Pipe Analysis by BEM

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Abstract: The BEM is applied to a mixed boundary value problem of linear elastostatics whose region is multiply connected. The sample problem is a hollow pipe on the ground, subjected to a singular, vertical force on the top. The formulation does not involve any singularity.

Key–Words: reciprocity theorem, boundary element method, integral equations, mixed boundary value problem, multiply-connected region

1 Introduction

In the presented study, a two-dimensional mixed boundary value problem is solved by boundary element method for a linear, homogeneous and isotropic material in a multiply connected region. It is known that the reciprocal theorem gives an integral equation which relates two different elastostatic states of the same body. The first elastostatic state in the expression of the reciprocal theorem, represents the problem to be solved, whereas the second one expresses the displacement and stress field in an unbounded medium due to an application of a point load. The second state is also named as a fundamental solution. The aim of the boundary element method is to reduce the problem to a system of linear algebraic equations. Boundary is idealized as a collection of segments. End points of these segments are named as nodal points. On each segment, any unknown function is selected as a certain function passing from the end points. Then the unknowns of the problem are reduced to the values of the displacement/stress components at nodal points. $2N$ integral equations, each one of them corresponding to a singular loading at a nodal point in one direction can be written. In these integral equations, integrals over the boundary are transformed to the summation of the integrals over the segments. The number of unknowns is twice times of the number of nodal points for the first and second boundary value-problems, but in the mixed boundary value problem, it is less than the twice of number of nodal points. At the boundary points, on which the values of surface traction in one side and displacement on the other side are known, an additional equation can be written between the components of the unknown surface traction vector due to symmetry of stress tensor. The construction of the system and the unknowns have been explained on a sample, mixed boundary-value problem which is an hollow pipe on the ground, subjected to a singular pressure force on the top.

2 Basic Formulation

The definition of an elastostatic state is summarized below:
A region B with interior volume V and boundary S is considered. The ordered triple $\mathcal{S}[\boldsymbol{u}(\boldsymbol{x}), \boldsymbol{\tau}(\boldsymbol{x}), \boldsymbol{f}(\boldsymbol{x})]$ defines an elastostatic state on $\overline{V}$, where $\overline{V}$ is the closure of V, $\boldsymbol{u}(\boldsymbol{x})$ is displacement vector and $\boldsymbol{x}$ denote the position vector of a point, $\boldsymbol{\tau}(\boldsymbol{x})$ is the stress tensor and $\boldsymbol{f}$ denotes body force. They satisfy the following relations:

$$\tau_{kj,j} + f_i = 0 \tag{1}$$

$$\tau_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij} \tag{2}$$

$$\epsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \tag{3}$$

where ϵ_{ij} is the strain tensor, δ_{ij} represents Kronecker's delta, λ and μ indicate Lamé's elastic coefficients.

The expression of the reciprocity theorem which is written between two elastostatic states

$\mathcal{S}^\star[\boldsymbol{u}^\star(\boldsymbol{x}),\boldsymbol{\tau}^\star(\boldsymbol{x}),\boldsymbol{f}^\star(\boldsymbol{x})]$ and $\mathcal{S}[\boldsymbol{u}(\boldsymbol{x}),\boldsymbol{\tau}(\boldsymbol{x}),\boldsymbol{f}(\boldsymbol{x})]$ of the same body is [1-4]

$$\int_S \boldsymbol{T}\cdot\boldsymbol{u}^\star dS + \int_V \boldsymbol{f}\cdot\boldsymbol{u}^\star\, dV$$

$$= \int_S \boldsymbol{T}^\star\cdot\boldsymbol{u}\, dS + \int_V \boldsymbol{f}^\star\cdot\boldsymbol{u}\, dV \tag{4}$$

$$T_i = \tau_{ij}\, n_j \quad , \quad T_i^\star = \tau_{ij}^\star\, n_j \tag{5}$$

