The stability of collocation methods for approximate solution of singular integro- differential equations.

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Abstract: In this article we demonstrate that the collocation methods are stable in according with the small perturbations of coefficients, kernels and right part of studied equations. We prove that the condition number of the approximate operator exist and bounded(for the numbers n large enough). This condition number of collocation methods is appropriated with condition number for exact singular integro- differential equations equations (SIDE).

Key–Words: condition number, stability, collocation methods, singular integro- differential equations equations (SIDE).

1 Introduction

The main results about of the stability of projection methods were obtained in S. G. Mihlin [1], [2] and G.M. Vainikko [3] for Hilbert spaces and B.G. Gabdulhaev [4] for Banach spaces.

The definition of condition number for system of linear algebraic equations was introduced for example in [5],[6], [7] and generalized for operators and operator equations in [4].

The convergence for collocation methods was proved in [14], [8]. Numerical results were obtained in [13].

2 The main definitions notations

In this section we introduce the main definitions from [4],[7], [6]. Let

$$Ax = y, \ (x \in X, y \in Y, \tag{1}$$

be an exact equation and

$$A_n x_n = y_n, \quad (x_n \in X_n, y_n \in Y_n), \tag{2}$$

be an approximative equation.

Let A and A_n be a linear operators which acting from Banach space X to the Banach space Y and from subspace $X_n \subset X$ to the subspace $Y_n \subset Y$.

In practice the approximative solution of equation (2) is solved approximative because of the elements of this equations are not defined exactly. It means that the equation (2) is changed by new one

$$B_n x_n = z_n \quad (x_n \in X_n, z_n \in Y_n), \tag{3}$$

where B_n is linear operator from X_n to Y_n and so A_n and B_n, as y_n and z_n are appropriated in some sense.

That is why we should study the stability of direct methods[1] for the solution of equation (1). So we study the error

$$\delta_n = ||x_n^{(*)} - x_n^{(1)}||, \tag{4}$$

where $x_n^{(*)}$ and $x_n^{(1)}$ are solutions (if the solutions exist) of equations (2) and (3), respectively.

We introduce the definition of condition number defined by operator.

The value $\eta = \eta(A) = ||A||||A^{-1}||$ is named the condition number of operator A and equation (1).

The operator A and the equation (1) are named well-conditioned if η is small, and ill-conditioned in another case.

The following equality was obtained in [7] :

$$\eta(A) = \sup_{x^*} \left\{ \sup_{y} \frac{||x^* - x_n^*||}{||x^*||} : \frac{||A(x^* - x_n^*)||}{||y||} \right\}, \tag{5}$$

where x^*, x_n^* are solutions of equations (1) and (2) and y is right part in (1).

[1]The collocation methods, mechanical quadrature- methods are direct methods.

We study the stability of collocation methods. We suppose that the operator A in (1) are invertible.

The following theorems holds: [8]

Theorem 1. *Let the following conditions be satisfied*

1) $dimX_n = dimY_n \ (= m(n) < \infty)$ *and* $Y_n = Q_nY$, *where* Q_n *is bounded projector for all* n;
2) *the operators* $A_n : X_n \to Y_n$ *are invertible and* $||A_n^{-1}||_{Y_n\to X_n} \le c_1 \ (< \infty)$;
3) $||A_n - B_n||_{X_n\to Y_n} = O(\varepsilon_n^{(1)})$;
4) $||y_n - z_n|| = O(\varepsilon_n^{(2)})$; $y_n, z_n \in Y_n$;
5) $\lim\limits_{n\to\infty} \varepsilon_n^{(1)} = \lim\limits_{n\to\infty} ||Q_n||\varepsilon_n^{(1)} = \lim\limits_{n\to\infty} \varepsilon_n^{(2)} = 0.$

