Correlation Sensitivity Analysis of Failure Modes

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Abstract: A numerical method of the correlation sensitivity analysis for nonlinear random vibration system is presented in this paper. Based on the first passage failure model, the probability finite element method is employed to determine the statistical characteristic of failure modes and the correlation between them. The sensitivity of correlation between failure modes with respect to random parameters is discussed in time domain.

Key-Words: Correlated failure modes; Sensitivity analysis; Nonlinear random vibration; Probability finite element method; The first passage failure model.

1 Introduction

Uncertainties in material properties and structural geometry are due to measurement inaccuracies or structure complexities. Structures with random parameters are complicated to analyze, since the response is statistically correlated to these random parameters. During the last two decades, there has been considerable development in the analysis for random vibration systems, by using of probability finite element method (PFEM), neural network method and the maximum entropy theory [1-3]. The safety of engineering structures is one of the major objectives for designer. Reliability analysis can help designer to establish acceptable tolerances on structures and determine the fluctuations of the system parameters for safe operations. Reasonably consideration of correaltion between different failure modes is very important. It is maybe lose of practical meaning if you could not handle them corretly[4]. Since different components have common source of excitations and the systems are characterized by one set of parameters, the responses must be mutually dependent[5-7]. From the theory of PFEM, the statistical properties of responses depend on random parameters. So, by using of dynamic sensitivity analysis method, the sensitivity of correlation between failure modes with respect to random parameters can be gotten. The result will be full of greatly significances for simplifying of correlated failure modes, the system reliability analysis with correlated failure modes and reliability-based optimum design. In this paper, on the basement of first passage failure model, PFEM is employed to determine the statistical characteristic of any failure mode and the correlation coefficient between them. The sensitivity of correlation between failure modes with respect to random parameters is discussed later.

2 The statistical characters of failure modes

The first passage problem for nonlinear vibration systems is defined as

$$\boldsymbol{g}(\boldsymbol{A},\boldsymbol{X}) = |\boldsymbol{A}| - |\boldsymbol{X}| \quad (1)$$

where $\boldsymbol{X}=[X_1, X_2,\dots, X_n]^{\mathrm{T}}$ is the response vector of the system, which can include any response of system, such as displacement, velocity or acceleration etc; and $\boldsymbol{A}=[A_1, A_2,\dots, A_n]^{\mathrm{T}}$ is the threshold vector of $\boldsymbol{X}$. $\boldsymbol{g}(\boldsymbol{A},\boldsymbol{X})$ represents the safe state and failure state and can be defined as

$$\begin{cases} \boldsymbol{g}(\boldsymbol{A},\boldsymbol{X}) > 0 & \text{Safe state} \\ \boldsymbol{g}(\boldsymbol{A},\boldsymbol{X}) \le 0 & \text{Failure state} \end{cases} \quad (2)$$

When $\boldsymbol{g}(\boldsymbol{A},\boldsymbol{X})=0$ is the limit-state equation, representing limit-state surface or failure surface. According to different failure situation, Eq.(2) can be expressed as the following equations

$$g_i(A_i, X_i) = |A_i| - |X_i| \quad (3)$$

where $g_i(A_i,X_i)$ is ith state function. In fact, the response vector of $\boldsymbol{X}$ and the threshold vector $\boldsymbol{A}$ can be considered to be mutually independent. By using of PFEM[1,3,8], the mean vector and standard variance vector of system responses and failure modes can be written as

$$\mathrm{E}[\boldsymbol{X}] = \frac{\partial \overline{\boldsymbol{X}}}{\partial [\mathrm{cs}(\boldsymbol{r})]^{\mathrm{T}}}[\mathrm{cs}(\boldsymbol{r}-\bar{\boldsymbol{r}})] + \frac{1}{2}\frac{\partial^2 \overline{\boldsymbol{X}}}{\partial [\mathrm{cs}(\boldsymbol{r})]^{\mathrm{T}2}}\{\mathrm{Var}[\mathrm{cs}(\boldsymbol{r})]\} \quad (4)$$