$\boldsymbol{T}$ and $\boldsymbol{T}^\star$ are surface traction vectors in two states, respectively, $\boldsymbol{n}$ is the outward normal of the surface S. It will be considered that an elastostatic state $\mathcal{S}[\boldsymbol{u}(\boldsymbol{x}),\boldsymbol{\tau}(\boldsymbol{x}),\boldsymbol{f}(\boldsymbol{x})]$ represents a problem to be solved on the region B of volume V bounded by surface S. This problem can be a first, second or mixed boundary-value problem [5]. The body force $\boldsymbol{f}$ will be neglected in the formulation. The second elastostatic state $\mathcal{S}^\star[\boldsymbol{u}^\star(\boldsymbol{x}),\boldsymbol{\tau}^\star(\boldsymbol{x}),\boldsymbol{f}^\star(\boldsymbol{x})]$, represents the displacement and stress fields in an unbounded elastic medium due to a singular point load $\boldsymbol{f}^\star$. The elastostatic state $\mathcal{S}^k[\boldsymbol{u}^k(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{\tau}^k(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{f}^k(\boldsymbol{x},\boldsymbol{y}]$, which have been given here, will be used as $\mathcal{S}^\star[\boldsymbol{u}^\star(\boldsymbol{x}),\boldsymbol{\tau}^\star(\boldsymbol{x}),\boldsymbol{f}^\star(\boldsymbol{x})]$ in Eq. (4) for the solutions of plane elasticity problems.

3 A Singular Elastostatic State for the Solutions of Plane Elasticity Problems

A body force in an infinite elastic medium having the same material with the problem to be solved is defined as

$$\boldsymbol{f}^k(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})\,\boldsymbol{e}_k \tag{6}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ represent the position vectors of an arbitrary point and a specific point of volume V, respectively. $\boldsymbol{e}_k$ $(k=1,2)$ represents a base vector in Cartesian coordinates. $\delta(\boldsymbol{x}-\boldsymbol{y})$ is a generalized function, which is known as Dirac delta function satisfying following property for an infinite volume V:

$$\int_V h(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y})\, dV_x = h(\boldsymbol{y}) \;\; for\ \boldsymbol{y}\in V$$

$$= 0 \qquad for\ \boldsymbol{y}\notin V \tag{7}$$

The displacement field $\boldsymbol{u}^k(\boldsymbol{x},\boldsymbol{y})$ due to this body force can be represented as

$$u_i^k(\boldsymbol{x},\boldsymbol{y}) =$$

$$-\frac{1}{8\pi\mu(1-\nu)}[(3-4\nu)\delta_{ik} ln(r) - \frac{x_k' x_i'}{r^2}] \tag{8}$$

where ν is the Poisson's ratio and

$$x_k' = x_k - y_k \;\; , \;\; r = \sqrt{x_i' x_i'}. \tag{9}$$

Using Eq. (3), the (ij)th component of strain tensor can be written as

$$\epsilon_{ij}^k(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{8\pi\mu(1-\nu)}\Big[-\delta_{ij}\frac{x_k'}{r^2}+$$

$$(1-2\nu)[\delta_{ik}\frac{x_j'}{r^2}+\delta_{jk}\frac{x_i'}{r^2}] + 2\frac{x_k' x_i' x_j'}{r^4}\Big] \tag{10}$$

And substituting Eq. (10) in Eq. (2), the (ij)th component of the stress tensor, $\boldsymbol{\tau}^k(\boldsymbol{x},\boldsymbol{y})$, can also be obtained as

$$\tau_{ij}^k(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{4\pi(1-\nu)}\Big[(1-2\nu)[-\frac{x_k'}{r^2}\delta_{ij}+\frac{x_j'}{r^2}\delta_{ik}$$

$$+\frac{x_i'}{r^2}\delta_{kj}] + 2\frac{x_i' x_j' x_k'}{r^4}\Big] \tag{11}$$

The expression of reciprocal identity (Eq. (4)) which is written between $\mathcal{S}[\boldsymbol{u}(\boldsymbol{x}),\boldsymbol{\tau}(\boldsymbol{x}),\boldsymbol{0}]$ and $\mathcal{S}^k[\boldsymbol{u}^k(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{\tau}^k(\boldsymbol{x},\boldsymbol{y}),\boldsymbol{f}^k(\boldsymbol{x},\boldsymbol{y})]$ is reduced to the following form:

$$\int_S u_i^k(\boldsymbol{x},\boldsymbol{y}).T_i(\boldsymbol{x})dS - \int_S T_i^k(\boldsymbol{x},\boldsymbol{y}).u_i(\boldsymbol{x})dS \tag{12}$$

$$= \begin{cases} u_k(\boldsymbol{y}) & (\boldsymbol{y}\in V) \\ 0 & (\boldsymbol{y}\notin V) \end{cases}$$

