A Solution to the Discrete Optimal Tracking Problem for Linear Systems

CORNELIU BOTAN, FLORIN OSTAFI
Dept. of Automatic Control and Industrial Informatics
"Gh. Asachi" Technical University of Iasi
Bd. Mangeron. 53A, 6600 Iasi
ROMANIA

Abstract: The paper establishes a new procedure to obtain the solution for the discrete optimal tracking problem based on dynamic programming. The optimal control refers to a quadratic criterion with finite final time, regarding a perturbed discrete invariant linear system. The proposed algorithm can be easier implemented by comparison with other procedures.

Key-Words: optimal control, linear quadratic, tracking problem

1 Introduction

A perturbed discrete linear invariant multivariable system is considered

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + w(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

where $x(k) = x(k\tau) \in \Re^n$ is the state vector (τ is the sampling period, $k \in Z$), $u(k) \in \Re^m$ is the control vector, $y(k) \in \Re^r$ is the output vector and $w(k) \in \Re^n$ is the disturbance vector.

The problem is to ensure that the output vector y(k) evolves near to a desired trajectory $z(k) \in \Re^r$ and that the energy consumption has a low level. For this purpose, it is introduced the criterion

$$\begin{aligned} I &= \frac{1}{2} e(k_f)^T S e(k_f) + \\ &+ \frac{1}{2} \sum_{k=k_0}^{k_f-1} \left[e^T(k) Q e(k) + u^T(k) P u(k) \right], \end{aligned} \tag{2}$$

(T denotes the transposition), where S≥0, Q≥0, P>0 are the weight matrices of appropriate dimensions and

$$e(k) = z(k) - y(k) = z(k) - Cx(k) \tag{3}$$

is the tracking error.

The optimal tracking problem refers to the system (1) and the criterion (2). If the pair (A,C) is completely observable, the problem can be reformulated as one referring to the state vector [1],[2] and thus the criterion is

$$\begin{aligned} J &= \frac{1}{2} x^T(k_f) S' x(k_f) + \frac{1}{2} z^T(k_f) S z(k_f) - \\ &- z^T(k_f) S C x(k_f) + \\ &+ \frac{1}{2} \sum_{k=k_0}^{k_f-1} \Big[x^T(k) Q' x(k) + z^T(k) Q z(k) - \\ &\qquad - 2 z^T(k) Q C x(k) + u^T(k) P u(k) \Big] \end{aligned} \tag{4}$$

where

$$\begin{aligned} S' &= C^T Q C \geq 0, \\ Q' &= C^T Q C \geq 0 \end{aligned} \tag{5}$$

Let us consider V(k,x(k)) the minimum value of the criterion (4) within the interval [k, k_f]. V(k,x(k)) is defined as:

$$V(k, x(k)) = \min_{u(k)} I(x(k), u(k)) \tag{6}$$

According to dynamic programming, the minimum cost function V(k,x(k)) may be written [2]

$$V(k,x(k)) = L(k,x(k),u(k)) + V(k+1,u(k+1)), \tag{7}$$

where L(.) denotes the expression included in square brackets in the criterion (4).
The optimal control vector is computed taking into account the following equation:

$$\begin{aligned} & u(k) = \underset{u}{\operatorname{argmin}} (V(k,x(k)) = \\ & \underset{u}{\operatorname{argmin}} [L(k, x(k), u(k)) + V(k+1, x(k+1)] \end{aligned} \tag{8}$$

If V(k,x(k)) is imposed as

$$V(k, x(k)) = \frac{1}{2} x^T(k) \tilde{R}(k) x(k) + x^T(k) \tilde{g}(k) \tag{9}$$

($\tilde{g}(k)$ depends on z(k) and w(k)), thus V(k,x(k)) is the solution to the equation (7) only if the symmetric matrix $\tilde{R}$ verifies the difference matriceal Riccati equation

$$\tilde{R}(k) = Q + A^T \tilde{R}(k+1)[I_n + BP^{-1}B^T R(k+1)]^{-1} A \tag{10}$$

with

$$\tilde{R}(k_f) = S' \tag{11}$$

and $\tilde{g}(k)$ satisfies a linear difference equation.

Matrix $\tilde{R}(k)$ is time variant even in the cases when the matrices of the system (1) and of the criterion (2) are constant. It means that the resulting optimal controller is time variant even in the case of an invariant linear quadratic tracking problem. Solving in inverse time of the equation (10) introduces a supplementary difficulty in the implementation of this controller.