$$\boldsymbol{\mu}_{\boldsymbol{g}} = \mathrm{E}[\boldsymbol{g}(\boldsymbol{A},\boldsymbol{X})] = \mathrm{E}[|\boldsymbol{A}|] - \mathrm{E}[|\boldsymbol{X}|] \quad (5)$$

$$\mathrm{Var}[\boldsymbol{X}] = \left[\frac{\partial \overline{\boldsymbol{X}}}{\partial [\mathrm{cs}(\boldsymbol{r})]^{\mathrm{T}}}\right]^{[2]}\{\mathrm{Var}[\mathrm{cs}(\boldsymbol{r})]\} \quad (6)$$

$$\boldsymbol{\sigma}_{\boldsymbol{g}}^2 = \mathrm{Var}[\boldsymbol{g}(\boldsymbol{A},\boldsymbol{X})] = \boldsymbol{\sigma}_{|\boldsymbol{A}|}^2 + \boldsymbol{\sigma}_{|\boldsymbol{X}|}^2 \quad (7)$$

where $\boldsymbol{r}$ is the random parameter matrix $(r_{ij})_{s\times t}$, which can include all of the systems probabilistic random variables. In most cases, the element of matrix $\boldsymbol{r}$ are considered to be as mutually independent. cs($\boldsymbol{r}$) is the column of $\boldsymbol{r}$. $[\bullet]^{[2]}=[\bullet]\otimes[\bullet]$, $\otimes$represents Kronecker product, which is defined as $(\boldsymbol{A})_{p\times q}\otimes(\boldsymbol{B})_{s\times t}=[a_{ij}\boldsymbol{B}]_{ps\times qt}$.

The combination vector of thresholds and responses is defined as $\boldsymbol{Y}=[A_1, A_2,\dots, A_n, X_1, X_2,\dots, X_n]^{\mathrm{T}}$. Then the covariance vector and correlation coefficient vector of failure modes can be written as

$$\mathrm{Cov}(\boldsymbol{g}) = \mathrm{E}\left[(\boldsymbol{g}-\boldsymbol{\mu}_{\boldsymbol{g}})(\boldsymbol{g}-\boldsymbol{\mu}_{\boldsymbol{g}})^{\mathrm{T}}\right] = \left[\frac{\partial \boldsymbol{g}}{\partial[\mathrm{cs}(\boldsymbol{Y})]^{\mathrm{T}}}\right]^{[2]}\{\mathrm{Cov}[\mathrm{cs}(\boldsymbol{Y})]\} = \left[\frac{\partial \boldsymbol{g}}{\partial[\mathrm{cs}(\boldsymbol{Y})]^{\mathrm{T}}}\right]^{[2]}\otimes\left[\frac{\partial \overline{\boldsymbol{Y}}}{\partial[\mathrm{cs}(\boldsymbol{r})]^{\mathrm{T}}}\right]^{[2]}\mathrm{Var}[\mathrm{cs}(\boldsymbol{r})] \quad (8)$$

$$\rho(\boldsymbol{g}) = \frac{\mathrm{E}\left[(\boldsymbol{g}-\boldsymbol{\mu}_{\boldsymbol{g}})(\boldsymbol{g}-\boldsymbol{\mu}_{\boldsymbol{g}})^{\mathrm{T}}\right]}{\boldsymbol{\sigma}_{\boldsymbol{g}}\boldsymbol{\sigma}_{\boldsymbol{g}}^{\mathrm{T}}} \quad (9)$$

From Eq.(8) and Eq.(9), we can determine the covariance and correlation coefficient of any failure mode.

3 The correlation sensitivity analysis of failure modes

Sensitivity analysis plays an important role in structural optimization, since most of optimization methods require the derivatives with respect to the design variables. The system reliability analysis with correlated failure modes is much more complicated than that of with independent failure modes. So far, the system reliability analysis with correlated failure modes has not been solved completely. On the other hand, the system responses are mutually dependent. For our knowledge, the failure mode is correlated, and in some cases, is strongly correlated.