Then for numbers n *large enough* $(n \ge N_0)$ *the operators* $B_n : X_n \to Y_n$ *are invertible and*

a) $||B_n^{-1}||_{Y_n\to X_n} \le c_2 \ (< \infty)$;
b) $\lim\limits_{n\to\infty} \delta_n = 0$, $\delta_n = ||x_n^* - x_n^{(1)}||$, *and* $\delta_n \le c_3||Q_n||_Y\varepsilon_n^{(1)} + c_4\varepsilon_m^{(2)}$;
c) $||x^* - x_n^{(1)}||_X \le ||x^* - x_n^*|| + ||Q_n||_Y O(\varepsilon_n^{(1)}) + O(\varepsilon_n^{(2)})$.

Theorem 1. determines the stability of methods in case when the projectors Q_n are bounded for all n and the following relation is true for these projectors. $\lim\limits_{n\to\infty} ||Q_n||\varepsilon_n^{(1)} = 0$. The correlation between the condition numbers of approximative and exact solutions are given in the following theorem.

Theorem 2. *Let the operators* A *and* A_n *are linear and invertible as operators acting from* X *to* Y *and from* X_n *to* Y_n *respectively where* $dimY_n \ (< \infty)$. *Let*

$$||A - A_n||_{X_n\to Y} = O(\varepsilon_n); \quad \lim_{n\to\infty} \varepsilon_n = 0, \tag{6}$$

then the condition numbers $\eta(A)$ *and* $\eta(A_n)$ *of operators* A *and* A_n *exist. The following relations hold:*

$$\eta(A_n) \le c\eta_A, \quad 1 \le c \le \frac{1+\varepsilon}{1-\varepsilon}$$
$$\textit{for}\, n \ge N_3(\varepsilon), \tag{7}$$

where ε *is is an arbitrary positive less the unity and*

$$\lim_{n\to\infty} \eta(A_n) = \eta(A). \tag{8}$$

So if the exact solutions of equation (1) are well conditioned then from the conditions of theorem 2. the approximative solution of (2) are also well conditioned

3 Numerical schemes of the collocation methods

The numerical schemes of collocation methods for the approximate solution of SIDE are presented in this section. The theorem of the convergence of the approximate solutions to the exact solution are formulated.[8], [14].

Let U_n be the Lagrange interpolating polynomial operator constructed on the points $\{t_j\}_{j=0}^{2n}$ (n is a natural number) for any continuous function on Γ

$$(U_ng)(t) = \sum_{j=0}^{2n} g(t_j)\cdot l_j(t), \quad t \in \Gamma,$$

where

$$l_j(t) = \left(\frac{t_j}{t}\right)^n \prod_{(k=0,k\ne j)}^{2n} \frac{t-t_k}{t_j-t_k} \equiv$$
$$\equiv \sum_{k=-n}^{n} \Lambda_k^{(j)} t^k, \quad t \in \Gamma. \tag{9}$$

By $H_\beta(\Gamma)$ we denote Hölder space with the exponent β $(0 < \beta < 1)$ and with norm

$$\|g\|_\beta = \|g\|_C + H(g;\beta),$$

$$H(g,\beta) = \sup_{t'\ne t''} \frac{\left|g(t'') - g(t')\right|}{|t' - t''|^\beta}, t', t'' \in \Gamma.$$

By $H_\beta^{(q)}(\Gamma)$ $q = 0, 1, \ldots,$ we denote the space of r times continuously- differentiable functions. The derivatives of the q-th order for these functions are elements of $H_\beta(\Gamma)$ ($g^{(q)} \in H_\beta(\Gamma)$.)