In order to avoid these difficulties, another way for optimal controller computing is proposed in this paper. A simple solution for the controller implementation in the time-invariant case is presented. The controller is built only with invariant blocks. Note that the presence of exogenous vectors z(k) and w(k) complicates the solution, since appear the additional terms.

2 Main Results

The following form for the minimum cost function V(k,x(k)) is proposed

$$\begin{aligned} & V(k, x(k)) = \frac{1}{2} x^T(k) R x(k) + x^T(k) v(k) + \eta(k) \\ & \quad v \in \Re^n, \quad \eta \in \Re \end{aligned} \tag{12}$$

with R constant symmetrical positive definite matrix.
From the transversality condition, yields

$$\begin{aligned} & V(k_f, x(k_f)) = \frac{1}{2} x^T(k_f) S' x(k_f) + \\ & \quad + \frac{1}{2} z^T(k_f) S z(k_f) - z^T(k_f) S C x(k_f) \end{aligned} \tag{13}$$

Introducing (12) and (1) in (7) one obtain

$$\begin{aligned} & V(k, x(k)) = \frac{1}{2} x^T(k)(Q' + A^T R A) x(k) + \\ & + \frac{1}{2} z^T(k) Q z(k) - z^T(k) Q C x(k) + \\ & + \frac{1}{2} u^T(k)(P + B^T R B) u(k) + u^T(k) B^T R A x(k) + \\ & + x^T(k) A^T R w(k) + u^T(k) B^T R w(k) + \\ & + x^T(k) A^T v(k+1) + u^T(k) B^T v(k+1) + \\ & + w^T v(k+1) + \frac{1}{2} w^T(k) R w(k) + \eta(k+1) \end{aligned} \tag{14}$$

The optimal control vector results from:

$$\frac{\partial V(k, x(k))}{\partial u(k)} = 0 \tag{15}$$

Replacing (14) in (15), yields

$$\begin{aligned} & \left(P + B^T R B\right) u(k) + B^T R A x(k) + \\ & + B^T R w(k) + B^T v(k+1) = 0 \end{aligned} \tag{16}$$

and optimal control becomes

$$\begin{aligned} & u(k) = -(P + B^T R B)^{-1} [B^T R A x(k) + \\ & + B^T v(k+1) + B^T R w(k)] \end{aligned} \tag{17}$$

One can write

$$u(k) = u_f(k) + u_c(k), \tag{18}$$

with

$$u_f(k) = K x(k), \tag{19}$$

$$K = -\bar{P}^{-1} B^T R A, \quad \bar{P} = P + B^T R B \tag{20}$$

and

$$u_c(k) = -\bar{P}^{-1}B^T v(k+1) - \bar{P}^{-1}B^T R w(k) \quad (21)$$

The feedback component $u_f(k)$ is identical with the one obtained in the similar optimization problem with infinite final time. The corrective component $u_c(k)$ ensures the coincidence with the solution obtained in the usual procedure, when the form (9) is used.
Introducing now (17), (18) and (19) in (14) one obtain

$$\begin{aligned} V(k,x(k)) &= \frac{1}{2}x^T(k)[Q + A^T RA + K^T \bar{P}K + \\ &+ 2K^T B^T RA]x(k) + x^T(k)[-C^T Qz(k) + \\ &+ K^T \bar{P}u_c(k) + A^T RBu_c(k) + A^T Rw(k) + \\ &+ K^T B^T Rw(k) + A^T v(k+1) + K^T B^T v(k+1)] + \\ &+ \frac{1}{2}z^T(k)Qz(k) + \frac{1}{2}u_c{}^T(k)\bar{P}u_c(k) + w^T v(k+1) + \\ &+ u_c{}^T(k)B^T v(k+1) + u_c{}^T(k)B^T Rw(k) + \\ &+ \frac{1}{2}w^T(k)Rw(k) + \eta(k+1) \end{aligned} \quad (22)$$

Comparing (22) and (12), yields:

$$R = Q' + A^T RA + K^T B^T RA \quad (23)$$

$$\begin{aligned} v(k) &= -C^T Qz(k) + K^T \bar{P}u_c(k) + A^T RBu_c(k) + \\ &+ A^T Rw(k) + K^T B^T Rw(k) + A^T v(k+1) + \\ &+ K^T B^T v(k+1) \end{aligned} \quad (24)$$

$$\begin{aligned} w(k) &= \frac{1}{2}z^T(k)Qz(k) + \frac{1}{2}u_c{}^T(k)\bar{P}u_c(k) + \\ &+ w^T v(k+1) + u_c{}^T(k)B^T v(k+1) + \\ &+ u_c{}^T(k)B^T Rw(k) + \frac{1}{2}w^T(k)Rw(k) + \eta(k+1) \end{aligned} \quad (25)$$

The equation (23) can be computed recurrently, starting from a certain positive defined initialisation for R.

