A Two-Dimensional Mixed Boundary-Value Problem by Boundary Element Method

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Abstract: A fundamental solution for the plane problems of linear elasticity is introduced. The reciprocal identity, gives an integral equation, is written between the fundamental solution and the problem to be solved. This integral equation has been solved by boundary element and algorithm of the BEM solution is explained on a sample, mixed boundary-value problem. The formulation is valid for the first, second and mixed boundary-value problems. The formulation does not involve any singularity.

Key–Words: reciprocity theorem, boundary element method, integral equations, mixed boundary value problem

1 Introduction

In the presented study, a two-dimensional mixed boundary value problem is solved by boundary element method for a linear, homogeneous and isotropic material. In the mixed boundary value problem, the surface tractions were known on a part of the boundary while displacement components on the other part. Reciprocity theorem is the starting point of the boundary element method. It is known that the reciprocal theorem gives an integral equation which relates two different elastostatic states of the same body. The first elastostatic state in the expression of the reciprocal theorem, represents the problem to be solved, whereas the second one expresses the displacement and stress field in an unbounded medium due to an application of a point load. The second state is also named as a fundamental solution. The aim of the boundary element method is to reduce the problem to a system of linear algebraic equations. Boundary is idealized as a collection of segments. End points of these segments are named as nodal points. On each segment, any unknown function is selected as a certain function passing from the end points. Then the unknowns of the problem are reduced to the values of the displacement/stress components at nodal points. $2N$ integral equations, each one of them corresponding to a singular loading at a nodal point in one direction can be written. In these integral equations, integrals over the boundary are transformed to the summation of the integrals over the segments. The construction and the unknowns of this system are different for the first, second and mixed boundary-value problems. There is no problem for the first boundary-value problems because it is easy to place the dominant terms on the main diagonal in the construction of the coefficient matrix of the system of the linear algebraic equations mentioned above. For a mixed boundary value problem, the order of the equations and the unknowns should be arranged so that dominant terms are on the main diagonal of the coefficients matrix. The number of unknowns is twice times of the number of nodal points for the first and second boundary value-problems, but in the mixed boundary value problem, it is less than the twice of number of nodal points. In the boundary points on which the values of surface traction in one side and displacement on the other side are known, an additional equation can be written between the components of the unknown surface traction vector due to symmetry of stress tensor. The construction of the system and the unknowns have been explained on a sample, mixed boundary-value problem which is a thick and wide concrete column subjected to a singular and eccentric normal force. The displacement components on a part of the boundary and the surface tractions on a second part of the boundary have been determined. Results have been compared by finite element solution of the same problem.

2 Basic Formulation

The definition of an elastostatic state is summarized below:
A region B with interior volume V and boundary S is considered. The ordered triple $\mathcal{S}[\boldsymbol{u}(\boldsymbol{x}), \boldsymbol{\tau}(\boldsymbol{x}), \boldsymbol{f}(\boldsymbol{x})]$ defines an elastostatic state on $\overline{V}$, where $\overline{V}$ is the closure of V, $\boldsymbol{u}(\boldsymbol{x})$ is displacement vector and $\boldsymbol{x}$ denote

the position vector of a point, $\boldsymbol{\tau}(\boldsymbol{x})$ is the stress tensor and $\boldsymbol{f}$ denotes body force. They satisfy the following relations:

$$\tau_{kj,j} + f_i = 0 \tag{1}$$

$$\tau_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} \tag{2}$$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3}$$

where ϵ_{ij} is the strain tensor, δ_{ij} represents Kronecker's delta, λ and μ indicate Lamé's elastic coefficients having the following relation between them

$$\frac{\lambda}{\mu} = \frac{2\nu}{1-2\nu}$$

here ν indicates the Poisson's ratio.

The expression of the reciprocity theorem which is written between two elastostatic states $\mathcal{S}^\star[\boldsymbol{u}^\star(\boldsymbol{x}), \boldsymbol{\tau}^\star(\boldsymbol{x}), \boldsymbol{f}^\star(\boldsymbol{x})]$ and $\mathcal{S}[\boldsymbol{u}(\boldsymbol{x}), \boldsymbol{\tau}(\boldsymbol{x}), \boldsymbol{f}(\boldsymbol{x})]$ of the same body is [1,2,3]

$$\int_S \boldsymbol{T}\cdot\boldsymbol{u}^\star dS + \int_V \boldsymbol{f}\cdot\boldsymbol{u}^\star\, dV$$

$$= \int_S \boldsymbol{T}^\star\cdot\boldsymbol{u}\, dS + \int_V \boldsymbol{f}^\star\cdot\boldsymbol{u}\, dV \tag{4}$$

$$T_i = \tau_{ij}\, n_j \quad , \quad T_i^\star = \tau_{ij}^\star\, n_j \tag{5}$$

$\boldsymbol{T}$ and $\boldsymbol{T}^\star$ are surface traction vectors in two states, respectively, $\boldsymbol{n}$ is the outward normal of the surface S. It will be considered that an elastostatic state $\mathcal{S}[\boldsymbol{u}(\boldsymbol{x}), \boldsymbol{\tau}(\boldsymbol{x}), \boldsymbol{f}(\boldsymbol{x})]$ represents a problem to be solved on the region B of volume V bounded by surface S. This problem can be a first, second or mixed boundary-value problem [4]. The body force $\boldsymbol{f}$ will be neglected in the formulation. The second elastostatic state $\mathcal{S}^\star[\boldsymbol{u}^\star(\boldsymbol{x}), \boldsymbol{\tau}^\star(\boldsymbol{x}), \boldsymbol{f}^\star(\boldsymbol{x})]$, represents the displacement and stress fields in an unbounded elastic medium due to a singular point load $\boldsymbol{f}^\star$. The elastostatic state $\mathcal{S}^k[\boldsymbol{u}^k(\boldsymbol{x},\boldsymbol{y}), \boldsymbol{\tau}^k(\boldsymbol{x},\boldsymbol{y}), \boldsymbol{f}^k(\boldsymbol{x},\boldsymbol{y}]$, which have been given here, will be used as $\mathcal{S}^\star[\boldsymbol{u}^\star(\boldsymbol{x}), \boldsymbol{\tau}^\star(\boldsymbol{x}), \boldsymbol{f}^\star(\boldsymbol{x})]$ in Eq. (5) for the solutions of plane elasticity problems.

