Fuzzy Logic Model for Soil Water Balance Problem

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Abstract: Groundwater systems behaviour is usually very hard to simulate due to the uncertainty of the hydraulic parameters involved. Fuzzy analysis is one of the available tools that can be used for such problems, involving uncertain data. The behaviour of hydrological systems is usually simulated using partial differential equations (PDE). The exact value of various coefficients (transmissivity, dispersion, etc.) of the PDE is often not accurately known. Many authors propose the use of fuzzy arithmetic to deal with uncertainties, occurring in water management problems. In such an approach, uncertain or vague parameters are defined as fuzzy numbers. A fuzzy analysis approach of water management problems usually involves the consideration of several a-level cuts and an explicit scheme approach for the PDE's discretization. Several application examples of this approach are listed in the literature, including uncertainty in transmissivities, porosities, dispersivities, and deoxygenation rate coefficient .

A methodology for the simulation of aquifers having vague values of hydraulic parameters is introduced in this paper, and an analytical solution for a two-dimensional application example is presented. The two-dimensional problem of drainage is addressed using fuzzy analysis by defining the hydraulic conductivity K as a Triangular Fuzzy Number (TFN). The analysis utilises the a-cut method for a 0 and 1 membership function and the results are compared with a-cut when the membership function is 0, 0.25, 0.5, 0.75 and 1.

Key-Words: Drainage, fuzzy numbers, fuzzy arithmetic, uncertainty, hydraulic conductivity.

1 Introduction

Fuzziness, as handled in fuzzy logic, can refer to various types of vagueness and uncertainty but particularly to the vagueness related to human linguistics and thinking, differing from the uncertainty of the Probabilistic Theory [10].

1.1. Definition 1. Fuzzy set.

If X is a collection of objects denoted generically by x, then a fuzzy set $\overline{A}$ in X is a set of ordered pairs: $\overline{A} = \{(x, \mu_{\overline{A}}(x)) \,|\, x \in X\}$. Where $\mu_{\overline{A}}(x)$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in $\overline{A}$ that maps X to the membership space (When M contains only the two points 0 and 1, $\overline{A}$ is no fuzzy set) [11].

1.2. Definition 2. α-level cut.

The α-level cut (α-level set) of the fuzzy subset A is the set of those elements, which have at least α membership:
$A_\alpha = \{x \,|\, \mu_A(x) \geq \alpha\}$. If $A'_\alpha = \{x \,|\, \mu_A(x) > \alpha\}$ is called "strong α-level cut".

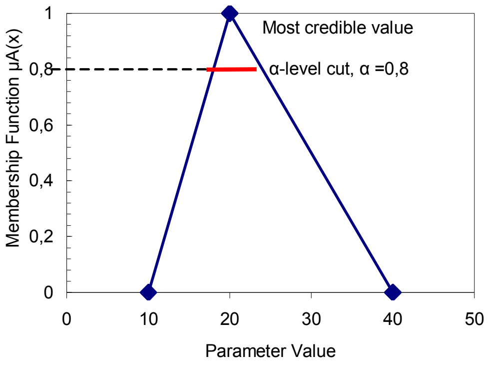

Fig. 1: A fuzzy number and an α-level cut

1.3. Definition 3. Convex fuzzy set.

A fuzzy set is convex if:

$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$,

$x_1, x_2 \in X, \lambda \in [0,1]$

Also a fuzzy set can be convex if all α-level sets are convex.

1.4. Definition 4. Fuzzy numbers.

A fuzzy number *M* is a convex normalized set *M* of the real line **R** such that:

1. It exist one $x_0 \in R$ with $\mu_M(x_0) = 1$ (x_0 is called the mean value of *M*)
2. $\mu_M(x) = 1$ is piecewise continuous.

When fuzzy set theory is used to solve real problems of realistic size, it is more efficient to use a special type of fuzzy numbers, the LR-type [11].

1.5. Definition 5. LR-type.

A fuzzy number *M* is of LR-type if there exist reference functions L (for left), R (for right), and scalars α>0, β>0 with

$$\mu_M(x) = \begin{cases} L\left(\dfrac{m-x}{\alpha}\right) & For\, x \leq m \\ R\left(\dfrac{x-m}{\beta}\right) & For\, x \geq m \end{cases}$$

m, is a real number called the mean value of *M*, and α and β are called the left and right spreads respectively. A triangular fuzzy number (TFN) is a special case of semisymetric LR fuzzy number [7], [11]. To specify a TFN we use the three values (α_1, α_2, α_3) of the triangle base, where $\alpha_1 \leq \alpha_2 \leq \alpha_3$.

