An exact solution of the plate on elastic foundation under impact loading

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Abstract: - An exact solution of an infinite plate on elastic foundation under impact loading is presented. The formulation is based on application of Laplace and Hankel integral transforms and Bessel functions' properties. Representative examples are studied and the obtained solutions are discussed.

Key-Words: - plate, impact, elastic foundation, Hankel transform, Laplace transform, axis-symmetric loading

1 Introduction

Plates on elastic foundation are often used in civil or mechanical engineering problems, such as building infrastructures, tanks or silos foundations, aerospace engineering etc. The reaction of the foundation at these problems is approximated to be proportional of the plates' deflection w at each point. Numerical procedures to solve such problems are mostly based on finite elements (Cheung and Zienkiewicz [1]), finite differences (Long and Alturi [2], Krysl and Belytschko [3]) or meshless methods (Van Daele et al. [4], Melerski [5]). An interesting hybrid procedure combining finite elements and analytical method to analyze annular plate-soil interaction is presented by Chandrashekhara and Antony [6]. An alternative numerical procedure for the circular plate on elastic foundation developed by Utku et al. [7] represents the considered plate as a series of simply supported annular plates resting on support springs along their common edges and obtains the stiffness coefficients by the classical thin plate theory. Most of the above numerical methods solve the case of the loading by static loads. Analytical solution of the above problem have been published recently by Pavlou [8] also for the case of static axi-symmetric loads. However, during earthquake or other dynamic loading conditions the plates on elastic foundation may subjected in dynamic loads. In the present work the improvement of the previous solution of Pavlou [8] is presented in order to cover the impact loading case. The proposed analytical method is based on Laplace and Hankel integral transforms as well as

on Bessel functions' properties. Using these transformations, the fourth order differential equation describing the deflection w of the plate is simplified into a simple algebraic one with respect of the Laplace-Hankel transformed deflection. The required solution is obtained using inverse Hankel and inverse Laplace transforms.

2 Formulation of the problem

An infinite elastic plate with thickness h is considered to be founded on Winkler type foundation. The plate is loaded by load $q^{**} = q - q^*$ where q is the external step impact load

$$q(r,t) = q_o \delta(t) [H(r) - H(r - r_o)]$$
⁽¹⁾

and $q^*(r)$ is the foundation reaction which is proportional to the vertical deflection w(r,t) of the plate, i.e.

$$q^*(r,t) = k_s w(r,t) \tag{2}$$

In eqs. (1) and (2), $\delta(t)$ is the Dirac delta function of the time t, H(r) is the Heavyside step function of the radius r and ks is the modulus of the Winkler foundation.

The equilibrium of the bending moments in an elementary part of the plate results (Timoshenko [9]) to:

$$-M_{r}rd\vartheta + q^{**}rd\vartheta dr\frac{dr}{2} + (M_{r} + dM_{r})(r + dr)d\vartheta - 2\left(M_{t}\frac{d\vartheta}{2}\right)dr + (3) + (Q + dQ)d\vartheta(r + dr)dr = 0$$

where Mr and Mt are bending moments per unit length along circumferential and radial sections of the plate respectively, while Q is shearing force per unit length of a cylindrical section of radius r. Neglecting the small quantities, above equation can be written:

$$Q = -\frac{M_r}{r} - \frac{dM_r}{dr} + \frac{M_t}{r}$$
(4)

Taking into consideration the well known (Timoshenko [9]) relations between the bending moments and the deflection

$$M_r = -D\left(w'' + \frac{v}{r}w'\right) \tag{5}$$

and

$$M_t = -D\left(v\,w'' + \frac{1}{r}\,w'\right) \tag{6}$$

where v is the Poisson ratio, E is the modulus of elasticity and D the flexural rigidity given by

$$D = \frac{Eh^3}{12(1-v^2)}$$
(7)

it can be written:

$$\frac{Q}{D} = w''' + \frac{1}{r} w'' - \frac{1}{r^2} w'$$
(8)

The equilibrium of the vertical forces in an elementary part of the plate results to

$$(r+dr)d\vartheta(Q+dQ) - q^{**}r \,d\vartheta\,dr - -r \,d\vartheta\,Q + h\,r\,d\vartheta\,dr\,\rho\frac{d^2w}{dt^2} = 0$$
⁽⁹⁾

where ρ is the density of the material. Neglecting the small quantities, above equation can be written:

$$q^{**} = \frac{dQ}{dr} + \frac{Q}{r} + \rho h \frac{d^2 w}{dt^2}$$
(10)