3 A Singular Elastostatic State for the Solutions of Plane Elasticity Problems

A body force in an infinite elastic medium having the same material with the problem to be solved is defined as

$$\boldsymbol{f}^k(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})\,\boldsymbol{e}_k \tag{6}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ represent the position vectors of an arbitrary point and a specific point of volume V, respectively. $\boldsymbol{e}_k$ $(k = 1, 2)$ represents a base vector in Cartesian coordinates. $\delta(\boldsymbol{x}-\boldsymbol{y})$ is a generalized function, which is known as Dirac delta function satisfying following property for an infinite volume V:

$$\int_V h(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{y})\, dV_x = h(\boldsymbol{y}) \;\; for\ \boldsymbol{y}\in V$$

$$= 0 \qquad for\ \boldsymbol{y}\notin V \tag{7}$$

The displacement field $\boldsymbol{u}^k(\boldsymbol{x},\boldsymbol{y})$ due to this body force can be represented as

$$u_i^k(\boldsymbol{x},\boldsymbol{y}) =$$

$$-\frac{1}{8\pi\mu_o(1-\nu)}[(3-4\nu)\delta_{ik}ln(r) - \frac{x'_k x'_i}{r^2}] \tag{8}$$

where

$$x'_k = x_k - y_k \;\; , \;\; r = \sqrt{x'_i x'_i} \tag{9}$$

Using Eq. (3), the (ij)th component of strain tensor can be written as

$$\epsilon_{ij}^k(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{8\pi\mu_o(1-\nu)}\Big[-\delta_{ij}\frac{x'_k}{r^2} +$$

$$(1-2\nu)[\delta_{ik}\frac{x'_j}{r^2} + \delta_{jk}\frac{x'_i}{r^2}] + 2\frac{x'_k x'_i x'_j}{r^4}\Big] \tag{10}$$

And substituting Eq. (10) in Eq. (2), the (ij)th component of the stress tensor, $\boldsymbol{\tau}^k(\boldsymbol{x},\boldsymbol{y})$, can also be obtained as

$$\tau_{ij}^k(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{4\pi(1-\nu)}\Big[(1-2\nu)[-\frac{x'_k}{r^2}\delta_{ij} + \frac{x'_j}{r^2}\delta_{ik}$$

$$+\frac{x'_i}{r^2}\delta_{kj}] + 2\frac{x'_i x'_j x'_k}{r^4}\Big] \tag{11}$$

The expression of reciprocal identity (Eq. (4)) which is written between $\mathcal{S}[\boldsymbol{u}(\boldsymbol{x}), \boldsymbol{\tau}(\boldsymbol{x}), \boldsymbol{f}(\boldsymbol{x})]$ and $\mathcal{S}^k[\boldsymbol{u}^k(\boldsymbol{x},\boldsymbol{y}), \boldsymbol{\tau}^k(\boldsymbol{x},\boldsymbol{y}), \boldsymbol{f}^k(\boldsymbol{x},\boldsymbol{y})]$ is reduced to the following form:

$$\int_S u_i^k(\boldsymbol{x},\boldsymbol{y}).T_i(\boldsymbol{x})dS + \int_V u_i^k(\boldsymbol{x},\boldsymbol{y}).f_i(\boldsymbol{x})dV$$

$$-\int_S T_i^k(\boldsymbol{x},\boldsymbol{y}).u_i(\boldsymbol{x})dS \tag{12}$$

$$= \begin{cases} u_k(\boldsymbol{y}) & (\boldsymbol{y}\in V) \\ 0 & (\boldsymbol{y}\notin V) \end{cases}$$

It is clear that if the boundary values of $\boldsymbol{T}(\boldsymbol{x})$ and $\boldsymbol{u}(\boldsymbol{x})$ are known on the boundary S displacement vector at an inner point $\boldsymbol{y}$ can be determined using Eq. (12). Besides the stress components can also be calculated at this point using Eq. (12) and Eqs. (2) and (3). This expression is given below:

$$\tau_{kl}(\boldsymbol{y}) = \int_S u_i^{kl}(\boldsymbol{x},\boldsymbol{y}).T_i(\boldsymbol{x})dS_x$$

$$+\int_V u_i^{kl}(\boldsymbol{x},\boldsymbol{y}).f_i(\boldsymbol{x})dV_x$$

$$-\int_S \tau_{ij}^{kl}(\boldsymbol{x},\boldsymbol{y})n_j.u_i(\boldsymbol{x})dS_x \tag{13}$$

where

$$u_i^{kl}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi(1-\nu)}$$

$$\Big[(1-2\nu)[\delta_{ik}\frac{x_l'}{r^2}+\delta_{il}\frac{x_k'}{r^2}-\delta_{kl}\frac{x_i'}{r^2}]+2\frac{x_i'x_k'x_l'}{r^4}\Big] \tag{14}$$

$$\tau_{ij}^{kl}(\boldsymbol{x},\boldsymbol{y}) = \frac{\mu}{2\pi(1-\nu)}$$

$$\Big[-(1-4\nu)\delta_{ij}\delta_{kl}\frac{1}{r^2}+(1-2\nu)[\delta_{il}\delta_{jk}\frac{1}{r^2}+\delta_{ik}\delta_{jl}\frac{1}{r^2}]$$

$$+2(1-2\nu)[\delta_{ij}\frac{x_k'x_l'}{r^4}+\delta_{kl}\frac{x_i'x_j'}{r^4}]+$$

$$2\nu[\delta_{ik}\frac{x_j'x_l'}{r^4}+\delta_{il}\frac{x_j'x_k'}{r^4}+\delta_{jl}\frac{x_i'x_k'}{r^4}+\delta_{jk}\frac{x_i'x_l'}{r^4}]$$

$$-8\frac{x_i'x_j'x_k'x_l'}{r^6}\Big] \tag{15}$$

4 Sample Problem

A vertical concrete column under an eccentrical normal force $P = 1000\ kN$ is considered acting at a point $\boldsymbol{y_3}$ as shown in Fig. 1. The body force will be neglected and the Poisson's ratio, ν is 0.2. The problem is considered as a plane stress problem. The third dimension of the column is $0.4m$.