From the Eq.(8) and Eq.(9), the covariance and correlation coefficient between failure modes g_p and g_q can be written as

$$\mathrm{Cov}(g_p,g_q) = \sum_{i=1}^{2n}\sum_{j=1}^{2n}\left(\frac{\partial g_p}{\partial y_i}\right)\left(\frac{\partial g_q}{\partial y_j}\right)\mathrm{Cov}(y_i,y_j) = \sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{k=1}^{s}\sum_{l=1}^{t}\left(\frac{\partial g_p}{\partial y_i}\right)\left(\frac{\partial g_q}{\partial y_j}\right)\left(\frac{\partial y_i}{\partial r_{kl}}\right)\left(\frac{\partial y_j}{\partial r_{kl}}\right)\mathrm{Var}(r_{kl}) \quad (10)$$

$$\mathrm{Var}(g_p) = \sum_{i=1}^{2n}\sum_{j=1}^{2n}\left(\frac{\partial g_p}{\partial y_i}\right)\left(\frac{\partial g_p}{\partial y_j}\right)\mathrm{Cov}(y_i,y_j) = \sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{k=1}^{s}\sum_{l=1}^{t}\left(\frac{\partial g_p}{\partial y_i}\right)\left(\frac{\partial g_p}{\partial y_j}\right)\left(\frac{\partial y_i}{\partial r_{kl}}\right)\left(\frac{\partial y_j}{\partial r_{kl}}\right)\mathrm{Var}(r_{kl}) \quad (11)$$

$$\mathrm{Var}(g_q) = \sum_{i=1}^{2n}\sum_{j=1}^{2n}\left(\frac{\partial g_q}{\partial y_i}\right)\left(\frac{\partial g_q}{\partial y_j}\right)\mathrm{Cov}(y_i,y_j) = \sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{k=1}^{s}\sum_{l=1}^{t}\left(\frac{\partial g_q}{\partial y_i}\right)\left(\frac{\partial g_q}{\partial y_j}\right)\left(\frac{\partial y_i}{\partial r_{kl}}\right)\left(\frac{\partial y_j}{\partial r_{kl}}\right)\mathrm{Var}(r_{kl}) \quad (12)$$

$$\rho(g_p,g_q) = \frac{\mathrm{Cov}(g_p,g_q)}{\sqrt{\mathrm{Var}(g_p)}\sqrt{\mathrm{Var}(g_q)}} \quad (13)$$

The sensitivity of $\rho(g_p,g_q)$ with respect to arbitrary element r_0 of random matrix $\boldsymbol{r}$ can be derived as

$$\frac{\partial\rho(g_p,g_q)}{\partial r_0} = \frac{\partial\mathrm{Cov}(g_p,g_q)/\partial r_0}{\sigma_{g_p}\sigma_{g_q}} - \mathrm{Cov}(g_p,g_q)\left(\frac{\partial\sigma_{g_p}/\partial r_0}{\sigma_{g_p}^2} + \frac{\partial\sigma_{g_q}/\partial r_0}{\sigma_{g_q}^2}\right) \quad (14)$$

where

$$\frac{\partial\mathrm{Cov}(g_p,g_q)}{\partial r_0} = \sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{k=1}^{s}\sum_{l=1}^{t}(A_1+B_1+C_1+D_1)\mathrm{Var}(r_{kl}) \quad (15)$$

$$A_1 = \left(\frac{\partial^2 g_p}{\partial^2 y_i}\frac{\partial y_i}{\partial r_0} + \frac{\partial^2 g_p}{\partial y_i\partial y_j}\frac{\partial y_j}{\partial r_0}\right)\left(\frac{\partial g_q}{\partial y_j}\right)\left(\frac{\partial y_i}{\partial r_{kl}}\right)\left(\frac{\partial y_j}{\partial r_{kl}}\right) \quad (16)$$

$$B_1 = \left(\frac{\partial g_p}{\partial y_i}\right)\left(\frac{\partial^2 g_q}{\partial y_i\partial y_j}\frac{\partial y_i}{\partial r_0} + \frac{\partial^2 g_q}{\partial^2 y_j}\frac{\partial y_j}{\partial r_0}\right)\left(\frac{\partial y_i}{\partial r_{kl}}\right)\left(\frac{\partial y_j}{\partial r_{kl}}\right) \quad (17)$$