The norm on $H_\beta^{(q)}(\Gamma)$ is given by formula

$$||g||_{\beta,q} = \sum_{k=0}^{q} ||g^{(k)}||_c + H(g^{(q)};\beta). \tag{10}$$

In the complex space $H_\beta(\Gamma)$ we consider the SIDE

$$(Mx \equiv) \sum_{r=0}^{\nu} [\tilde{A}_r(t)x^{(r)}(t) + \tilde{B}_r(t)\frac{1}{\pi i}\int_\Gamma \frac{x^{(r)}(\tau)}{\tau - t}d\tau +$$
$$+\frac{1}{2\pi i}\int_\Gamma K_r(t,\tau)x^{(r)}(\tau)d\tau] = f(t), t \in \Gamma, \tag{11}$$

where $\tilde{A}_r(t), \tilde{B}_r(t)$, $h_r(t,\tau)(r = \overline{0,\nu})$ and $f(t)$ are known functions which belong to $H_\beta(\Gamma)$, $x^{(0)}(t) =$

$x(t)$ is the unknown function from $H_\beta(\Gamma)$, and $x^{(r)}(t) = \dfrac{d^r x}{dt^r}$, $r = \overline{1,\nu}$, ν is positive integer.

We assume that the function $x^{(\nu)}(t)$ belongs to $H_\beta(\Gamma)$, then

$$x^{(k)}(t) \in H_\beta(\Gamma), \quad k = \overline{0,\nu-1}.$$

We search for a solution of equation (11) in the class of functions, satisfying the condition

$$\int_\Gamma x(\tau)\tau^{-k-1}d\tau = 0, \quad k = \overline{0,\nu-1}. \tag{12}$$

Equation (11) with conditions (12) will be denoted as "problem (11), (12) "

We search for the approximate solutions of problem (11), (12) in the polynomial form

$$x_n(t) = \sum_{k=0}^{n} \alpha_k^{(n)} t^{k+\nu} + \sum_{k=-n}^{-1} \alpha_k^{(n)} t^k, \quad t \in \Gamma, \tag{13}$$

where $\alpha_k^{(n)} = \alpha_k$ $(k = \overline{-n,n})$ are unknowns; we note that the function $x_n(t)$, constructed by formula (13), obviously, satisfies the conditions (12).

According to the collocation methods, we determine unknowns α_k $k = \overline{-n,n}$ from the condition

$$(Mx_n)(t_j) - f(t_j) = 0, \tag{14}$$

in $2n+1$ different points $t_j \in \Gamma$, $j = \overline{0,2n}$.

As a result we will obtain a system of linear algebraic equations (SLAE):

$$\sum_{k=-n}^{n}\sum_{r=0}^{\nu}\Big\{\frac{(k+\nu)!}{(k+\nu-r)!}\cdot sign(k)[A_r(t_j)t_j^{k+\nu-r} + \int_\Gamma h_r(t_j,\tau)\cdot\tau^{k+\nu-r}d\tau] + \frac{(k+r-1)!}{(k-1)!}sign(-k)\cdot[(-1)^r B_r(t_j)t_j^{-k-r} + \int_\Gamma K_r(t_j,\tau)\tau^{-k-r}d\tau]\Big\}\alpha_k = f(t_j), \quad j = \overline{0,2n}, \tag{15}$$

where $A_r(t) = \tilde{A}_r(t) + \tilde{B}_r(t)$, $B_r(t) = \tilde{A}_r(t) - \tilde{B}_r(t)$, $r = \overline{0,\nu}$, sign$(k) = 1$, $k \geq 0$, sign$(k) = -1$, $k < 0$.

Let $\overset{o}{H}{}_\beta^{(\nu)}(\Gamma)$ is a subspace of $H_\beta^{(\nu)}(\Gamma)$ space. The elements of $\overset{o}{H}{}_\beta^{(\nu)}(\Gamma)$ are satisfied the condition (12) with the norm as in $H_\beta(\Gamma)$.