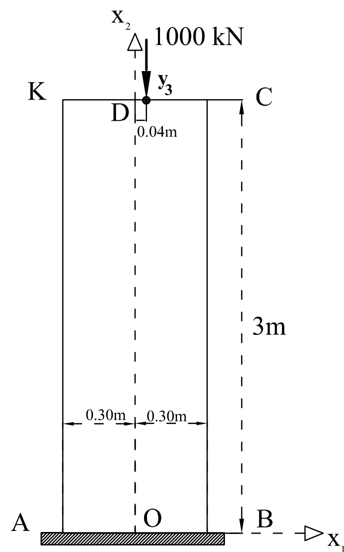

Fig. 1. Sample problem

Then body force $\boldsymbol{f}$ will be defined as

$$\boldsymbol{f} = \boldsymbol{0} \tag{16}$$

The boundary conditions of the problem are as follows:

The surface tractions on the BCDKA part of the boundary can be defined as

$$\boldsymbol{T} = -P\delta(\boldsymbol{x}-\boldsymbol{y}_3)\boldsymbol{e}_2 \tag{17}$$

The displacement components on AB part of the boundary can be written as

$$u_1(\boldsymbol{x}) = u_2(\boldsymbol{x}) = 0 for\ x_2 = 0, x_1 \in [-0.3, 0.3] \tag{18}$$

Since the problem is a plane problem, volume V and surface S came out to be a planar area and the summation of plane lines, respectively. And the integrals over boundary are reduced to line integrals. BCKA and AB parts of the boundary are named as L_1 and L_2, respectively. The surface tractions are known on L_1 while displacements are known on L_2. Because of these, problem is a mixed boundary-value problem. From now on, $\mathcal{S}[\boldsymbol{u}(\boldsymbol{x}),\boldsymbol{\tau}(\boldsymbol{x}),\boldsymbol{0})]$ will represent the problem mentioned above. Substituting Eqs. (16) to (18) in Eq. (12), the following two integral equations given below are found:

$$-u_2^k(\boldsymbol{y_3},\boldsymbol{y})P + \int_{L_2} u_i^k(\boldsymbol{x},\boldsymbol{y}).T_i(\boldsymbol{x})dL_2$$

$$-\int_{L_1} T_i^k(\boldsymbol{x},\boldsymbol{y}).u_i(\boldsymbol{x},t)dL_1 =$$

$$\begin{cases} u_k(\boldsymbol{y})(if\ \boldsymbol{y}\ is\ an\ inner\ point\ of\ BCKA\ plane \\ region) \qquad\qquad (k=1,2) \\ 0\ (if\ \boldsymbol{y}\ is\ outside\ of\ BCKA\ plane\ region) \end{cases} \tag{19}$$

From now on, it will be considered that point $\boldsymbol{y}$ is outside the planar surface. Then the unknowns of the problem become the surface traction vector $\boldsymbol{T}(\boldsymbol{x})$ on L_2 and the displacement vector $\boldsymbol{u}(\boldsymbol{x})$ on L_1. Besides during integrations over the boundary, one must keep the region on the left.

The procedure which will be used to solve these unknowns by Boundary Element Method has been explained below step by step:

The total boundary $(L_1 + L_2)$ is idealized as a collection of line segments which named as boundary elements. If the number of these line segments is N, the number of the end points, named as nodal points, is also N. The starting and end points of Jth element are $\boldsymbol{x}_{(J)}$ and $\boldsymbol{x}_{(J+1)}$. It is assumed that the variation of any displacement or stress component on the Jth line segment has the following form.

$$u_k(s) = u_k(J)[1 - \frac{s}{l_{(J)}}] + u_k(J{+}1)[\frac{s}{l_{(J)}}] \;\; (k = 1, 2) \quad (20)$$

$$T_k(s) = T_k(J)[1 - \frac{s}{l_{(J)}}] + T_k(J{+}1)[\frac{s}{l_{(J)}}] \;\; (k = 1, 2) \quad (21)$$

where s is the distance from $\boldsymbol{x}(J)$ to any point between $\boldsymbol{x}(J)$ and $\boldsymbol{x}(J{+}1)$. After these definitions the unknowns of the problem will be reduced to the nodal values of displacement components on L_1 and the surface traction vectors on L_2. Both BC and KA lines have been divided to $N1$ intervals while CK and AB divided to $N2$. Then the number of the nodal points becomes $N{=}2N1{+}2N2$. And point B is selected as the last nodal point having the nod number N. After this selection the nodal numbers of C, K, A points become $N1$, $(N1{+}N2)$ and $(2N1{+}N2)$ respectively, (Fig. 2).