$$C_1 = \left(\frac{\partial g_p}{\partial y_i}\right)\left(\frac{\partial g_q}{\partial y_j}\right)\left(\frac{\partial^2 y_i}{\partial r_{kl}\partial r_0}\right)\left(\frac{\partial y_j}{\partial r_{kl}}\right) \quad (18)$$

$$D_1 = \left(\frac{\partial g_p}{\partial y_i}\right)\left(\frac{\partial g_q}{\partial y_j}\right)\left(\frac{\partial y_i}{\partial r_{kl}}\right)\left(\frac{\partial^2 y_j}{\partial r_{kl}\partial r_0}\right) \quad (19)$$

$$\frac{\partial\sigma_{g_p}}{\partial r_0} = \sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{k=1}^{s}\sum_{l=1}^{t}(A_2+B_2+C_2+D_2)\mathrm{Var}(r_{kl}) \quad (20)$$

$$A_2 = \left(\frac{\partial^2 g_p}{\partial^2 y_i}\frac{\partial y_i}{\partial r_0} + \frac{\partial^2 g_p}{\partial y_i\partial y_j}\frac{\partial y_j}{\partial r_0}\right)\left(\frac{\partial g_p}{\partial y_j}\right)\left(\frac{\partial y_i}{\partial r_{kl}}\right)\left(\frac{\partial y_j}{\partial r_{kl}}\right) \quad (21)$$

$$B_2 = \left(\frac{\partial g_p}{\partial y_i}\right)\left(\frac{\partial^2 g_p}{\partial y_i\partial y_j}\frac{\partial y_i}{\partial r_0} + \frac{\partial^2 g_p}{\partial^2 y_j}\frac{\partial y_j}{\partial r_0}\right)\left(\frac{\partial y_i}{\partial r_{kl}}\right)\left(\frac{\partial y_j}{\partial r_{kl}}\right) \quad (22)$$

$$C_2 = \left(\frac{\partial g_p}{\partial y_i}\right)\left(\frac{\partial g_p}{\partial y_j}\right)\left(\frac{\partial^2 y_i}{\partial r_{kl}\partial r_0}\right)\left(\frac{\partial y_j}{\partial r_{kl}}\right) \quad (23)$$

$$D_2 = \left(\frac{\partial g_p}{\partial y_i}\right)\left(\frac{\partial g_p}{\partial y_j}\right)\left(\frac{\partial y_i}{\partial r_{kl}}\right)\left(\frac{\partial^2 y_j}{\partial r_{kl}\partial r_0}\right) \quad (24)$$

$$\frac{\partial\sigma_{g_q}}{\partial r_0} = \sum_{i=1}^{2n}\sum_{j=1}^{2n}\sum_{k=1}^{s}\sum_{l=1}^{t}(A_3+B_3+C_3+D_3)\mathrm{Var}(r_{kl}) \quad (25)$$

$$A_3 = \left(\frac{\partial^2 g_q}{\partial^2 y_i}\frac{\partial y_i}{\partial r_0} + \frac{\partial^2 g_q}{\partial y_i\partial y_j}\frac{\partial y_j}{\partial r_0}\right)\left(\frac{\partial g_q}{\partial y_j}\right)\left(\frac{\partial y_i}{\partial r_{kl}}\right)\left(\frac{\partial y_j}{\partial r_{kl}}\right) \quad (26)$$

$$B_3 = \left(\frac{\partial g_q}{\partial y_i}\right)\left(\frac{\partial^2 g_q}{\partial y_i \partial y_j}\frac{\partial y_i}{\partial r_0} + \frac{\partial^2 g_q}{\partial^2 y_j}\frac{\partial y_j}{\partial r_0}\right)\left(\frac{\partial y_i}{\partial r_{kl}}\right)\left(\frac{\partial y_j}{\partial r_{kl}}\right) \quad (27)$$

$$C_3 = \left(\frac{\partial g_q}{\partial y_i}\right)\left(\frac{\partial g_q}{\partial y_j}\right)\left(\frac{\partial^2 y_i}{\partial r_{kl}\partial r_0}\right)\left(\frac{\partial y_j}{\partial r_{kl}}\right) \quad (28)$$

$$D_3 = \left(\frac{\partial g_q}{\partial y_i}\right)\left(\frac{\partial g_q}{\partial y_j}\right)\left(\frac{\partial y_i}{\partial r_{kl}}\right)\left(\frac{\partial^2 y_j}{\partial r_{kl}\partial r_0}\right) \quad (29)$$

