LONGITUDINAL DYNAMICS ANALYSIS of BOEING 747-400

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Abstract: - In this paper, the equations of motion are derived and longitudinal stability equations are found and linearized. Following this, it is dedicated to stability derivatives of longitudinal dynamic model of Boeing 747-400.

Key-Words: - Short Period and Phugoid Approximation, Boeing 747-400, Longitudinal Dynamics

1 Introduction

In this paper longitudinal stability of a commercial airplane, Boeing 747-400 is analyzed. First of all an approach to the stability concept is done. Then by the help of *Newton's Second Law,* equations of motion are derived, subsequently longitudinal stability equations are found and linearized. Afterwards the equations are used for the stability analysis of Boeing 747-400. In this analysis, *MATHEMATICA* programming is also used; transfer functions are plotted with *MATLAB*.

The objective of *Stability and Control* is to develop fundamental understanding on the subject of stability, control and flight mechanics. The study of flying and handling characteristics is called stability and control **[1]**.

Starting from known forces and moments generated on a given wing, fuselage and tail configuration, it will be developed static and dynamic model of the aircraft to study its behavior under different flight regimes.

Concepts of static stability and dynamic stability will be introduced in next parts of the paper. General equations of motion for a rigid-body aircraft are derived. Basic motions of the aircraft separated into longitudinal modes are discussed in details. Laplace transform techniques are used in the analysis and the solution of the longitudinal equations.

2 Longitudinal Dynamics

As an introduction to longitudinal dynamics in order to obtain the transfer function of the aircraft, it is first necessary to obtain the equations of motion for the aircraft. The equations of motion are derived by applying Newton's Laws of motion which relate to the summation of the external forces and moments to the linear and angular accelerations of the system or body. Certain assumptions must be made to do this application. By the way, the application is done according to **[2]**.

Furthermore in longitudinal dynamics in order to get the linearized and Laplace transformed equations of motion, stability derivatives have to be also calculated.

Then the related force term and moment term are handled, the longitudinal equations of motion for the aircraft are written as;

$$\left(\frac{mU}{Sq}\dot{u}-C_{X_U}u\right)+\left(-\frac{c}{2U}C_{X_{\dot\alpha}}-C_{X_\alpha}\alpha\right)+\left[-\frac{c}{2U}C_{X_q}\dot\theta-C_W(\cos\theta)\theta\right]=C_{F_{X_\alpha}}$$

$$-\left(C_{Z_U}u\right)+\left[\left(\frac{mU}{Sq}-\frac{c}{2U}C_{Z_{\dot\alpha}}\right)\dot\alpha-C_{Z_\alpha}\alpha\right]+\left[\left(-\frac{mU}{Sq}-\frac{c}{2U}C_{Z_q}\right)\dot\theta-C_W(\sin\theta)\theta\right]=C_{F_{Z_\alpha}}$$

$$\left(-C_{m_u}u\right)+\left(-\frac{c}{2U}C_{m_{\dot\alpha}}\dot\alpha-C_{m_\alpha}\alpha\right)+\left(\frac{I_y}{Sqc}\ddot\theta-\frac{c}{2U}C_{m_q}\dot\theta\right)=C_{m_a}$$

(2.1)

These equations assume that:

1. The X and Z axes lie in the plane of symmetry and the origin of the axis system is at the center of the gravity of the aircraft.
2. The mass of the aircraft is constant.
3. The aircraft is a rigid body.
4. The earth is an inertial reference.
5. The perturbations from equilibrium are small.
6. The flow is quasi-steady.

In solving the equations of motions it is necessary to obtain the transient solution, which is obtained from homogenous equations, that is, with no external inputs $C_{m\alpha}=C_{Fza}=C_{Fxa}=0$. Taking the Laplace Transform (discussed before) of Equation 2.1 with the initial conditions zero and neglecting $C_{X_{\dot\alpha}}, C_{X_q}, C_{m_u}$ yields **[2]**:

$$\left(\frac{mU}{Sq}s - C_{X_U}\right)\grave{u}(s) - C_{X_\alpha}\grave{\alpha}(s) - C_W(\cos\theta)\theta(s) = 0$$

$$-C_{Z_U}\grave{u}(s) + \left[\left(\frac{mU}{Sq} - \frac{c}{2U}C_{Z_{\dot{\alpha}}}\right)s - C_{Z_\alpha}\right]\grave{\alpha}(s) + \left[\left(-\frac{mU}{Sq} - \frac{c}{2U}C_{Z_q}\right)s - C_W(\sin\theta)\right]\theta(s) = 0$$

$$\left(-\frac{c}{2U}C_{m_{\dot{\alpha}}}s - C_{m_\alpha}\right)\grave{\alpha}(s) + \left(\frac{I_y}{Sqc}s^2 - \frac{c}{2U}C_{m_q}s\right)\theta(s) = 0 \quad \textbf{(2.2)}$$

3 Calculation of the Stability Derivatives for the Aircraft

The selected aircraft ***Boeing 747-400*** is flying in straight level flight at **20000 ft** with a velocity of **673 ft**/s and the compressibility effects are neglected. For this aircraft the values **[3]** are given like in the table below.