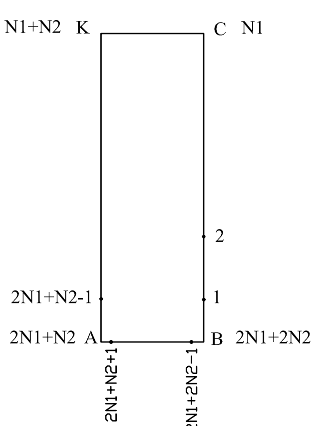

Fig. 2. Nodal points

Depending upon these, the numbers of the unknowns and their order can be expressed as follows:

First $(2N1{+}N2{-}1)$ unknowns are the horizontal displacement components on nodal points of L_1 starting from $u_1(1)$. The following $(2N1{+}N2{-}1)$ unknowns of the problem are the vertical displacement components on nodal points L_1 starting from $u_2(1)$. The third group of unknowns will be the horizontal component of the surface traction vector on nodal points on L_2. The values of this quantity are equal to zero for this problem at both A and B points. Then the first and last elements of this group, having $N2{-}1$ unknowns, will be $T_1(2N1{+}N2{+}1)$ and $T_1(2N1{+}2N2{-}1)$. The last group of $N2{+}1$ unknowns are the vertical components of the surface traction vector on nodal points on L_2 starting from $T_2(2N1{+}N2)$. And the total numbers of the unknowns becomes $M = 4N1{+}4N2{-}2$. To determine these unknowns M equations is necessary. Any of these equations can be written selecting loading point $\boldsymbol{y}$ to be any nodal point $\boldsymbol{x}(I)$ and k being equal to 1 or 2 in Eq. (19). But $\boldsymbol{x}(I)$ is a boundary point of the planar region. Because of this an artificial boundary including all of the line segments but not the nodal point $\boldsymbol{x}(I)$, will be defined for a singular loading on that nodal point. Around $\boldsymbol{x}(I)$ a small circular arc L_ϵ, with radius ϵ which leaves this nodal point outside the region is added to complete this artificial boundary [1,2], (Fig. 3).

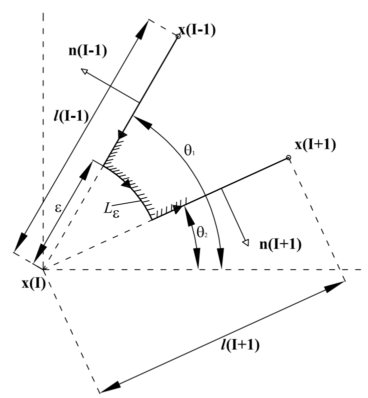

Fig. 3. Artificial boundary

It is assumed that any displacement component is being equal to $u_k(\boldsymbol{x}(I))$ and any component of surface traction vector is zero over this circular arc. As a consequence of the definition of the artificial boundary, when the loading point is $\boldsymbol{x}(I)$, right side of Eq. (19) becomes zero because $\boldsymbol{x}(I)$ is not a point in the region bounded by this artificial boundary. After necessary calculations, the radius ϵ will be shrunk to the nodal point $\boldsymbol{x}(I)$. The first assumption on circular arc, L_ϵ, means that any displacement component at a nodal point is single valued. The second assumption is that there is not a singular force acting at that nodal point. Then if a singular force exists at a point of the boundary, this point must not be selected as a nodal point either. After these Eq. (19) takes the following form

$$\int_{L_1} T_i^k(\boldsymbol{x}, \boldsymbol{x}(\text{I})).u_i(\boldsymbol{x})dL_1 +$$

$$\lim_{\epsilon \to 0}(\int_{L_\epsilon} T_i^k(\boldsymbol{x}, \boldsymbol{x}(\text{I})).u_i(\boldsymbol{x}(\text{I}))dL_\epsilon)$$

$$-\int_{L_2} u_i^k(\boldsymbol{x}, \boldsymbol{x}(\text{I})).T_i(\boldsymbol{x})dL_2 = -Pu_2^k(\boldsymbol{y_3}, \boldsymbol{x}(\text{I})) \quad (22)$$

Where k represents the direction of the loading and when $k = 1$ this direction coincides with the direction of x_1 axis while $k = 2$ indicates the loading direction to be the direction of x_2 axis. As it is mentioned above, M equations, each of these corresponding to a singular loading at a nodal point in any direction, are necessary. The order of these loadings is as below:

For the first (2N1+N2-1) equations, loading points are the nodal points on L_1 and the loading index k is one and for the second (2N1+N2-1) equations, loading points are the same but index k is two. For the following (N2-1) equations, k is one and the loading points are the nodal points on L_2 except A and B points. And in the last (N2+1) equations, loading points are the nodal points on L_2 either but including A and B points and k is two.

After writing the necessary M equations and substituting Eqs. (5), (11), (20) and (21) in Eq. (22), the following system of linear algebraic equations, given in partitioned form, is obtained.

$$\left[\boldsymbol{A}^{\text{M}\times(\text{4N1+2N2-2})}, \boldsymbol{B}^{\text{M}\times(\text{2N2})}\right]\boldsymbol{X}^{\text{K}\times 1} = \boldsymbol{C}^{1\times\text{M}} \quad (23)$$

Where $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are constant matrices. The components of these matrices are given as follows:

$$A(\text{I}, \text{J}) = \delta_{\text{IJ}} AD11(\text{I}) +$$

$$\int_0^{l(\text{J})} \{\tau_{1i}^1(\boldsymbol{x}, \boldsymbol{x}(\text{I}))n_i[1 - \frac{s}{l(\text{J})}]\}\, ds$$

$$+\int_0^{l(\text{J-1})} \{\tau_{1i}^1(\boldsymbol{x}, \boldsymbol{x}(\text{I}))n_i[\frac{s}{l(\text{J-1})}]\}\, ds$$

$$A(\text{I}, \text{J+2N1+N2-1}) = \delta_{\text{IJ}} AD12(\text{I}) +$$

$$\int_0^{l(\text{J})} \{\tau_{2i}^1(\boldsymbol{x}, \boldsymbol{x}(\text{I}))n_i[1 - \frac{s}{l(\text{J})}]\}\, ds$$

$$+\int_0^{l(J-1)} \{\tau_{2i}^1(\boldsymbol{x}, \boldsymbol{x}(\text{I}))n_i[\frac{s}{l_{(\text{J-1})}}]\}\, ds$$

$$A(\text{I+2N1+N2-1}, \text{J}) = \delta_{\text{IJ}} AD21(\text{I}) +$$

$$\int_0^{l(\text{J})} \{\tau_{1i}^2(\boldsymbol{x}, \boldsymbol{x}(\text{I}))n_i[1 - \frac{s}{l_{(\text{J})}}]\}\, ds$$

