A Numerical Analysis of the Burgers-Poisson (BP) Equation Using Variational Iteration Method

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Abstract: - In this paper, the numerical analysis of dispersive type wave equation (Burgers–Poisson Equation) is studied. The analyses are carried out by using both finite difference method (FDM) and variational iteration methods (VIM). The stability and consistency of the obtained finite difference equation is provided. The available exact solution and the result of these two numerical methods are compared to see the accuracy for a specified length and time interval.

Key-Words: - Burgers–Poisson (BP) equation, Variational iteration method, Finite difference method.

1 Introduction

For shallow water, a dispersive model equation was introduced by Whitnam (1974). A number of formal properties (existence, variance of entropy) were also derived in Whitnam (1974). A rescaled form of the Whitnam model was extended by Fellner and Schmeisser (2004) as Burgers-Poisson equation (BP). The difference of this BP equation with Camassa-Holm equation (1993) and Benjamin-Ona equation (1967) was studied by Fellner and Schmeisser (2004). In these equations, global existence of smooth solutions were obtained under certain conditions of the initial data, and some comparable results were not true. As a result, a dispersive wave model was used to get smooth solutions for water waves. The model equation (BP equation) for the unidirectional propagation of long waves in a dispersive media for the water waves can be given in the following form:

$$u_t + u_x + uu_x - u_{xxt} - 3(u_x u_{xx} + uu_{xxx}) = 0 \qquad (1)$$

where $u(x,t)$ is a function of spatial coordinate x and time t, and the subscripts t and x denote partial differentiation with respect to t and x. An initial condition for u can be denoted by the following expression

$$u(x,0) = f(x) \qquad (2)$$

An analytical group invariant solution to the equation (1) was given by Turgay and Hizel (2007). Two different numerical methods, FDM and VIM, will be used for the numerical analysis of the BP equation. It is well known that FDM has been widely used in applied sciences for solutions of nonlinear differential equations.

The variational iteration method (VIM) introduced by He in the references (6-9) has been used extensively in the numerical solution of nonlinear differential equations for a decade. It was proved its efficiency and accuracy in the recent studies and it will be used to compare FDM and analytical results.

2 Problem Formulation

a) FINITE DIFFERENCE METHOD (FDM)

In this study, the forward finite difference scheme will be used for the analysis of BP equation, in order to solve equation (1) both space and time are divided into a larger numbers of grid points and derivatives are replaced by finite difference equation. Let h and k are the step sizes in coordinate and time grids and n the number of grid points in coordinate grid.

In this case, $x_i = ih$ for $i = 0,1,\ldots,n$ and $t_j = jk$, for $j = 0,1,\ldots,m$ (3)

The method is based on replacing u_t, u_x, u_{xxt}, u_{xx}, u_{xxx} by the means of the finite difference representation, one obtains the solutions of equation (1) as

$$\begin{aligned}
&u_{i+2,j+1} - 2u_{i+1,j+1} + (1-h)u_{i,j+1} = \\
&h^2(u_{i,j}) + (u_{i+1,j} - u_{i,j})ht(1+u_{i,j}) \\
&-3(u_{i+1,j} - u_{i,j})(u_{i+2,j} - 2u_{i+1,j} + u_{i,j})r \\
&-u_{i,j}(u_{i+3,j} - 3u_{i+2,j} + 3u_{i+1,j} - u_{i,j})r
\end{aligned} \qquad (4)$$

where $k/h = r$. By applying the Von Neumann stability method to above equation, one can obtain stable solutions for $-1 \le k/h^2 \le 1$. The above finite difference equation is also consistent, since the limiting value of the local truncation error tends to zero when *h* and *k* goes to zero.

b) VARIATIONAL ITERATION METHOD (VIM)

Let consider the differential equation

$$Lu + Nu = f(t) \tag{5}$$

where *L* and *N* are linear and nonlinear operators, respectively, and *f(t)* is the inhomogeneous term. In the references (6-9), a correction functional for Eq. (5) can be written as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(\xi) + N\tilde{u}_n(\xi) - f(\xi))d\xi \tag{6}$$

where λ is a general Lagrange's multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. The successive approximations u_{n+1}, $n > 0$, of the solution u will be readily obtained upon using the determined Lagrangian multiplier and any selective function u_0. Therefore, the solution is given by an infinite series of the form

$$u = \lim_{n\to\infty} u_n \tag{7}$$

The resulting successive approximations applied to correction functional, eq (6), gives exact solution for infinite number of iterations.

3 Problem Solution

By considering the BP equation (Eq. 1), an initial condition that obeys the exact solution can be written in the following form:

$$u_t + u_x + uu_x - u_{xxt} - 3(u_x u_{xx} + uu_{xxx}) = 0,$$

and

$$u(x,0) = x \tag{8}$$

and the exact solution was given by Turgay and Hizel (2007) as

$$u(x,t) = \frac{x+1}{1+t} - 1 \tag{9}$$

The VIM can be applied to equation (6) in the form

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) \\ &+ \int_0^t \lambda(u_t(x,\xi) + \tilde{u}_x(x,\xi) + u\tilde{u}_x(x,\xi) \\ &- \tilde{u}_{xxt}(x,\xi) - 3u_x\tilde{u}_{xx}(x,\xi) - u\tilde{u}_{xxx}(x,\xi))d\xi \end{aligned} \tag{10}$$

The stationary condition were obtained for

$$\lambda' = 0 \tag{11a}$$

$$1 + \lambda = 0 \tag{11b}$$

where λ yields to -1.