From Eq.(14)~Eq.(29), we can determine the sensitivity of $\rho(g_p,g_q)$ with respect to arbitrary element r_0 of random matrix $\boldsymbol{r}$.

4 Numerical example

A two story frame with material nonlinearities is shown in Fig.1. The deterministic masses m_1 and m_2 are 3.7kg and 1.5kg, respectively. The random spring stuffiness k_1 and k_2 are normally distributed with the coefficient of variation equal to 0.01. The mean values of spring stuffiness are both of 45×10^6N/cm. The system failure model is considered to be the first passage model, i.e. the failure of the first floor occurred when its displacement is beyond the limits of threshold A_{th1}; and so does the second floor failure. The random threshold A_{th1} and A_{th2} also are normally distributed with the coefficient of variation equal to 0.01. The mean value of A_{th1} and A_{th2} are 1.2cm and 2.1cm, respectively.

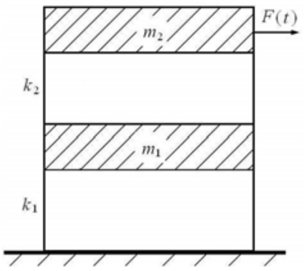

Fig.1 System model

The nonlinear differential equation is represented as

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} + \boldsymbol{K}_\varepsilon(\boldsymbol{x}\otimes\boldsymbol{x}\otimes\boldsymbol{x}) = \boldsymbol{F}(t) \quad (30)$$

with initial conditions $\dot{\boldsymbol{x}}(0)=0$, $\boldsymbol{x}(0)=0$.

where

$$\boldsymbol{K}_\varepsilon = \varepsilon\begin{bmatrix} k_1+k_2 & -k_2 & -k_2 & k_2 & -k_2 & k_2 & k_2 & -k_2 \\ -k_2 & k_2 & k_2 & -k_2 & k_2 & -k_2 & -k_2 & k_2 \end{bmatrix}$$

$$\varepsilon = 0.5, \quad \boldsymbol{F}(t) = \begin{bmatrix} 0.0 \\ 25.0\times10^6\sin(2000t) \end{bmatrix}.$$

The integration of Eq. (30) has been carried out using a forth Runge-Kutta routine. From Eq.(7)~Eq.(9), the standard variance of g_1 and g_2, the covariance and correlation coefficient of g_1 and g_2 can be obtained. In order to verify the effectiveness of the method proposed in this paper, the results are compared with Monte Carlo simulations and are shown in Fig.2~Fig.5.