Theorem 3. *Let $\Gamma \in \Lambda$ and the following conditions are satisfied:*

1. *the functions $\tilde{A}_r(t), \tilde{B}_r(t), K_r(t,\tau), (r = \overline{0,\nu})$ and $f(t)$ belong to the space $H_\alpha^{(r)}(\Gamma)$; $0 < \alpha < 1, r \geq 0$;*
2. *$A_\nu(t) \cdot B_\nu(t) \neq 0, t \in \Gamma$;*
3. *the index of function $t^\nu B_\nu^{-1}(t) A_\nu(t)$ is equal to zero;*
4. *the operator $V : \overset{o}{H}{}_\beta^{(\nu)}(\Gamma) \to H_\beta(\Gamma)$ is linear and invertible;*
5. *the points $t_j (j = \overline{0,2n})$ form a system of Fejér points [11], [12] on Γ:*

$$t_j = \psi\left[exp\left(\frac{2\pi i}{2n+1}(j-n)\right)\right], \; j = \overline{0,2n};$$

6. $0 < \beta < \alpha < 1$.

Then, beginning with $n \geq n_1$, SLAE (15) has the unique solution α_k, $k = \overline{-n,n}$. The approximate solution $x_n(t)$, constructed by formula (13,) converges when $n \to \infty$ in according to the norm of space $H_\beta(\Gamma)$ to the exact solution $x(t)$ of the problem (11), (12). The following estimation of convergence speed holds:

$$||x - x_n||_{\beta,\nu} = \frac{d_1 + d_2 \ln n}{n^{r+\alpha-\beta}} H(x^{(r)}, \alpha).$$

4 Stability of collocation methods. Condition numbers

Theorem 4. *In conditions of the theorem 3 the collocation methods for the approximate solution of SIDE (11) is stable in Hölder spaces from different of small variations in approximative equations.*

Proof of theorem. From the demonstration of the theorem 3 we obtained that approximative collocation operator A_n starting from the numbers $n \geq n_1$, is invertible as operator acting from $\overset{o}{X}_n$ to X_n, where X_n and $\overset{o}{X}_n$ are defined in [14], [8]

$$||A_n^{-1}|| = O(1), \quad A_n : \overset{o}{X}_n \to X_n.$$

From proof of theorem 3 we have that the operators U_n is bounded in H_β and $X_n = U_n H_\beta$. Using the theorem 1. in conditions $A = M$, $X_n = \overset{o}{X}_n$, $Y_n = U_n H_\beta$; $Q_n = U_n$, $\varepsilon_n^{(1)} = \varepsilon_n^{(2)} = \dfrac{\ln n}{n^{\alpha-\beta}}$, we have the collocation operator A_n. Theorem 4. is proved

Theorem 5. *Let the conditions of theorem 3. be satisfied. Then beginning with the number $n \geq N_1$ exist a condition numbers $\eta(A_n)$ for approximative equations of collocation methods and $\eta(A_n) \leq c \cdot \eta(M)$, $1 \leq c \leq \dfrac{1+\varepsilon}{1-\varepsilon}$, $\varepsilon(>0)$ is an arbitrary small number $n \geq N_1(\varepsilon)$:*

$$\lim_{n\to\infty}(A_n) = \eta(M).$$

From theorem 3 we have,

$$||A_n - M||_{X_n} = const\frac{\ln n}{n^{\alpha-\beta}}.$$

The conditions (6) of the theorem 2 Now the theorem 5. followed by from the relations (7) and (8).

5 Stability of exact SIDE

In this section we study the stability of SIDE exacts in Hölder spaces $H_\beta(\Gamma), \Gamma \in \Lambda$.

Using the Riesz operators we rewrite the SIDE (11) in the form:

$$(Mx \equiv) \sum_{s=0}^{\nu} A_s(t)(Px^{(s)})(t) + B_s(t)(Qx^{(s)})(t) +$$

$$+\frac{1}{2\pi i}\int_\Gamma K_s(t,\tau)x^{(s)}(\tau)d\tau = f(t), \quad t \in \Gamma, \quad (16)$$

where $P = \frac{1}{2}(I+S)$; $Q = I - S$; I identic operator and S is an singular operator.