Strictly speaking, these special cases of fuzzy numbers are fuzzy intervals. So every a-level cut, actually, gives an interval number. Disposing various a-level cuts we can construct a fuzzy number in discrete form. Finally, if we want to use fuzzy sets in applications, we will have to deal with interval number operations [11], [3].

If * is one of the symbols +, -, ·, /, we define arithmetic basic operations on interval number by $[a,b]*[c,d] = \{x*y \mid a \leq x \leq b, c \leq y \leq d\}$ except that we do not define [a, b]/[c, d] if 0 ∈ [c, d]. Specifically, [a, b]+[c, d] = [a + c, b + b], [a, b]-[c, d] = [a - d, b - c]
[a, b] · [c, d] = [min(ac, ad, bc, db), max(ac, ad, bc, db)], [a, b]/[c, d] = [a, b] · [1/d, 1/c] if 0 ∉ [c, d]. [8]

2 Problem Formulation

In this article, a two-dimensional analytical solution for obtaining the groundwater level is presented. Examples of real applications are: Drainage of an area, by drainpipes or natural ground configurations, drainage systems with subsequent well points, which are applied in soil operations in dry conditions, on grounds with a rich groundwater supply.

The analytical solution of the two-dimensional problem is derived from the two-dimensional linearized Boussinesq equation.

$$\frac{\partial H}{\partial t} = a\left\{\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right\} \qquad (1)$$

Where $a = \dfrac{K \times \overline{H}}{S}$, t denotes time, x, y Cartesian coordinates, h water table elevation, K the hydraulic conductivity, $\overline{H}$ the average value of the water table elevation and S is the Specific Yield of the soil.

The following one-dimensional analytical solution for a drainage problem is obtained [12], [5].

$$\left.\begin{aligned} &\frac{H(y,t) - H_\alpha}{H_1 - H_\alpha} = \\ &= \sum_{n=1}^{\infty} \frac{2}{k_n}(-1)^{n+1} \cos\left(k_n \frac{2x}{R_1}\right) e^{-k_n^2 \frac{at}{R_1^2}} = \\ &= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \cos\frac{(2n-1)\pi x}{2R_1} e^{-a(2n-1)^2 \pi^2 t / 4R_1^2} \\ &\left(k_n = (2n-1)\frac{\pi}{2}\right) \end{aligned}\right\} \qquad (2)$$

Where t denotes time, x denotes distance from the drainpipe and R_1 is the half of the drain spacing.

Using Fourier series approach and the above equations, the solution of following two-dimensional drainage problem is obtained. [13], [5].

$$\left.\begin{aligned} &\frac{H(x,y,t)-H_\alpha}{H_1-H_\alpha}= \\ &\left(\frac{4}{\pi}\right)^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n}{(2n+1)}\frac{(-1)^m}{(2m+1)}\cos(Nx)\cos(My)\, e^{-w} \\ &Nx=\left(\frac{(2n+1)\pi x}{2R_x}\right),\qquad My=\left(\frac{(2m+1)\pi y}{2R_y}\right) \\ &w=\frac{\alpha\pi^2 t}{4}\left\{\frac{(2n-1)^2}{R_x^2}+\frac{(2m-1)^2}{R_y^2}\right\} \end{aligned}\right\}\quad(3)$$

3 Problem Solution

Equation (3) is applied to a subsurface drainage system for a homogeneous and isotropic soil. A rectangular area with a pipe drainage system is assumed, figure 4. The drainpipes are located at the four sides of the rectangular and a Cartesian axes system is referenced to the centre of the rectangular area.

The initial condition is the saturated status of the area and at time t=0 the ground water starts to flow to the surrounding drainpipes. The phenomenon is symmetric; therefore the theoretical solution can be applied just to the quarter of the study area.