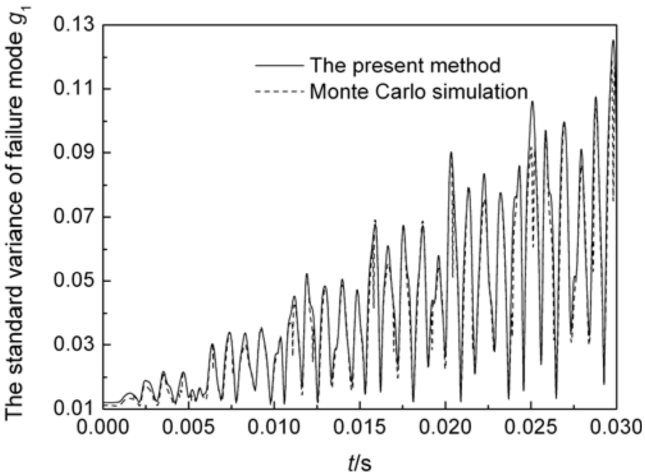

Fig.2 The standard variance of failure mode g_1

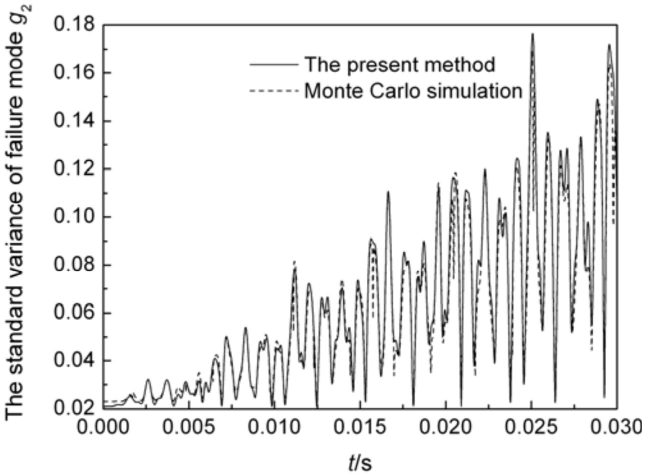

Fig.3 The standard variance of failure mode g_2

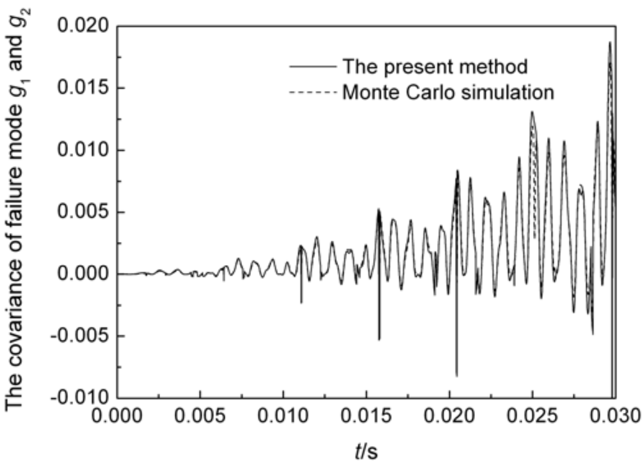

Fig.4 The covariance of failure mode g_1 and g_2

From Fig.2~Fig. 5, it can be seen that the failure modes g_1 and g_2 are correlated; the independent of them is only true at some discrete point during the period; and in most of cases, the strength of correlation is strong ($\left|\rho_{g_1g_2}(t)\right| \geq 0.66$). Therefore, the correlated failure is the main cause of system failure.

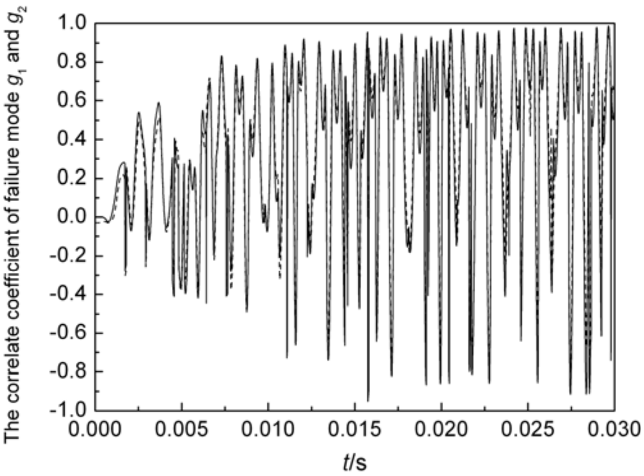

Fig.5 The correlation coefficient of failure mode g_1 and g_2

From Eq.(14)~Eq.(29), the sensitivity of $\rho(g_1,g_2)$ with respect to the random parameters k_1 and k_2 are obtained and shown in Fig.6 ~ Fig.7.