$$+\int_0^{l(J-1)} \{\tau_{1i}^2(\boldsymbol{x}, \boldsymbol{x}(\text{I}))n_i[\frac{s}{l_{(\text{J-1})}}]\}\, ds$$

$$A(\text{I+2N1+N2-1}, \text{J+2N1+N2-1}) = \delta_{\text{IJ}} AD22(\text{I}) +$$

$$\int_0^{l(\text{J})} \{\tau_{2i}^2(\boldsymbol{x}, \boldsymbol{x}(\text{I}))n_i[1 - \frac{s}{l_{(\text{J})}}]\}\, ds +$$

$$\int_0^{l(\text{J-1})} \{\tau_{2i}^2(\boldsymbol{x}, \boldsymbol{x}(\text{I}))n_i[\frac{s}{l_{(\text{J-1})}}]\}\, ds$$

$$for \;\; (\text{I=1}\; to \;\text{2N1+N2-1},\; \text{J=1}\; to\; \text{2N1+N2-1}) \quad (24)$$

where δ_{IJ} is the Kronecker's delta. Additional matrices $\boldsymbol{AD11}$, $\boldsymbol{AD12}$, $\boldsymbol{AD21}$ and $\boldsymbol{AD22}$ which correspond to the second term in Eq. (22) can be expressed, in terms of θ_1 and θ_2 angles are shown in Fig. (3), as

$$AD11(\text{I}) = \frac{-1}{4\pi(1-\nu)}[2(1-\nu)(\theta_2 - \theta_1) - n_1(\text{I})n_2(\text{I})$$

$$+n_1(\text{I-1})n_2(\text{I-1})]$$

$$AD12(\text{I}) = -\frac{1}{4\pi(1-\nu)}[-n_2(\text{I})n_2(\text{I}) + n_2(\text{I-1})n_2(\text{I-1})]$$

$$AD21(\text{I}) = -\frac{1}{4\pi(1-\nu)}[n_1(\text{I})n_1(\text{I}) - n_1(\text{I-1})n_1(\text{I-1})]$$

$$AD22(\text{I}) = -\frac{1}{4\pi(1-\nu)}[2(1-\nu)(\theta_2 - \theta_1) +$$

$$n_1(\text{I})n_2(\text{I}) - n_1(\text{I-1})n_2(\text{I-1})] \quad (25)$$

The remaining terms of the matrix $\boldsymbol{A}$ are given as follows:

$$A(\text{I+4N1+2N-2}, \text{J}) = \int_0^{l(\text{J})} \{\tau_{1i}^1(\boldsymbol{x}, \boldsymbol{x}(\text{I+2N1+N2}))n_i[1 - \frac{s}{l(\text{J})}]\}\, ds$$

$$+\int_0^{l(\text{J-1})}\{\tau_{1i}^1(\boldsymbol{x},\boldsymbol{x}(\text{I+2N1+N2}))n_i[\frac{s}{l(\text{J-1})}]\}\,ds$$

$$A(\text{I+4N1+2N2-2 , J+2N1+N2-1}) =$$

$$\int_0^{l(\text{J})}\{\tau_{2i}^1(\boldsymbol{x},\boldsymbol{x}(\text{I+2N1+N2}))n_i[1-\frac{s}{l(\text{J})}]\}\,ds$$

$$+\int_0^{l(\text{J-1})}\{\tau_{2i}^1(\boldsymbol{x},\boldsymbol{x}(\text{I+2N1+N2}))n_i[\frac{s}{l(\text{J-1})}]\}\,ds$$

$$for\ \ (\text{I=1}\ to\ \text{N2-1},\ \text{J=1}\ to\ \text{2N1+N2-1}) \qquad (26)$$

$$A(\text{I+4N1+3N2-3 , J}) = \int_0^{l(\text{J})}\{\tau_{1i}^2(\boldsymbol{x},\boldsymbol{x}(\text{I+2N1+N2-1}))n_i[1-\frac{s}{l(\text{J})}]\}\,ds$$

$$+\int_0^{l(\text{J-1})}\{\tau_{1i}^2(\boldsymbol{x},\boldsymbol{x}(\text{I+2N1+N2-1}))n_i[\frac{s}{l(\text{J-1})}]\}\,ds$$

$$A(\text{I+4N1+3N2-3 , J+2N1+N2-1}) =$$

$$\int_0^{l(\text{J})}\{\tau_{2i}^2(\boldsymbol{x},\boldsymbol{x}(\text{I+2N1+N2-1}))n_i[1-\frac{s}{l(\text{J})}]\}\,ds$$

$$+\int_0^{l(\text{J-1})}\{\tau_{2i}^2(\boldsymbol{x},\boldsymbol{x}(\text{I+2N1+N2-1}))n_i[\frac{s}{l(\text{J-1})}]\}\,ds$$

$$for\ \ (\text{I=1}\ to\ \text{N2+1},\ \text{J=1}\ to\ \text{2N1+N2-1}) \qquad (27)$$

The elements of the matrix $\boldsymbol{B}$ are

$$B(\text{I, J}) = -\int_0^{l(\text{J+2N1+N2})}\{u_1^1(\boldsymbol{x},\boldsymbol{x}(\text{I}))[1-\frac{s}{l(\text{J+2N1+N2})}]\}ds$$

$$-\int_0^{l(\text{J+2N1+N2-1})}\{u_1^1(\boldsymbol{x},\boldsymbol{x}(\text{I}))[\frac{s}{l(\text{J+2N1+N2-1})}]\}ds$$

$$B(\text{I+2N1+N2-1, J}) = -\int_0^{l(\text{J+2N1+N2})}\{u_1^2(\boldsymbol{x},\boldsymbol{x}(\text{I}))[1-\frac{s}{l(\text{J+2N1+N2})}]$$

$$-\int_0^{l(\text{J+2N1+N2-1})}\{u_1^2(\boldsymbol{x},\boldsymbol{x}(\text{I}))[\frac{s}{l(\text{J+2N1+N2-1})}]\}ds$$