With the aid of eq.(8) and eq.(2), above equation results to

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$$\nabla^4 w(r,t) + \lambda w(r,t) + \rho h \frac{d^2 w(r,t)}{dt^2} = \frac{q(r,t)}{D} \quad (11)$$

where q(r,t) is the loading given by eq.(1) while λ is given by

$$\lambda = \frac{k}{D} \tag{12}$$

and ∇^4 is the differential operator given by

$$\nabla^4 w(r) = \nabla^2 \nabla^2 w(r) =$$

$$= \left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right] \left[\frac{d^2 w(r)}{dr^2} + \frac{1}{r} \frac{dw(r)}{dr} \right]$$
(13)

3 Analytical solution

To solve the differential equation (11) the Laplace and Hankel integral transforms and their inverse forms will be used. The definitions of these integral transformations are:

Laplace and Inverse Laplace transform:

$$f^{*}(r,p) = L\{f(r,t); p\} = \int_{0}^{\infty} e^{-pt} f(r,t) dt \qquad (14)$$

and

$$f(r,t) = L^{-1} \{ f^*(r,p); t \} =$$

= $\frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} f^*(r,p) e^{pt} dp$ (15)

where L and L-1 are Laplace and inverse Laplace transform operator respectively.

Hankel and Inverse Hankel transform:

$$\overline{f}_{n}(\xi) = H_{n}\left\{f(r);\xi\right\} = \int_{0}^{\infty} rf(r)J_{n}(\xi r)dr \qquad (16)$$

and

$$f(r) = H^{-1} \left\{ \overline{f}_n(\xi); r \right\} = \int_0^\infty \xi \overline{f}_n(\xi) J_n(\xi r) d\xi \quad (17)$$
where $I_n(r)$ is the *n* th Bassel function and

where $J_n(x)$ is the n-th Bessel function and

 $H_n\{f(r);\xi\}, H^{-1}_n\{\overline{f}_n(\xi);r\}$ are the Hankel and inverse Hankel transform operator respectively.

Considering the following properties (Sneddon^(a) [10]) of Laplace transform:

$$L\left\{\frac{d^{2}w(r,t)}{dt^{2}};p\right\} = p^{2}w^{*}(r,p)$$
(18)

and

$$L\{\delta(t); p\} = 1 \tag{19}$$

and taking the operator L in eq. (11) it can be written:

$$\nabla^{4}w^{*}(r,p) + \lambda w^{*}(r,p) + \rho h p^{2} w^{*}(r,p) =$$

$$= \frac{q_{o}}{D} [H(r) - H(r - r_{o})]$$
or
$$\nabla^{4}w^{*}(r,p) + (\lambda + \rho h p^{2})w^{*}(r,p) =$$

$$= \frac{q_{o}}{D} [H(r) - H(r - r_{o})]$$
(21)

Taking the operator H_0 to eq. (21) it can be written:

$$H_{0} \{ \nabla^{4} w^{*}(r, p); \xi \} + (\lambda + \rho h p^{2}) H_{0} \{ w^{*}(r, p); \xi \} = -\frac{q_{o}}{D} J_{1}(\xi r_{o})$$
⁽²²⁾

Considering the substitution

$$\nabla^2 w^*(r, p) = f(r, p) \tag{23}$$

the transformation $H_0\{\nabla^4 w^*(r, p); \xi\}$ can be written:

$$H_0\{\nabla^4 w^*(r,p);\xi\} = H_0\{\nabla^2 f(r,p);\xi\}$$
(24)

or

$$H_{0}\{\nabla^{4}w^{*}(r,p);\xi\} =$$

= $H_{0}\{\frac{d^{2}f(r,p)}{dr^{2}} + \frac{1}{r}\frac{df(r,p)}{dr};\xi\}$ (25)

According to Sneddon^(a) [11], the following property of the Hankel transform will be used:

$$H_{n}\{B_{n}f;\xi\} = -\xi^{2}H_{n}\{f;\xi\}$$
(26)

where

$$B_n = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}$$
(27)

Taking into account eqs. (26), (27) and putting n=0, it can be written:

$$H_{0}\left\{\nabla^{2}f(r,p);\xi\right\} = -\xi^{2}H_{0}\left\{f(r,p);\xi\right\}$$
(28)