We consider the SIDE (16) as exact equation.

We suppose that equation (16) has an unique solution. The coefficients, nuclei and right part have small perturbations.

$$||A_s - \hat{A}_s||_c < \varepsilon,\ ||B_s - \hat{B}_s||_c < \varepsilon,$$

$$||f - \hat{f}||_c < \varepsilon,\ ||K_s(t,\tau) - \hat{K}_s(t,\varepsilon)||_c < \varepsilon,$$

$$(t, \tau \in \Gamma, \varepsilon < 1), \quad s = 0, \ldots \nu. \quad (17)$$

The following question appears: if the unique solution $x_\varepsilon(t)$ exists for equation

$$(M_1 x \equiv) \sum_{s=0}^{\nu} \hat{A}_s(t)(Px^{(s)})(t) + \hat{B}_s(t)(Qx^{(s)})(t) +$$

$$+\frac{1}{2\pi i}\int_\Gamma \hat{K}_s(t,\tau)x^{(s)}(\tau)d\tau = \hat{f}(t), \quad t \in \Gamma, \quad (18)$$

if yes we should study the error $\delta_n^{(1)} = ||x^*(t) - x_\varepsilon(t)||$, where $x^*(t)$ is an unique solution for equation (16) and $x_\varepsilon(t)$ is an unique solution for (18)?

Suppose $A_s(t), B_s(t), f(t)$ and $K_s(t,\tau) \in H_\alpha^r(\Gamma)$, $r = 0,1,2,\ldots$, $s = 0,\ldots\nu$ (by both variables).

As It was proved in [9], for small ε the coefficients $\hat{A}_s(t), \hat{B}_s(t)$ şi $\hat{K}_s(t,\tau)$, $s = 0,\ldots\nu$, belong to the $H_\alpha^r(\Gamma)$, $r = 0,1,2,\ldots, s = 0,\ldots\nu$.

We estimate the function norm ΔMx,

$$\Delta Mx \overset{df}{=} (M - M_1)x,$$

in $H_\beta(\Gamma)$ $(0 < \beta < \alpha)$:

$$\Delta Mx = \sum_{s=0}^{\nu}\{[A_s(t) - \hat{A}_s(t)](Px^{(s)})(t) +$$

$$[B_s(t) - \hat{B}_s(t)](Qx^{(s)})(t) +$$

$$\frac{1}{2\pi i}\int_\Gamma [K_s(t,\tau) - \hat{K}_s(t,\tau)]x^{(s)}(\tau)d\tau\}, t \in \Gamma. \quad (19)$$

It is enough to estimate $||\Delta Mx||_c$ and $H(\Delta Mx;\beta)$.

a) $|\Delta Mx|(t)|$:

$$|\Delta Mx)(t)| \leq \sum_{s=0}^{\nu} |[A_s(t) - \hat{A}_s(t)](Px^{(s)})(t)| +$$

$$\sum_{s=0}^{\nu} |[B_s(t) - \hat{B}_s(t)](Px^{(s)})(t)| +$$

$$+\frac{1}{2\pi i}\sum_{s=0}^{\nu}\{\int_\Gamma |K_s(t,\tau) - \hat{K}_s(t,\tau)|$$

$$|x(\tau)||d\tau|\} = M_1 + M_2 + M_3.$$

Taking into consideration that the operators P, Q is bounded in Hölder spaces, (17) and evident equality $||\cdot||_c \leq ||\cdot||_\beta$ for M_1 and M_2, we obtain.

$$M_1 \leq \sum_{s=0}^{\nu}\{|A_s(t) - \hat{A}_s(t)||(Px^{(s)})(t)|\} \leq$$

$$\varepsilon\sum_{s=0}^{\nu}||Px^{(s)}||_\beta \leq \varepsilon||P||_\beta||x||_{\beta,\nu};$$

$$M_2 \leq \sum_{s=0}^{\nu}\{|B_s(t) - \hat{B}_s(t)||(Qx^{(s)})(t)|\}$$

$$\leq \varepsilon \sum_{s=0}^{\nu} ||Qx^{(s)}||_\beta \leq \varepsilon ||Q||_\beta ||x||_{\beta,\nu}.$$