$$B(\text{I, J+N2}) = -\int_0^{l(\text{J+2N1+N2})}\{u_2^1(\boldsymbol{x},\boldsymbol{x}(\text{I}))[1-\frac{s}{l(\text{J+2N1+N2})}]\}ds$$

$$-\int_0^{l(\text{J+2N1+N2-1})}\{u_2^1(\boldsymbol{x},\boldsymbol{x}(\text{I}))[\frac{s}{l(\text{J+2N1+N2-1})}]\}ds$$

$$B(\text{I+2N1+N2-1, J+N2}) = -\int_0^{l(\text{J+2N1+N2})}\{u_2^2(\boldsymbol{x},\boldsymbol{x}(\text{I}))[1-\frac{s}{l(\text{J+2N1+N2})}]$$

$$-\int_0^{l(\text{J+2N1+N2-1})}\{u_2^2(\boldsymbol{x},\boldsymbol{x}(\text{I}))[\frac{s}{l(\text{J+2N1+N2-1})}]\}ds$$

$$for\ \ (\text{I=1}\ to\ \text{2N1+ N2-1},\ \text{J=1}\ to\ \text{N2-1}) \qquad (28)$$

$$B(\text{I, N2}) = -\int_0^{l(\text{2N1+N2})}\{u_2^1(\boldsymbol{x},\boldsymbol{x}(\text{I}))[1-\frac{s}{l(\text{2N1+N2})}]\}ds$$

$$B(\text{I+2N1+N2-1, N2}) = -\int_0^{l(\text{2N1+N2})}\{u_2^2(\boldsymbol{x},\boldsymbol{x}(\text{I}))[1-\frac{s}{l(\text{2N1+N2})}]\}ds$$

$$B(\text{I, 2N2}) = -\int_0^{l(\text{2N1+N2-1})}\{u_2^1(\boldsymbol{x},\boldsymbol{x}(\text{I}))[\frac{s}{l(\text{2N1+N2})}]\}ds$$

$$B(\text{I+2N1+N2-1, 2N2}) = -\int_0^{l(\text{2N1+2N2-1})}\{u_2^2(\boldsymbol{x},\boldsymbol{x}(\text{I}))[\frac{s}{l(\text{2N1+2N2-1})}]\}ds$$

$$for\ \ (\text{I=1}\ to\ \text{2N1+ N2-1}) \qquad (29)$$

$$B(\text{I+4N1+2N2-2, J}) =$$

$$-\int_0^{l(\text{2N1+N2+J})}\{u_1^1(\boldsymbol{x},\boldsymbol{x}(\text{I+2N1+N2}))[1-\frac{s}{l(\text{J+2N1+N2})}]\}ds$$

$$-\int_0^{l(\text{2N1+N2+J-1})}\{u_1^1(\boldsymbol{x},\boldsymbol{x}(\text{I+2N1+N2}))[\frac{s}{l(\text{J+2N1+N2-1})}]\}ds$$

$$B(\text{I+4N1+3N2-2, J}) =$$

$$-\int_0^{l(\text{2N1+N2+J})}\{u_1^2(\boldsymbol{x},\boldsymbol{x}(\text{I+2N1+N2}))[1-\frac{s}{l(\text{J+2N1+N2})}]\}ds$$

$$-\int_0^{l(\text{2N1+N2+J-1})}\{u_1^2(\boldsymbol{x},\boldsymbol{x}(\text{I+2N1+N2}))[\frac{s}{l(\text{J+2N1+N2-1})}]\}ds$$

$$B(\text{I+4N1+2N2-2, J+N2}) =$$

$$-\int_0^{l(\mathrm{2N1+N2+J})}\{u_2^1(\boldsymbol{x},\boldsymbol{x}(\mathrm{I+2N1+N2}))[1-\frac{s}{l(\mathrm{J+2N1+N2})}]\}ds$$

$$-\int_0^{l(\mathrm{2N1+N2+J\text{-}1})}\{u_2^1(\boldsymbol{x},\boldsymbol{x}(\mathrm{I+2N1+N2}))[\frac{s}{l(\mathrm{J+2N1+N2\text{-}1})}]\}ds$$

$$B(\mathrm{I+4N1+3N2\text{-}2,\ J+N2})=$$

$$-\int_0^{l(\mathrm{2N1+N2+J})}\{u_2^2(\boldsymbol{x},\boldsymbol{x}(\mathrm{I+2N1+N2}))[1-\frac{s}{l(\mathrm{J+2N1+N2})}]\}ds$$

$$-\int_0^{l(\mathrm{2N1+N2+J\text{-}1})}\{u_2^2(\boldsymbol{x},\boldsymbol{x}(\mathrm{I+2N1+N2}))[\frac{s}{l(\mathrm{J+2N1+N2\text{-}1})}]\}ds$$

$$for\quad(\mathrm{I{=}1}\ to\ \mathrm{N2\text{-}1},\ \mathrm{J{=}1}\ to\ \mathrm{N2\text{-}1})\tag{30}$$

$$B(\mathrm{I+4N1+2N2\text{-}2,\ N2})=$$

$$-\int_0^{l(\mathrm{2N1+N2})}\{u_2^1(\boldsymbol{x},\boldsymbol{x}(\mathrm{I+2N1+N2}))[1-\frac{s}{l(\mathrm{2N1+N2})}]\}ds$$

$$B(\mathrm{I+4N1+2N2\text{-}2,\ 2N2})=$$

$$-\int_0^{l(\mathrm{2N1+2N2\text{-}1})}\{u_2^1(\boldsymbol{x},\boldsymbol{x}(\mathrm{I+2N1+N2}))[\frac{s}{l(\mathrm{2N1+2N2\text{-}1})}]\}ds$$

$$B(\mathrm{I+4N1+3N2\text{-}2,\ N2})=$$

$$-\int_0^{l(\mathrm{2N1+N2})}\{u_2^2(\boldsymbol{x},\boldsymbol{x}(\mathrm{I+2N1+N2}))[1-\frac{s}{l(\mathrm{2N1+N2})}]\}ds$$