In above equation, if $w^*(r,p)$ is inserted instead of f(r,p) the following form will be resulted:

$$H_0\left\{\nabla^2 w^*(r,p);\xi\right\} = -\xi^2 H_0\left\{w(r,p);\xi\right\}$$
(29)

Considering eqs. (23, 29) the eq. (28) takes the form:

$$H_{0} \left\{ \nabla^{2} \nabla^{2} w^{*}(r, p); \xi \right\} =$$

= $-\xi^{2} H_{0} \left\{ \nabla^{2} w^{*}(r, p); \xi \right\}$ (30)

This equation with the aid of eq. (29) results:

$$H_0\left\{\nabla^4 w^*(r,p);\xi\right\} = \xi^4 H_0\left\{w^*(r,p);\xi\right\}$$
(31)

Then, eq. (22) gives:

$$\xi^{4}H_{0}\{w^{*}(r,p);\xi\} + (\lambda + \rho h p^{2})H_{0}\{w^{*}(r,p);\xi\} = -\frac{q_{o}}{D}J_{1}(\xi r_{o})$$
(32)

or

$$H_0\{w^*(r,p);\xi\} = -\frac{q_o}{D} \frac{J_1(\xi r_o)}{(\xi^4 + \lambda + \rho h p^2)}$$
(33)

Considering eq. (33), the analytical solution of the differential equation (21) can be written:

$$w^{*}(r,p) = -\frac{q_{o}}{D} H_{0}^{-1} \left\{ \frac{J_{1}(\xi r_{o})}{(\xi^{4} + \rho h p^{2} + \lambda)}; r \right\} (34)$$

Considering the definition of the inverse Hankel transform given in eq.(17) it can be written:

$$w^{*}(r,p) = -\frac{q_{o}}{D} \int_{0}^{\infty} \frac{\xi J_{1}(\xi r_{o})}{(\xi^{4} + \rho h p^{2} + \lambda)} J_{o}(\xi r) d\xi$$
(35)

Taking the operator L-1 in above equation the required solution w(r,t) can be obtained:

$$w(r,t) = -\frac{q_o}{D} \cdot \int_0^\infty \xi J_1(\xi r_o) J_0(\xi r) L^{-1} \left\{ \frac{1}{(\xi^4 + \rho h p^2 + \lambda)}; t \right\} d\xi$$
(36)
or

$$w(r,t) = -\frac{q_o}{D\rho h} \cdot \int_{0}^{\infty} \xi J_1(\xi r_o) J_0(\xi r) L^{-1} \left\{ \frac{1}{(p^2 + c^2)}; t \right\} d\xi$$
(37)

where

$$c^2 = \frac{\lambda + \xi^4}{\rho h} \tag{38}$$

Taking into account the equation (Prudnikov et. al. [12]):

$$L^{-1}\left\{ \left(p^{2} + c^{2}\right)^{-1}; t \right\} = \frac{1}{c}\sin(ct)$$
(39)

The final solution can be written

a

$$w(r,t) = -\frac{q_o}{D\sqrt{\rho h}} \cdot \int_0^\infty \frac{\xi}{\sqrt{\xi^4 + \lambda}} J_1(\xi r_o) J_0(\xi r) \sin\left[\sqrt{\frac{\xi^4 + \lambda}{\rho h}} t\right] d\xi$$
(40)

4 Example: Solution of infinite plate on elastic foundation under impact uniform load q_0 lying in the finite area $0 < r < r_0$

An infinite plate on elastic foundation with geometric and mechanical parameters $\lambda = 1$, $\rho h = 1$ is considered. An impulse uniform loading $q_o / D = 1000$ acting at time t=0 and lying

in the finite area $0 < r < r_0$, where $r_0 = 5$, is applied. With the aid of the well known software "Mathematica" the wave propagation w(r,t) for the times t=0.005 (Fig.1), t=0.007 (Fig.2) and t=0.010 (Fig.3) is calculated by the eq.(40). The results for these times are demonstrated in the following figures indicating the decreasing of the deflection as the time is increased within the time interval [0.005, 0,010].



Fig.3 Plate deformation at t=0.010

5 Conclusions

A new analytical method based on Laplace and Hankel integral transforms and Bessel functions' properties was derived to solve the problem of infinite plate on elastic foundation under impact loading. This solution can be used as a Green's function in order to solve boundary-value problems of finite circular or annular plates on elastic foundation under impact axisymmetric loads. Some real examples ware solved indicating the wave propagation for several values of the time.

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