It is clear that if the boundary values of $\boldsymbol{T}(\boldsymbol{x})$ and $\boldsymbol{u}(\boldsymbol{x})$ are known on the boundary S, displacement vector at an inner point $\boldsymbol{y}$ can be determined using Eq. (12). Besides the stress components can also be calculated at this point using Eq. (12) and Eqs. (2) and (3). This expression has been given below:

$$\tau_{kl}(\boldsymbol{y}) = \int_S u_i^{kl}(\boldsymbol{x},\boldsymbol{y}).T_i(\boldsymbol{x})dS_x$$

$$-\int_S \tau_{ij}^{kl}(\boldsymbol{x},\boldsymbol{y})n_j.u_i(\boldsymbol{x})dS_x \tag{13}$$

where

$$u_i^{kl}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi(1-\nu)}$$

$$\Big[(1-2\nu)[\delta_{ik}\frac{x'_l}{r^2} + \delta_{il}\frac{x'_k}{r^2} - \delta_{kl}\frac{x'_i}{r^2}] + 2\frac{x'_i x'_k x'_l}{r^4}\Big] \quad (14)$$

$$\tau_{ij}^{kl}(\boldsymbol{x},\boldsymbol{y}) = \frac{\mu}{2\pi(1-\nu)}$$

$$\Big[-(1-4\nu)\delta_{ij}\delta_{kl}\frac{1}{r^2} + (1-2\nu)[\delta_{il}\delta_{jk}\frac{1}{r^2} + \delta_{ik}\delta_{jl}\frac{1}{r^2}]$$

$$+2(1-2\nu)[\delta_{ij}\frac{x'_k x'_l}{r^4} + \delta_{kl}\frac{x'_i x'_j}{r^4}] +$$

$$2\nu[\delta_{ik}\frac{x'_j x'_l}{r^4} + \delta_{il}\frac{x'_j x'_k}{r^4} + \delta_{jl}\frac{x'_i x'_k}{r^4} + \delta_{jk}\frac{x'_i x'_l}{r^4}]$$

$$-8\frac{x'_i x'_j x'_k x'_l}{r^6}\Big] \quad (15)$$

4 Sample Problem

An hollow pipe settled on the ground, subjected to a singular vertical pressure force acting at the point $\boldsymbol{y_3}$, $P = 702.69\ N$, is considered as shown in Fig. 1 [6]. The Poisson's ratio, ν is 0.4. The problem is considered as a plane stress problem. The third dimension of the column is $152.4mm$.

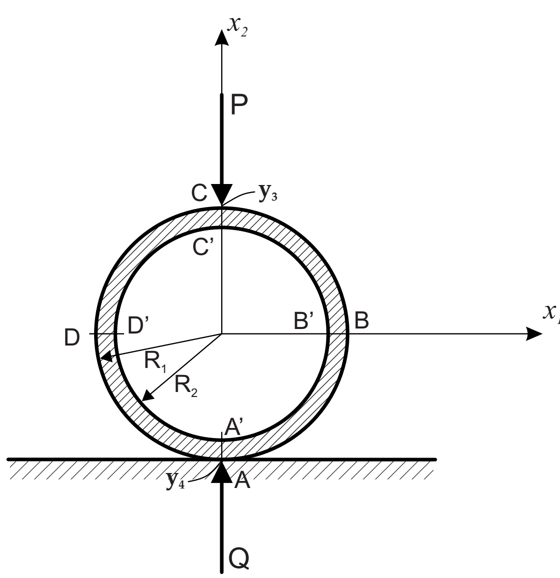

Fig. 1. Sample problem

The boundary conditions of the problem are as follows:

The surface tractions on the ABCD part of the boundary can be defined as

$$\boldsymbol{T} = [-P\delta(\boldsymbol{x}-\boldsymbol{y}_3) + Q\delta(\boldsymbol{x}-\boldsymbol{y}_4)]\boldsymbol{e}_2 \quad (16)$$

The displacement components on A and C points are

$$u_1(\boldsymbol{y_4}) = u_2(\boldsymbol{y_4}) = 0 \quad and \quad u_1(\boldsymbol{y_3}) = 0 \quad (17)$$

And the integrals over boundary are reduced to line integrals. ABCD and A′ D′ C′ B′ parts of the boundary are named as L_1 and L_2, respectively. The surface tractions are known on L_1 and L_2 while displacements are known on $\boldsymbol{y_4}$. And, the problem is symmetric with respect to x_2 axis. From now on, $\mathcal{S}[\boldsymbol{u}(\boldsymbol{x}), \boldsymbol{\tau}(\boldsymbol{x}), \boldsymbol{0}]$ will represent the problem mentioned above. Substituting Eqs. (5) and (16) in Eq. (12), the following two integral equations given below are found:

$$\int_{L_1} T_i^k(\boldsymbol{x},\boldsymbol{y}) u_i(\boldsymbol{x},t) dL_1 + \int_{L_2} T_i^k(\boldsymbol{x},\boldsymbol{y}) u_i(\boldsymbol{x},t) dL_2$$

$$= P[u_2^k(\boldsymbol{y_4},\boldsymbol{y}) - u_2^k(\boldsymbol{y_3},\boldsymbol{y})] \quad (k=1,2) \quad (18)$$

From now on, it will be considered that point $\boldsymbol{y}$ is outside the planar surface. Then the unknowns of the problem become the displacement vector $\boldsymbol{u}(\boldsymbol{x})$ on L_1 and L_2. Besides during integrations over the boundary, one must keep the region on the left.
The procedure which will be used to solve these unknowns by Boundary Element Method has been explained below step by step:
The total boundary $(L_1 + L_2)$ is idealized as a collection of line segments which named as boundary elements. If the number of these line segments is N, the number of the end points, named as nodal points, is also N. The starting and end points of Jth element are $\boldsymbol{x}_{(J)}$ and $\boldsymbol{x}_{(J+1)}$. It is assumed that the variation of any displacement component on the J'th line segment has the following form.

$$u_k(s) = u_k(_J)[1 - \frac{s}{l_{(J)}}] + u_k(_{J+1})[\frac{s}{l_{(J)}}] \quad (k=1,2) \quad (19)$$

Where s is the distance from $\boldsymbol{x}(_J)$ to any point between $\boldsymbol{x}(_J)$ and $\boldsymbol{x}(_{J+1})$. After these definitions the unknowns of the problem will be reduced to the nodal values of displacement components on L_1 and L_2. Both L_1 and L_2 parts have been divided to N1 intervals. Then the number of the nodal points becomes N=2N1. The selection of the nodal points are shown in Fig. 2.

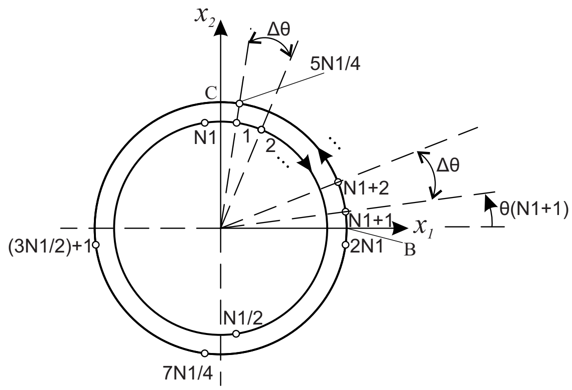

Fig. 2. Nodal points

It must be emphasized that none of the A′, A, C′, C points are not selected as nodal points. On inner and outer boundaries, the position of a nodal point, $\boldsymbol{x}(\mathrm{I})$, on a circle can be defined by an angle $\theta(\mathrm{I})$ and $|\theta(\mathrm{I+1}) - \theta(\mathrm{I})| = 2\pi/\mathrm{N1}$. Depending upon these, the numbers of the unknowns and their order can be expressed as follows:

First $(\mathrm{N1/2})$ unknowns are the horizontal displacement components on nodal points on L_2 starting from $u_1(\mathrm{1})$.