$$B(\mathrm{I+4N1+3N2\text{-}2,\ 2N2})=$$

$$-\int_0^{l(\mathrm{2N1+2N2\text{-}1})}\{u_2^2(\boldsymbol{x},\boldsymbol{x}(\mathrm{I+2N1+N2}))[\frac{s}{l(\mathrm{2N1+2N2\text{-}1})}]\}ds$$

$$for\quad(\mathrm{I{=}1}\ to\ \mathrm{N2\text{-}1})\tag{31}$$

$$B(\mathrm{4N1+3N2\text{-}2,\ J})=$$

$$-\int_0^{l(\mathrm{2N1+N2+J})}\{u_1^2(\boldsymbol{x},\boldsymbol{x}(\mathrm{2N1+N2}))[1-\frac{s}{l(\mathrm{2N1+N2+J})}]\}ds$$

$$-\int_0^{l(\mathrm{2N1+N2+J\text{-}1})}\{u_1^2(\boldsymbol{x},\boldsymbol{x}(\mathrm{2N1+N2}))[\frac{s}{l(\mathrm{2N1+N2+J\text{-}1})}]\}ds$$

$$B(\mathrm{4N1+3N2\text{-}2,\ J+N2})=$$

$$-\int_0^{l(\mathrm{2N1+N2+J})}\{u_2^2(\boldsymbol{x},\boldsymbol{x}(\mathrm{2N1+N2}))[1-\frac{s}{l(\mathrm{2N1+N2+J})}]\}ds$$

$$-\int_0^{l(\mathrm{2N1+N2+J\text{-}1})}\{u_2^2(\boldsymbol{x},\boldsymbol{x}(\mathrm{2N1+N2}))[\frac{s}{l(\mathrm{2N1+N2+J\text{-}1})}]\}ds$$

$$B(\mathrm{4N1+4N2\text{-}2,\ J})=$$

$$-\int_0^{l(\mathrm{2N1+N2+J})}\{u_1^2(\boldsymbol{x},\boldsymbol{x}(\mathrm{2N1+2N2}))[1-\frac{s}{l(\mathrm{2N1+N2+J})}]\}ds$$

$$-\int_0^{l(\mathrm{2N1+N2+J\text{-}1})}\{u_1^2(\boldsymbol{x},\boldsymbol{x}(\mathrm{2N1+2N2}))[\frac{s}{l(\mathrm{2N1+N2+J\text{-}1})}]\}ds$$

$$B(\mathrm{4N1+4N2\text{-}2,\ J+N2})=$$

$$-\int_0^{l(\mathrm{2N1+N2+J})}\{u_2^2(\boldsymbol{x},\boldsymbol{x}(\mathrm{2N1+2N2}))[1-\frac{s}{l(\mathrm{2N1+N2+J})}]\}ds$$

$$-\int_0^{l(\mathrm{2N1+N2+J\text{-}1})}\{u_2^2(\boldsymbol{x},\boldsymbol{x}(\mathrm{2N1+2N2}))[\frac{s}{l(\mathrm{2N1+N2+J\text{-}1})}]\}ds$$

$$for\quad(\mathrm{J{=}1}\ to\ \mathrm{N2\text{-}1})\tag{32}$$

$$B(\mathrm{4N1+3N2\text{-}2,\ N2})=$$

$$-\int_0^{l(\mathrm{2N1+N2})}\{u_2^2(\boldsymbol{x},\boldsymbol{x}(\mathrm{2N1+2N2}))[1-\frac{s}{l(\mathrm{2N1+N2})}]\}ds\tag{33}$$

$$B(\mathrm{4N1+3N2\text{-}2,\ 2N2})=$$

$$-\int_0^{l(\mathrm{2N1+2N2\text{-}1})}\{u_2^2(\boldsymbol{x},\boldsymbol{x}(\mathrm{2N1+2N2}))[\frac{s}{l(\mathrm{2N1+2N2\text{-}1})}]\}ds\tag{34}$$

$$B(\mathrm{4N1+4N2\text{-}2,\ N2})=$$

$$-\int_0^{l(\mathrm{2N1+N2+J})}\{u_1^2(\boldsymbol{x},\boldsymbol{x}(\mathrm{2N1+2N2}))[1-\frac{s}{l(\mathrm{2N1+N2})}]\}ds$$

$$B(\text{4N1+4N2-2}, \text{2N2}) =$$

$$-\int_0^{l(\text{2N1+2N2-1})} \{u_2^2(\boldsymbol{x}, \boldsymbol{x}(\text{2N1+2N2}))[\frac{s}{l(\text{2N1+2N2-1})}]\}ds \quad (35)$$

where

$$\boldsymbol{X}(\text{I}) = u_1(\text{I}) \;\; for \;\; \text{I} = \text{1, 2N1+N2-1}$$

$$\boldsymbol{X}(\text{I+2N1+N2-1}) = u_2(\text{I}) \;\; for \;\; \text{I} = \text{1, 2N1+N2-1}$$

$$\boldsymbol{X}(\text{I+4N1+2N2-2}) = T_1(\text{I+2N1+N2}) \;\; for \;\; \text{I} = \text{1, N2-1}$$

$$\boldsymbol{X}(\text{I+4N1+3N2-2}) = T_2(\text{I+2N1+N2-1}) \;\; for \;\; \text{I} = \text{1, N2+1} \quad (36)$$

$$C(\text{I}) = -P\, u_2^1(\boldsymbol{y_3}, \boldsymbol{x}(\text{I}))$$

$$C(\text{I+2N1+N2-1}) = -P\, u_2^2(\boldsymbol{y_3}, \boldsymbol{x}(\text{I}))$$

$$for \;\; \text{I} = \text{1, 2N1+N2-1} \quad (37)$$

$$C(\text{I+4N1+2N2-2}) = -P\, u_2^1(\boldsymbol{y_3}, \boldsymbol{x}(\text{I+2N1+N2}))$$