One can substitute λ and the initial approximation to equation (10) to get the successive approximation by the following expressions,

$$\begin{aligned} &u_0(x,0) = x \\ &u_1(x,t) = (1-t)(1+x) - 1 \\ &u_2(x,t) = (1-t+t^2)(1+x) - 1 - t^3(1+x)/3 \\ &u_3(x,t) = (1-t+t^2-t^3)(1+x) - 1 + \text{other terms} \\ &\cdot \\ &\cdot \\ &\cdot \\ &u_n(x,t) = (1-t+t^2-t^3+t^4-t^5+\ldots)(1+x) - 1 \end{aligned} \tag{12}$$

Recalling equation (7), one can obtain the exact solution immediately same as equation (9),

$$u(x,t) = \frac{x+1}{1+t} - 1 \quad \text{for} \quad |t| < 1 \tag{13}$$

In the figure 1, the variation of u in time at $x=0.5$ for FDM and VIM are plotted.

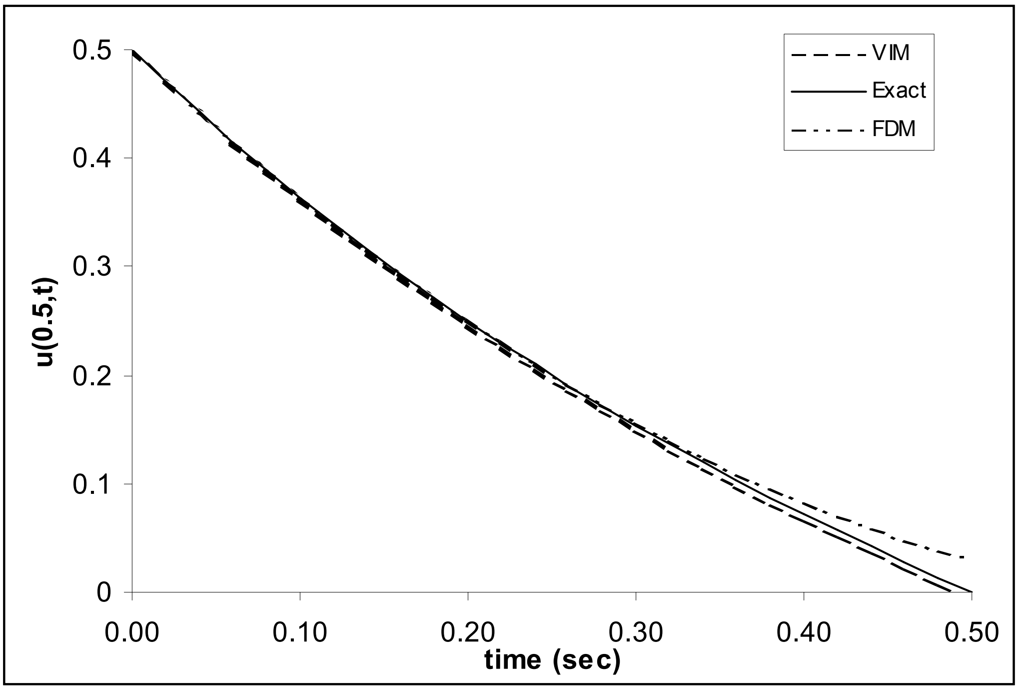

FIGURE 1. Variation of $u(x,t)$ for x=0.5 by FDM, VIM and exact solution

In the Table 1, the comparison of exact solution are done with the finite difference and variational iteration methods at time $t=0.5$ sec. For the FDM analysis h and k are selected as 0.1 and 0.02, respectively. The results are obtained for $0<x<1$.

TABLE 1. Comparison of exact and numerical methods for $u(x,t)$ at time t = 0.5

x	Exact	%Error with FDM	% Error with VIM
0	-.33333	.1101E+01	-.3906E+00
0.1	-.26667	.1506E+01	-.5371E+00
0.2	-.20000	.2214E+01	-.7813E+00
0.3	-.13333	.3517E+01	-.1270E+01
0.4	-.06667	.7918E+01	-.2734E+01
0.6	.06667	-.9359E+01	.3125E+01
0.7	.13333	-.4200E+01	.1660E+01
0.8	.20000	-.3871E+01	.1172E+01
0.9	.26667	-.1943E+01	.9277E+00
1.0	.33333	-.2736E+01	.7812E+00

In this table, the results of the VIM was calculated for the first eight terms of the series , i.e. $u_8(x,t)$.

4 Conclusion

The numerical solution of the Burgers-Poisson equation was studied by using both finite difference and variational iteration methods and results were compared with exact solution. He's VIM gives several successive approximations trough using the iteration of correctional functional.
VIM reduces the volume of calculations by not requiring many time steps and space intervals as in the FDM.
It was observed that the use of VIM provided a better accuracy and a lesser number of steps to calculate *u(x,t)*.

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