The minimum cost function V(k,x(k)) may be written in the form (22), where R is solution to the algebraic Riccati equation (23). Moreover, v(k) and w(k) satisfy equations (24) and (25).
Introducing (21) in (24) one obtain:

$$v(k) = Zv(k+1) + h(k), \quad (26)$$

with

$$Z = A^T(I - RB\bar{P}^{-1}B^T) \quad (27)$$

$$h(k) = ZRw(k) - C^T Qz(k) \quad (28)$$

Taking into account equations (13) and (12) for $k=k_f$, yields

$$\begin{aligned} &\frac{1}{2}x^T(k_f)Rx(k_f) + x^T(k_f)v(k_f) + w(k_f) = \\ &= \frac{1}{2}x^T(k_f)S'x(k_f) + \frac{1}{2}z^T(k_f)Sz(k_f) \\ &\quad - z^T(k_f)SCx(k_f) \end{aligned} \quad (29)$$

This condition is satisfied if one chooses:

$$\begin{aligned} \eta(k_f) &= -\frac{1}{2}x^T(k_f)(S' - R)x(k_f) + \\ &\quad + \frac{1}{2}z^T(k_f)Sz(k_f) \end{aligned} \quad (30)$$

and

$$v(k_f) = (S' - R)x(k_f) - C^T Sz(k_f) \quad (31)$$

From (26) one can obtain

$$v(k) = Z^{k_f - k}v(k_f) + \sum_{j=1}^{k_f - k} Z^{k_f - k - j}h(k_f - j) \quad (32)$$

It is not possible to compute the vector v(k) in real-time because the final condition $v(k_f)$ given by (31) is not a priori known. Therefore, the final value $v(k_f)$ must be expressed as a function of $x(k_0)$.
If the control vector given by (18) is replaced in (1), the system equation becomes:

$$x(k+1) = Fx(k) + Bu_c(k) + w(k) \quad (33)$$

with

$$F = A + BK \quad (34)$$

The solution to the discrete equation (33) is

$$x(k) = F^{-(k_f-k)}x(k_f) + \\ + \sum_{j=1}^{k_f-k} F^{-(k_f-k-j+1)}[Bu_c(k_f-j)+w(k_f-j)] \tag{35}$$

From (35) for $k=k_0$, (21) and (26) yields

$$x(k_0) = F^{-(k_f-k_0)}x(k_f) - \\ -F^{-(k_f-k_0)}\sum_{j=1}^{k_f-k_0} F^{j-1}\{B\bar{P}^{-1}B^TZ^{-1}[v(k_f-j) - \\ -h(k_f-j)] + Z^TA^{-1}w(k_f-j)]\} \tag{36}$$

Replacing $v(k_f\text{-}j)$ from (32) in (36) and considering

$$M = F^{-(k_f-k_0)} - \\ -F^{-(k_f-k_0)}\sum_{j=1}^{k_f-k_0} F^{j-1}B\bar{P}^{-1}B^TZ^{j-1}(S'-R), \tag{37}$$

and

$$g = -F^{-(k_f-k_0)}\sum_{j=1}^{k_f-k_0}\{F^{j-1}B\bar{P}^{-1}B^TZ^{-1}[-Z^jC^TSz(k_f) + \\ +\sum_{i=1}^{j-1}Z^{j-i}h(k_f-i)] + F^{j-1}Z^TA^{-1}w(k_f-j)\} \tag{38}$$

yields

$$x(k_f) = M^{-1}[x(k_0)-g] \tag{39}$$

Taking into account the above quations, the final result for v(k) is:

$$v(k) = Z^{k_f-k}\{(S'-R)M^{-1}[x(k_0)-g] - \\ -C^TSz(k_f)\} + \sum_{j=1}^{k_f-k} Z^{k_f-k-j}h(k_f-j) \tag{40}$$

Remark 1: The expression of *g* given by (38) can be computed only if the exogenous vectors z(k) and w(k) are beforehand known. Therefore, the problem can be solved only under this assumption, or, at least, the shape of these vectors is known and their amplitude is estimated at the beginning of the optimization process.