Analog, using (17), we obtain $M_3 \leq \frac{l}{2\pi}\varepsilon ||x||_{c,\nu}$ $\leq \frac{l}{2\pi}\varepsilon ||x||_{\beta,\nu}$ (where l is length of contour Γ).
So,

$$|(\Delta M x)(t)| \leq \varepsilon(||P||_\beta + ||Q||_\beta + \frac{l}{2\pi})||x||_{\beta,\nu}. \quad (20)$$

b) $H(\Delta M x;\beta)$. Let t' and $t'' \in \Gamma$. Then

$$|(\Delta M x)(t') - (\Delta M x)(t'')| \leq$$
$$\sum_{s=0}^{\nu} |[A_s(t') - \hat{A}_s(t')](Px^{(s)})(t') -$$
$$-[A_s(t'') - \hat{A}_s(t'')](Px^{(s)})(t'')| +$$
$$\sum_{s=0}^{\nu} |[B_s(t') - \hat{B}_s(t')](Qx^{(s)})(t') -$$
$$-[B_s(t'') - \hat{B}_s(t'')](Qx^{(s)})(t'')| +$$
$$\sum_{s=0}^{\nu} \frac{1}{2\pi i}\int_\Gamma |[K_s(t',\tau) - \hat{K}_s(t',\tau)] -$$
$$-[K_s(t'',\tau) - \hat{K}_s(t'',\tau)]|x^{(s)}(\tau)||d\tau| = M_4 + M_5 + M_6.$$

We estimate M_4 and M_5.
Let $|t' - t''| \geq \varepsilon$. Then from (17) we have

$$M_4 \leq \sum_{s=0}^{\nu}\{|[A_s(t') - \hat{A}_s(t')]|(Px^{(s)})(t')| +$$
$$+|A_s(t'') - \hat{A}_s(t'')||(Px^{(s)})(t'')|\} \leq$$
$$\leq 2\varepsilon \sum_{s=0}^{\nu} ||Px^{(s)}||_\beta \leq$$
$$2\varepsilon^{1-\beta}\varepsilon^\beta ||P||_\beta ||x||_{\beta,\nu} \leq$$
$$2\varepsilon^{1-\beta}||P||_\beta ||x||_{\beta,\nu}|t' - t''|^\beta.$$

If $|t' - t''| < \varepsilon$, then

$$M_4 \leq \sum_{s=0}^{\nu} |[A_s(t') - \hat{A}_s(t')][(Px^{(s)})(t') -$$
$$(Px^{(s)})(t'')]| +$$
$$+\sum_{s=0}^{\nu} |(Px^{(s)})(t'')[A_s(t') - \hat{A}_s(t') -$$
$$\hat{A}(t'') + \hat{A}_s(t'')]| \leq$$
$$\leq \varepsilon \sum_{s=0}^{\nu} H(Px^{(s)};\beta) + \sum_{s=0}^{\nu} ||Px^{(s)}||_c[H(A_s;\alpha) +$$
$$H(\hat{A}_s,\alpha)]|t' - t''|^\alpha \leq \varepsilon ||P||_\beta ||x||_{\beta,\nu} +$$
$$||P||_\beta ||x||_{\beta,\nu}[H(A_s;\alpha) + H(\hat{A}_s;\alpha)]|t' - t''|^\beta \varepsilon^{\alpha-\beta}.$$