The following $(\mathrm{N1/2})$ unknowns of the problem are the vertical displacement components on nodal points L_2 starting from $u_2(\mathrm{1})$.

The third group of unknowns will be the horizontal components of the displacement vector on nodal points of BC. Starting from $u_1(\mathrm{N1+1})$ till $u_1(\mathrm{N1+(N1/4)})$.

The fourth group of the unknowns are horizontal components of the displacements vector on DA starting from $u_1(\mathrm{N1+(N1/2)+1})$ till $u_1(\mathrm{N1+(3N1/4)})$.

The fifth group of unknowns will be the vertical components of the displacement vector on nodal points of CB. Starting from $u_2(\mathrm{N1+1})$ till $u_2(\mathrm{N1+(N1/4)})$.

The last group of the unknowns are vertical components of the displacements vector on DA starting from $u_2(\mathrm{N1+(N1/2)+1})$ till $u_2(\mathrm{N1+(3N1/4)})$.

Here, it must be also emphasized that $(\mathrm{N1})$ is selected to be an integer having 4 as a factor. And, the total numbers of the unknowns become $\mathrm{M} = \mathrm{2N1\text{-}1}$. Since $\boldsymbol{u(y_4)} = 0$ and we use line elements the last unknown of the last group vanishes. To determine these unknowns $\mathrm{M{=}2N1\text{-}1}$ equations is necessary. Any of these equations can be written selecting loading point $\boldsymbol{y}$ to be any nodal point $\boldsymbol{x}(\mathrm{I})$ and k being equal to 1 or 2 in Eq. (18). But $\boldsymbol{x}(\mathrm{I})$ is a boundary point of the planar region. Because of this an artificial boundary including all of the line segments but not the nodal point $\boldsymbol{x}(\mathrm{I})$, will be defined for a singular loading on that nodal point. Around $\boldsymbol{x}(\mathrm{I})$ a small circular arc L_ϵ, with radius ϵ which leaves this nodal point outside the region is added to complete this artificial boundary [1,4], (Fig. 3).

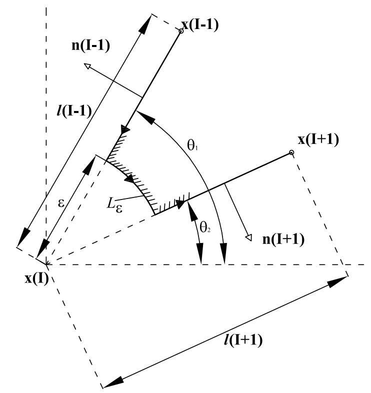

Fig. 3. Artificial boundary

It is assumed that any displacement component is being equal to $u_k(\boldsymbol{x}(\mathrm{I}))$ and any component of surface traction vector is zero over this circular arc. As a consequence of the definition of the artificial boundary, $\boldsymbol{x}(\mathrm{I})$ is not a point of the region bounded by this artificial boundary. After necessary calculations, the radius ϵ will be shrunk to the nodal point $\boldsymbol{x}(\mathrm{I})$. The first assumption on circular arc, L_ϵ, means that any displacement component at a nodal point is single valued. The second assumption is that there is not a singular force acting at that nodal point. Then, if a singular force exists at a point of the boundary, this point must not be selected as a nodal point either. After these Eq. (18) takes the following form

$$\begin{aligned} &\int_{L_1} T_i^k(\boldsymbol{x}, \boldsymbol{x}(\mathrm{I})).u_i(\boldsymbol{x})dL_1 \\ &+ \int_{L_2} T_i^k(\boldsymbol{x}, \boldsymbol{x}(\mathrm{I})).u_i(\boldsymbol{x})dL_2 \\ &+ \lim_{\epsilon \to 0}\Big(\int_{L_\epsilon} T_i^k(\boldsymbol{x}, \boldsymbol{x}(\mathrm{I})).u_i(\boldsymbol{x}(\mathrm{I}))dL_\epsilon\Big) \\ &= P[u_2^k(\boldsymbol{y_4}, \boldsymbol{x}(\mathrm{I})) - u_2^k(\boldsymbol{y_4}, \boldsymbol{x}(\mathrm{I}))] \quad (k = 1, 2) \quad (20) \end{aligned}$$