$$for \;\; \text{I} = \text{1, N2-1} \quad (38)$$

$$C(\text{I+4N1+3N2-3}) = -P\, u_2^2(\boldsymbol{y_3}, \boldsymbol{x}(\text{I+2N1+N2-1}))$$

$$for \;\; \text{I} = \text{1, N2+1} \quad (39)$$

Performing the integrals, the entries of $\boldsymbol{A}$ and $\boldsymbol{B}$ constant matrices are calculated. But it must be indicated that there are singular terms during calculation of these integrals on the boundary elements having the numbers I-1 and I for a loading point $\boldsymbol{x}(\text{I})$. Integrals over these elements are analytically calculated to eliminate these singularities. The singularities which arise in the integrals of $u_i^k(\boldsymbol{x}, \boldsymbol{y})$ functions have the type of

$$S_1 = \lim_{s \to 0} s\, ln(s) \quad (40)$$

which is zero. But the singularities arising in the integrals of $\tau_{ij}^k(\boldsymbol{x}, \boldsymbol{y})$ functions have the types of

$$S_2 = \lim_{s \to 0} ln(s) \quad (41)$$

but the singular terms having this form eliminates each other during construction of the entries of the matrix $\boldsymbol{A}$. Then, a system of linear algebraic equations, whose unknowns have been defined in Eqs. (36), is found.

The variations of the horizontal and vertical components of surface traction and displacement vectors are given in Figs. 4-11, for different lines.

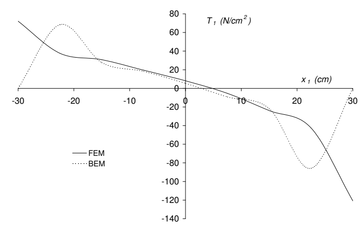

Fig. 4. Variation of the horizontal component of surface traction vector on line AB

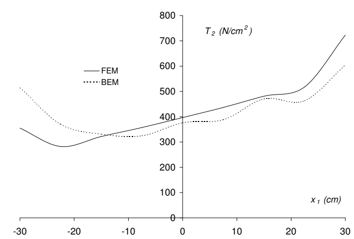

Fig. 5. Variation of the vertical component of surface traction vector on line AB

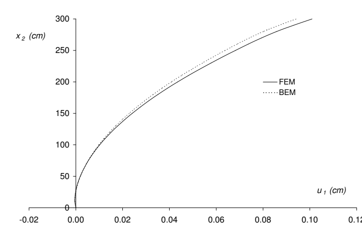

Fig. 6. Variation of the horizontal component of displacement vector on line AK

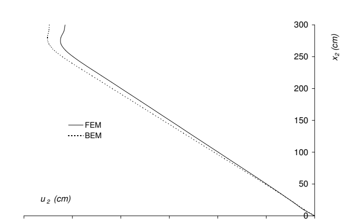

Fig. 7. Variation of the vertical component of displacement vector on line AK

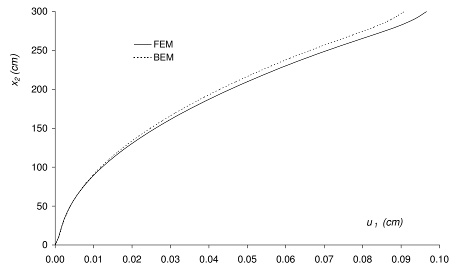

Fig. 8. Variation of the horizontal component of displacement vector on line CB

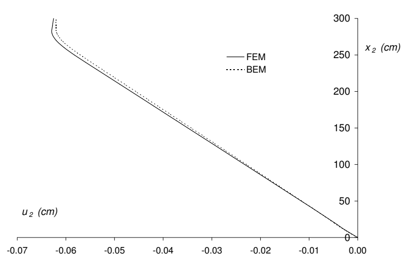

Fig. 9. Variation of the vertical component of displacement vector on line CB

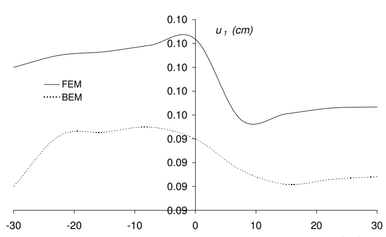

Fig. 10. Variation of the horizontal component of displacement vector on line KC

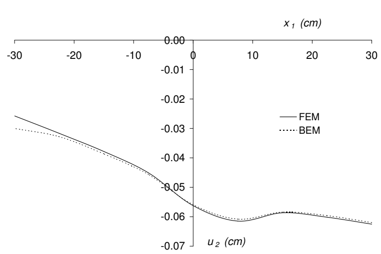

Fig. 11. Variation of the vertical component of displacement vector on line KC

5 Conclusions and Discussion

A solution method of plane problems of linear elasticity has been explained on a sample mixed-boundary value problem for a specific material. The reciprocity theorem provides a relation between displacements, traction components and body forces for two loading states of the same body and this relation gives a boundary integral equation for unknown fields on the boundary, complementary to the applied fields. This integral equation has been solved numerically. The selected approximations for unknowns are linear and the integral equation is reduced to a system of algebraic equations. Of course, higher order polynomials can be selected for a better approximation but it must be emphasized that dominant terms of the coefficients matrix are heavily dependent to the constant additional matrices $\boldsymbol{AD11}$, $\boldsymbol{AD12}$, $\boldsymbol{AD21}$, $\boldsymbol{AD22}$, and the constant terms in approximation polynomials while the effects of linear terms are secondary. The same elastostatic problem has also been solved by FEM (ANSYS 10.0) for checking.

Results are nearly the same for displacements while element numbers are quite different. In BEM, 28 elements are used while 38 elements(PLANE82) and 147 nodes in FEM. Relative errors are calculated by equilibrium equations. And the relative errors are 0.003868 and 0.00947 in horizontal and vertical directions respectively for BEM while 0.0086 and 0.0122 in FEM. The increment of the element number slightly affects the error after 28 in BEM. As an example, relative errors are 0.003499 and 0.00227 for 38 elements.

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