Remark 2: The above relations allow implementation of the optimal controller using only invariant blocks. These relations are rather complicated, but most of them are computed off-line, in the design stage of the controller. The real-time computation implies the finding of a usual state feedback component $u_f(k)$ (19) and the computation of the corrective component $u_c(k)$. The last component can be computed from (21) and (26) as

$$u_c(k) = -\bar{P}^{-1}B^TZ^{-1}[v(k)-h(k)] - \bar{P}^{-1}B^TRw(k) \tag{41}$$

The corrective component depends on v(k) given by (40).

Remark 3: Using (20), equation (23) can be written

$$R = Q' + A^TRA - A^TRB\bar{P}^{-1}B^TRA \tag{42}$$

In a previous paper [4] starting from the necessary minimum condition referring to the Hamiltonian, a following Riccati equation was obtained:

$$R = Q' + A^T(R^{-1} + BP^{-1}B^T)^{-1}A \tag{43}$$

instead of the equation (42). Both results are similar. Indeed, using the identity:

$$(R^{-1} + BP^{-1}B^T)^{-1} = R - RB\bar{P}^{-1}B^TR, \tag{44}$$

one can observe that equation (42) is the same as (43). The form (43) is more advantageous than (42) in the case m<<n, because it implies to compute a reduced order inverse matrix.

3 Numerical Example

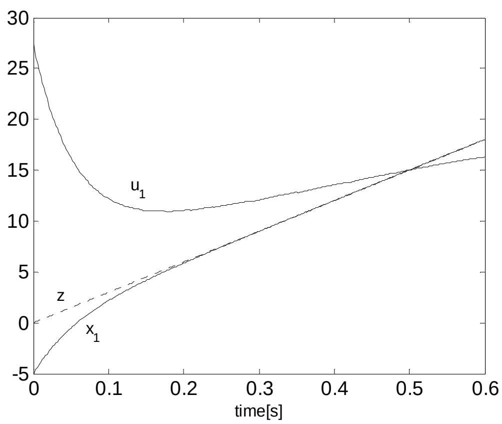
The behaviour of the optimal system was simulated for different system equations and weight matrices in the criterion. An example for a 4th order system with two control variables is presented in the following. We start from the next continuous time system and the equation (1) is obtained via discretization.

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 2 & 4 & -1 & 0 \\ 4 & 2 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 4 & 0 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x(t)$$

The sampling period is τ=0.002 seconds and the final time corresponds to k_f=300; the matrices in (2) are chosen as follows: S=1.5, Q=20, p=diag(1,1). The initial state vector is considered to be $x(0)=\begin{bmatrix}-5 & 5 & -8 & -4\end{bmatrix}^T$. The desired trajectory for the x_1 output is z(k)=30k.

The behaviour of the optimal control system is presented in the next figure.



These results were comparatively verified with the ones obtained by using a classical method and are nearly related to the continuous time case [5].

4 Conclusions

The optimal tracking problem for a discrete linear invariant system is studied, tacking into account the presence of the disturbances.

The proposed algorithm are more convenient for implementation by comparing with the usual procedures.

An efficient possibility of the implementation for the optimal controller in a time-invariant problem, using only time-invariant blocks is also proposed. The control consists in a state feedback and a correction depending on the initial state and on exogenous vectors.

References

[1] B.D.O. Anderson, J.B. Moore, *Optimal Control. Linear Quadratic Methods*, Prentice Hall, 1990.

[2] B.C. Kuo, *Digital Control System*, Saunders College Publishing, Philadelphia, 1992.

[3] C. Botan, A. Onea, F. Ostafi, A Solution to the LQ-Problem with Finite Final Time, *WSES Int. Conf. on Automation and Information (AITA 2001), Skiatos, Greece, in Advanced in Automation, Multimedia and Video Systems, and Modern Computer Science,* Ed. Kulev V.V., D'Attellis C.E., Mastorakis N.E.,WSES Press, 2001, pp.62-66.

[4] C. Botan, C. Belta, Some Aspects Concerning the Controller Implementation for the Discrete LQ Problem, *Bul. Inst.Politehnic Iaşi*, Tom XLIII (XLVII), Fasc. 1-4, Seria IV, 1997, pp 11-16.

[5] C. Boţan, F. Ostafi, A. Onea, A Solution to the Optimal Tracking Problem for Linear Systems, *WSEAS Transactions on Systems*, Vol. 1, ISSN 1109-2777, 2002, pp. 185-189.