Where k represents the direction of the loading. At first an $4N1 \times (4N1 + 1)$ augmented matrix will be constructed as follows:

I'th of the first N1 equations represents an horizontal loading at $\boldsymbol{x}(\mathrm{I})$ on L_2 while I'th of the second N1 equations corresponds to a vertical loading at the same point. Changing the loading points to the points of L_1 from the points of L_2 the third and the fourth N1 equations are also constructed. This augmented matrix can be reduced another $2N1 \times (2N1 + 1)$ augmented matrix considering symmetry with respect to x_2 axis and deleting the last row and the $2N1$'th column of it, the last augmented matrix, having the order $(2N1 - 1) \times (2N1)$, is found. The solution of it gives the displacement components of the nodal points on inner and outer boundaries. The variation of the horizontal and vertical of displacement vector on

L_1 and L_2 versus polar angle θ are given in Figs. 4-7.

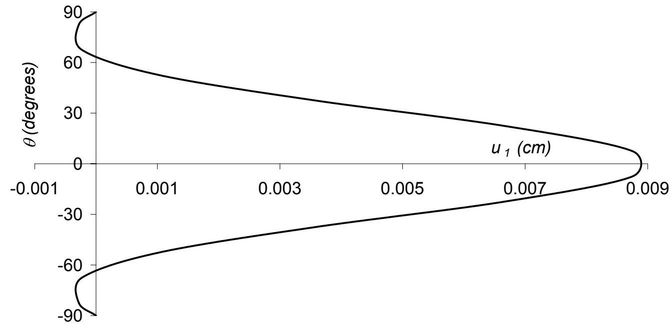

Fig. 4. Variation of the horizontal component of displacement vector on ABC versus polar angle, θ

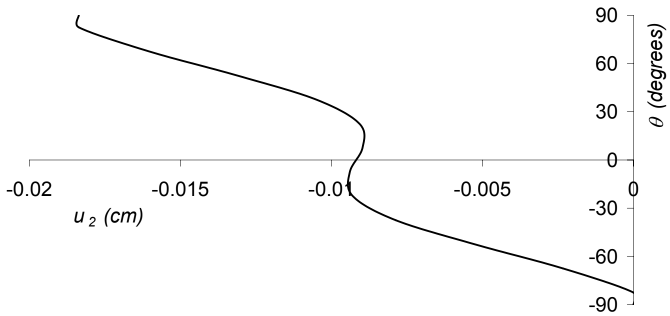

Fig. 5. Variation of the vertical component of displacement vector on ABC versus polar angle, θ

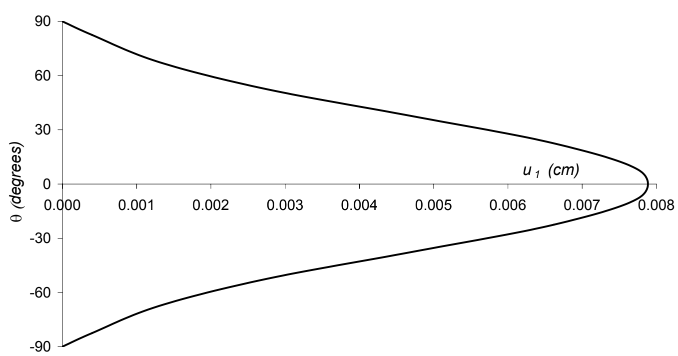

Fig. 6. Variation of the horizontal component of displacement vector on A′B′C′ versus polar angle, θ

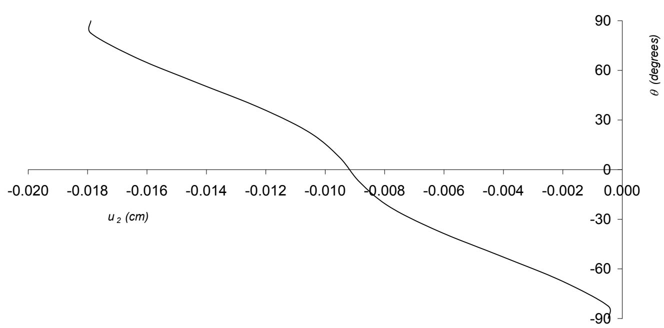

Fig. 7. Variation of the vertical component of displacement vector on A′B′C′ versus polar angle, θ

Using Eq. (13), the stress components are also calculated on a circle with radius $R = 16.64cm$, which is the mean radius, versus polar angle θ and the results have been given in Figs. 8-10.