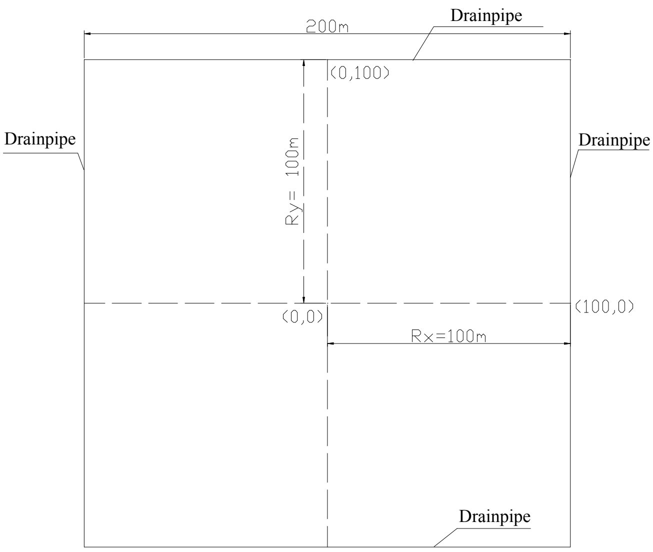

Fig 2: Area with a pipe drainage system

The equation (3) gives the temporal and spatial value of the water table height in every position given by the combination of the spatial steps dx, dy in every time step dt.

Settlement of the problem (figure 3):

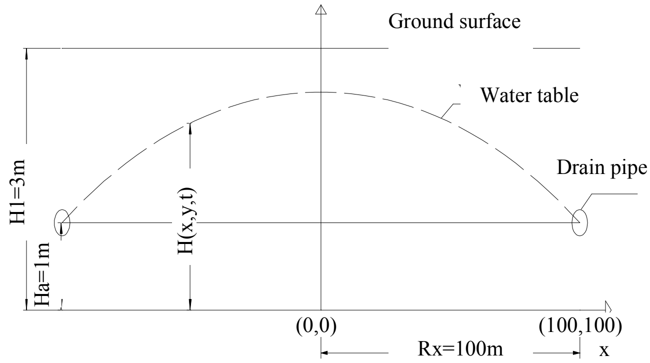

Fig 3: Cross section of the drainage system

Initial conditions:	t=0, Ha=H1=3m
Boundary conditions:	x=Rx=100m, Hx=Ha=1m, y=Ry=100m, Hy=Ha=1m
Spatial step:	dx=dy=10m
Time step:	dt=1day
Specific yield	S = 0,2

The uncertain parameter of the problem is the hydraulic conductivity K. A well-known value of K is 10m/day and cannot be greater than 20 m/day and less than 1 m/day. Those three values create a Triangular Fuzzy Number (TFN) (1,10,20).

When calculations include at least a fuzzy number, then the solution becomes a fuzzy number also [Dou, Woldt, Bogardi, Dahab 1995]. Consequently, for every time step and every spatial step, the water depth, which is a fuzzy number, is found.

Initially we use several a-cut levels (0.0, 0.25, 0.5, 0.75, 1.0) of the K (TFN) to obtain the solution of equation (3) and then from the intervals of the solutions we construct the fuzzy solution. So we use 5 a-cuts to obtain the solution

The second approach was to apply directly the three values of the K (TFN) to the equation (3). The three values of the solution of the equation (3), constitute the (TFN) of the solution. So we use 2 a-cuts to obtain the solution

Figure 4 shows both fuzzy solutions of equation (3) at point (0,0) for time t=10days. Figure 5 shows both fuzzy solutions at point (50,50) for time t=10days.

At the figures 4 and 5 "G025" is the gravity centre of the trapezium between the intervals at a-cut 0 and a-cut 0.25. Respectively "G050" is the gravity centre of the trapezium between the intervals at a-cut 0.25 and a-cut 0.50, "G075" is the gravity centre of the trapezium between the intervals at a-cut 0.50 and a-cut 0.75 and "G1" is the gravity centre of the triangle between the intervals at a-cut 0.75 and a-cut 1. Also "G_triangle" is the gravity centre of the TFN and "G total a-cut" is the gravity centre of the 5 a-cut shape solution

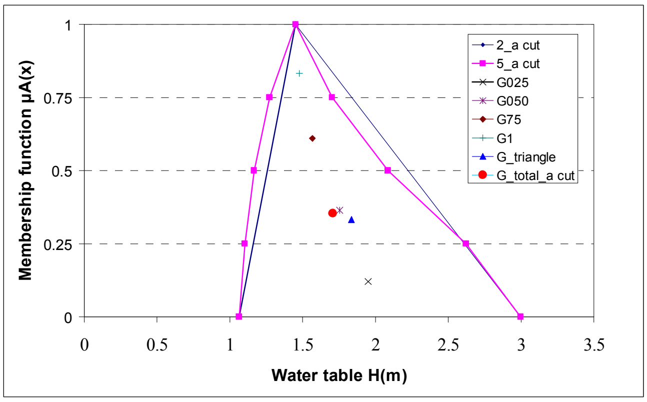

Fig 4. Fuzzy value of water table level H at position (0,0) after 10-day drainage