The analog estimations are true for M_5 changing $||P||$ by $||Q||$ and functions $A_s(t)$, $\hat{A}_s(t)$ by $B_s(t)$ and $\hat{B}_s(t)$, $s = \overline{0,\nu}$. So in both cases

$$\sum_{s=0}^{\nu} \frac{|[A_s(t') - \hat{A}_s(t')](Px^{(s)})(t') -}{|t' - t''|^\beta}$$
$$\frac{[A_s(t'') - \hat{A}_s(t'')](Px^{(s)})(t'')|}{|t' - t''|^\beta} \leq c_1\varepsilon^\delta ||x||_{\beta,\nu},$$
$$\sum_{s=0}^{\nu} \frac{|[B_s(t') - \hat{B}_s(t')](Qx^{(s)})(t')}{|t' - t''|^\beta} -$$
$$\frac{[B_s(t'') - \hat{B}_s(t'')](Qx^{(s)})(t'')|}{|t' - t''|^\beta} \leq c_2\varepsilon^\delta ||x||_{\beta,\nu}, \quad (21)$$

where
$$\delta = \min(\beta;\alpha - \beta). \quad (22)$$

For M_6, in similar way we will consider the case $|t' - t''| \geq \varepsilon$. Then

$$M_6 \leq \sum_{s=0}^{\nu}\{\frac{1}{2\pi i}\int_\Gamma |K_s(t',\tau) -$$
$$\hat{K}_s(t',\tau)||x^{(s)}(\tau)|\}|d\tau| +$$
$$+\frac{1}{2\pi i}\sum_{s=0}^{\nu}\{\int_\Gamma |K_s(t'',\tau) -$$
$$-\hat{K}_s(t'',\tau)||x^{(s)}(\tau)||d\tau|\} \leq$$
$$\leq \frac{\varepsilon}{\pi}||x||_{c,\nu} l \leq \frac{\varepsilon^{1-\beta}}{\pi} l ||x||_{\beta,\nu}|t' - t''|^\beta.$$

We used the fact that for functions $K_s(t,\tau)$ and $\hat{K}_s(t,\tau)$ the relation (17) holds.
If $|t' - t''| < \varepsilon$, then

$$M_6 \leq \frac{1}{2\pi i}\sum_{s=0}^{\nu}\int_\Gamma |K_s(t',\tau) -$$
$$K_s(t'',\tau)||x^{(s)}(\tau)||d\tau| +$$

$$+\frac{1}{2\pi i}\sum_{s=0}^{\nu}\int_{\Gamma}|\hat{K}_s(t^{'},\tau)-$$

$$\hat{K}_s(t^{''},\tau)||x^{(s)}(\tau)||d\tau|\leq$$

$$\leq\frac{1}{2\pi i}\sum_{s=0}^{\nu}\{||x^{(s)}||(H(K_s;\alpha)+$$

$$+H(\hat{K}_s;\alpha))\}|t^{'}-t^{''}|^{\alpha}\leq$$

$$\frac{1}{2\pi i}||x||_{c,\nu}\varepsilon^{\alpha-\beta}\sum_{s=0}^{\nu}(H(K_s;\alpha)+$$

$$+H(\hat{K}_s;\alpha))|t^{'}-t^{''}|^{\beta}.$$

From estimations of M_6, from (18) and (22) we obtain

$$||\Delta M||_{H_{\beta}^{(l)}(\Gamma)}\leq c\cdot\varepsilon^{\delta};\quad \delta=\min(\beta;\alpha-\beta). \quad (23)$$

From relation (23) we have for ε enough small the equation (18) has unique solution $x_{\varepsilon}^{*}(t)$.

Using the theory of operator perturbation ([10]) and the relations (23) we can determine the relations between exact solutions $x^{*}(t)$ and $x_{\varepsilon}(t)$ of equations (16) and (18) in spaces $H_{\beta}(\Gamma)$

Taking into consideration the definition of norm in Hölder spaces we obtain

$||x^{*}-x_{\varepsilon}^{*}||_{\beta}=O(\varepsilon^{\delta})$;

Remark The same results we can obtain for Lebesgue and Generalized Hödler spaces.

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