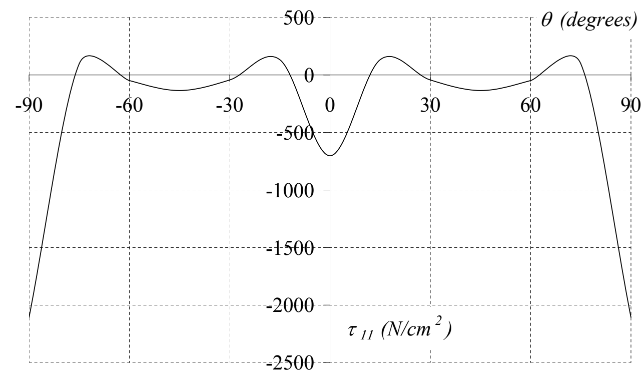

Fig. 8. Variation of stress component, τ_{11} versus angle θ on the mean radius

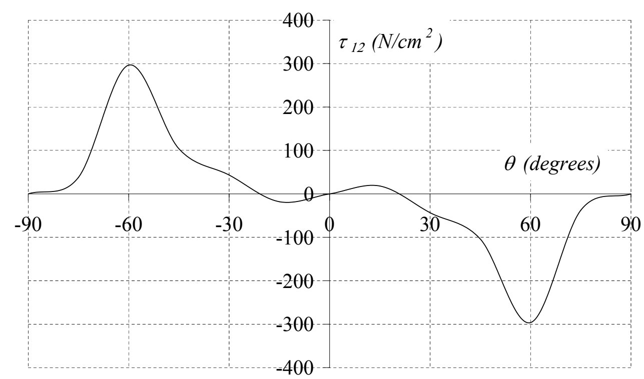

Fig. 9. Variation of stress component, τ_{12} versus angle θ on the mean radius

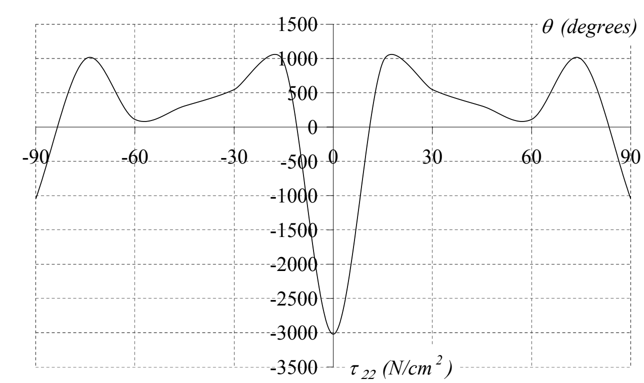

Fig. 10. Variation of stress component, τ_{22} versus angle θ on the mean radius

5 Conclusions and Discussion

A solution method of plane problems of linear elasticity has been explained on a sample mixed-boundary value problem. This problem has been considered as a plane stress problem. The reciprocity theorem provides a relation between displacements, traction components and body forces for two loading states of the same body and this relation gives a boundary integral equation for unknown fields on the boundary, complementary to the applied fields. This integral equation has been solved numerically. The selected approximations for unknowns are linear and the integral equation is reduced to a system of algebraic equations. Of course, higher order polynomials can be selected for a better approximation but it must be emphasized that the solution is heavily dependent to the diagonal terms of the coefficient matrix which remain constant for every approximation.

Here, the element number is selected to be 48 on total boundary. The increment of the element number slightly affect the result after 48. It is assumed that the contact between the pipe and ground is valid only at point A in the presented problem. In the following study, the contact line will be taken as another unknown of the problem and also the variation of the surface traction vector on this contact line will also produce another group of unknowns.

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