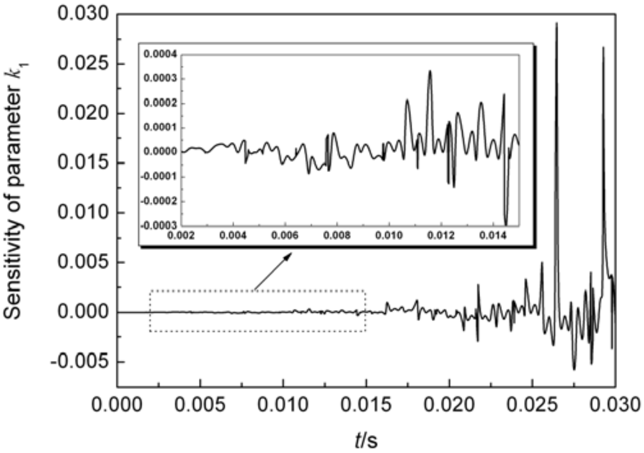

Fig.6 The sensitivity of $\rho(g_1,g_2)$ with respect to k_1

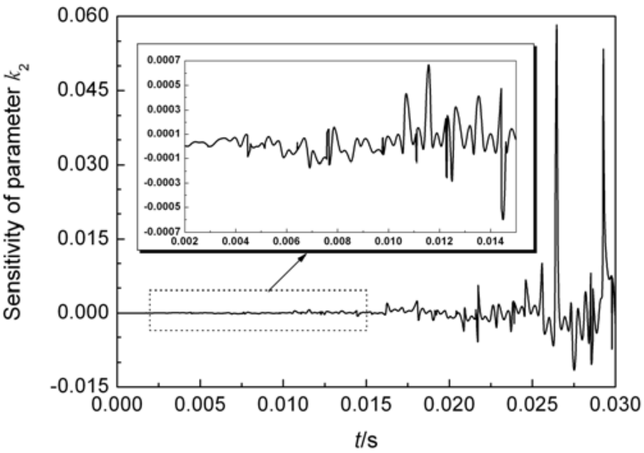

Fig.7 The sensitivity of $\rho(g_1,g_2)$ with respect to k_2

Form Fig.6 and Fig.7, it can be obtained that the sensitivity curves of $\rho(g_1,g_2)$ with respect to spring stiffness k_1 and k_2 have nearly the same tendency and increase remarkablely with the time. However, because of some inherent feathers of systme model, such as out exciting force, the senstivity of $\rho(g_1,g_2)$ with respect to k_2 is almost two times bigger than that of k_1.

4 Conclusions

A numerical method of the correlation sensitivity analysis for nonlinear random vibration system is presented in this paper. Based on the first passage failure model, the probability finite element method is employed to determine the statistical characteristic of failure modes and correlation coefficient between them. In most of cases, the system failure modes are strongly correlated, and the correlated failure is the main cause of system failure. The sensitivity of correlation between failure modes with respect to random parameters is discussed.

Acknowledgements

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References:

[1] Zhang Yimin, Wen Bangchun, Chen Suhuan. PFEM formalism in Kronecker notation, *MATH MECH SOLIDS*, Vol.1, No.4, 1996, pp.445-461.

[2]Zhang Yimin, Zhang Lei. Structural reliability optimum design besed on MCS-NN method, *Chinese Journal of Applied Mechanics*, Vol.22, No.1, 2005, pp.49-54

[3]Zhang Yimin, Wang Shun, Liu Qiaoling, Wen Bangchun. Reliability analysis of MDOF nonlinear random structure vibration systems with correlate failure modes, *SCIENCE IN CHINA (Series E)*,Vol.33,No.9,2003, pp.804-812

[4]Wang Guangyuan, Practical methods of optimum seismic design for engineering structures, *China Architecture & Building Press*, 1999

[5]Wen Bangchun, Zhang Yimin, Liu Qiaoling. Response of uncertain nonlinear vibration systems with 2D matrix functions, *NONLINEAR DYNAM*, Vol.15, No.2, 1998, pp.179-190.

[6]Zhang Yimin, Chen Suhuan, etc. Stochastic perturbation finite elements, *COMPUT STRUCT*, Vol.59, No.3, 1996, pp.425-429

[7]Zhang Yimin, Liu Qiaoling, Wen Bangchun. Dynamic research of a nonlinear stochastic vibratory machine, *SHOCK VIB*, Vol.9, No.6, 2002, pp.277-281

[8]Vetter. Matrix calculus operations and Taylor expansions, SIAM Review, Vol.15, No2, 1973, pp.352-369