}\dot{x} &= -ax - by - bz - y^2\\ \dot{y} &= -ay - bz - bx - z^2\\ \dot{z} &= -az - bx - by - x^2.\end{aligned} \tag{20}$$

5 Conclusion

The results, explained in the previous sections, show that the proposed quantifier is suitable for dynamical systems with large attractors. The principle itself is very simple and can be easily implemented. The significant speed-up of the computation is obvious because of reduction of one degree of freedom. The advantage of this method is also, it can be used for a time-series recognition. Some well known systems of dynamical equations were tested. The result were compared with the LE. The preliminary results showed, that the method can be used for chaotic motion recognition. If the reader has more questions, please do not hesitate to contact the authors.

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