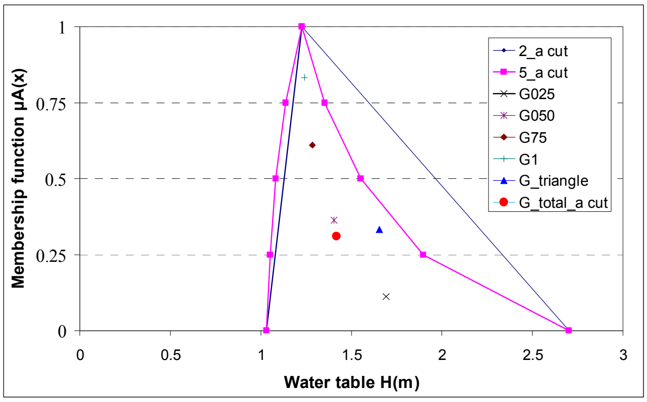

Fig 5. Fuzzy value of water table level H at position (50,50) after 10-days drainage

In this paper the solution was approached with an explicit fine element solution.

Specifically we applied an analytical solution and we defuzzified the two different fuzzy solutions just to compare the result.

For the defuzzification we used the method of fuzzy mean, or center of gravity. We calculate correspondingly the center of gravity of the TFN and from the shapes that are formed from the successive a-cuts and we use it as final result of our application. Thus the comparison of the results ends up being the comparison of the values of the centers of gravity.

The centre of gravity of the complex form is calculated by first calculating the centre of gravity of each of the three trapeziums and of the triangle which are generated from the a-cuts. Finally the resultant coordinates are calculated.

The mean square error (mse) is used in order to compare the defuzzified values of the two fuzzy solutions and the results are shown in table 1.

Table 1.

	TFN (2 a-cut level values)		**5 a-cut level values**	
x,y,t (m,m,days)	**XG2**	**YG2**	**XG5**	**YG5**
0,0,1	2.9662	0.3333	2.9785	0.3102
0,0,10	1.8365	0.3333	1.7061	0.3528
0,0,20	1.6552	0.3333	1.3414	0.2622
50,50,1	2.9662	0.3333	2.9785	0.3102
50,50,10	1.6534	0.3333	1.4167	0.3104
50,50,20	1.4251	0.3333	1.1925	0.2405
	h'	a	b	msr(%)
x,y,t (m,m,days)	max (XG2, XG5)	XG2/ h'	XG5/ h'	$(a-b)^2*100$
0,0,1	2.9785	0.9959	1.0000	0.0017
0,0,10	1.8365	1.0000	0.9290	0.5039
0,0,20	1.6552	1.0000	0.8104	**3.5933**
50,50,1	2.9785	0.9959	1.0000	0.0017
50,50,10	1.6534	1.0000	0.8569	2.0481
50,50,20	1.4251	1.0000	0.8367	**2.6655**
	h'	a	b	msr
x,y,t (m,m,days)	max (YG2, YG5)	YG2/ h'	YG5/ h'	$(a-b)^2*100$
0,0,1	0.3333	1.0000	0.9307	0.4803
0,0,10	0.3528	0.9449	1.0000	0.3039
0,0,20	0.3333	1.0000	0.7867	**4.5516**
50,50,1	0.3333	1.0000	0.9307	0.4803
50,50,10	0.3333	1.0000	0.9312	0.4739
50,50,20	0.3333	1.0000	0.7214	**7.7595**

4 Conclusions

The analytical solution for two-dimensional drainage problem was used for the investigation of the uncertainty of the aquifer's hydraulic conductivity on the numerical results, using two different techniques of fuzzy logic analysis.

With this analytical solution, the direct solution of the problem concerning the two-dimensional groundwater flow was achieved, without iterative calculations, for every point and at any time.

The two different fuzzy solutions were compared by using the mean square error (mse). The results that are shown at table 1 indicate that the two solutions are practically the same for drainage duration up to 10 days, since they differ less than 2.1% and the center of gravity of a 5 a-level cut is always included in the area of 2 α-level cut (TFN) application.

The restrictions of the iterative calculations, truncations errors or logic errors, are removed by the analytical solutions which permit simpler solving techniques.

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