Table 3.1 Stability Characteristics of the Aircraft

Aircraft	747-400
Parameters	
Altitude (ft)	20,000
Mach	0.650
True Speed (ft/s)	673
Dynamic Pressure (lb/ft^2)	287.2
Weight (lb)	636,636
Wing Area-S-(ft^2)	5,500
Wing Span-b-(ft)	196
Wing Chord-c-(ft)	27.3
C.G.(x c)	0.25
Trim AOA (deg)	2.5
I_{xxs}(slugs-ft^2)	$1.82x10^7$
I_{yys}(slugs-ft^2)	$3.31x10^7$
I_{zzs}(slugs-ft^2)	$4.97x10^7$
I_{xzs}(slugs-ft^2)	$-4.05x10^5$
Longitudinal Derivatives	
X_u (1/s)	-0.0059
X_α (ft/s^2)	15.9787
Z_u (1/s)	-0.1104
Z_α (ft/s^2)	-353.52
M_u (1/ft.s)	0
M_α ($1/s^2$)	-1.3028
$M_{\acute{\alpha}}$ (1/s)	-0.1057
M_q (1/s)	-0.5417
$X_{\delta e}$ (ft/s^2)	0.0000
$Z_{\delta e}$ (ft/s^2)	-25.5659
$M_{\delta e}$ ($1/s^2$)	-1.6937

Before calculations, some additional coefficients must be found; here below table contains the coefficients from Roskam J., *Aircraft Flight Dynamics and* Automatic Flight Controls **[4]**. Furthermore they are based on computer models rather than wind-tunnel tests or other real-word observations, and use stability axes.

Table 3.2 Additional Coefficients for the Aircraft

S	5500 ft^2	**C_{Du}**	0
$\bar{c}$	27.3 ft	**$C_{D\alpha}$**	0.2
b	196 ft	**C_{TXu}**	-0.055
h	20000 ft	**C_{Lo}**	0.21
M	0.65	**C_{Lu}**	0.13
U_1	673 fps	**$C_{L\alpha}$**	4.4
$\bar{q}$	287.2 lb/ft^2	**$C_{L\acute{\alpha}}$**	7
CG	0.25 % $\bar{c}$	**C_{Lq}**	6.6
α_1	2.5 deg	**C_{mo}**	0
W	636636 lb/ft^2	**C_{mu}**	0.013
I_{xx}	18200000 Slug ft^2	**$C_{m\alpha}$**	-1
I_{yy}	33100000 Slug ft^2	**$C_{m\acute{\alpha}}$**	-4
I_{zz}	49700000 Slug ft^2	**C_{mq}**	-20.5
I_{xz}	970000 Slug ft^2	**C_{mTu}**	0
C_{L1}	0.4	**$C_{mT\alpha}$**	0

C_{D1}	0.025	C_{DDe}	0
C_{TX1}	0.025	C_{LDe}	0.32
C_{m1}	0	C_{mDe}	-1.3
C_{MT1}	0	C_{Dih}	0
C_{Do}	0.0164	C_{lih}	0.7
C_{mih}	-2.7		

For low cruise condition, stability derivatives are calculated by the help of *MATHEMATICA* and solved for transfer functions, damping ratio and natural frequency for nonzero solution also both for short period and phugoid approximation. Calculations are shown;
To obtain nonzero solution, the values in coefficients matrix A are calculated as in the below table;

Table 3.3 The Values in Coefficients Matrix A

X_u	-0.005930	Z_{δ_e}	-25.5453
X_{Tu}	-0.005930	M_u	0.0000251658
X_α	15.9658	M_{Tu}	0
X_{δ_e}	0	M_α	-1.30281
Z_u	-0.110314	$M_{T\alpha}$	0
Z_α	-355.239	$M_{\dot{\alpha}}$	-0.105696
$Z_{\dot{\alpha}}$	-11.3338	M_q	-0.541693
Z_q	-10.6862	M_{δ_e}	-1.69366

4 The Nonzero Solution of the Longitudinal Equations

$$\frac{\theta(s)}{\delta_e(s)} = \frac{-1.68971(0.0119211+s)(0.486136+s)}{(0.00465356+0.00453985s+s^2)(1.5423+1.16507s+s^2)}$$

The nonzero solution of longitudinal equations of motion in matrix form is:

$$\begin{pmatrix} (s-X_u-X_{T_u}) & -X_\alpha & gCos[\theta] \\ -Z_u & (s(U-Z_{\dot{\alpha}})-Z_\alpha) & (-(Z_q+U)s+gSin[\theta]) \\ -(M_u+M_{T_u}) & -(M_{\dot{\alpha}}s+M_\alpha+M_{T_\alpha}) & (s^2-M_q s) \end{pmatrix} \begin{pmatrix} 'u(s) \\ '\alpha(s) \\ \theta(s) \end{pmatrix} = \begin{pmatrix} 0 \\ C_{Z\delta_e} \\ C_{M\delta_e} \end{pmatrix}$$

The nonzero solution of longitudinal equations of motion is:

$$\begin{pmatrix} 0.00652392+s & -15.96582 & 32.174 \\ 0.110314 & 355.2394+684.334s & -662.314s \\ -0.0000251658 & 1.302818+0.105696196s & 0.541693s+s^2 \end{pmatrix} \begin{pmatrix} 'u(s) \\ '\alpha(s) \\ \delta_e(s) \end{pmatrix} = \begin{pmatrix} 0 \\ -25.5453 \\ -1.69366 \end{pmatrix}$$

The lonely nonzero solution of these simultaneous equations requires that the determinant of the coefficients be zero;

$$A=\begin{pmatrix} 0.00652392+s & -15.96582 & 32.174 \\ 0.110314 & 355.2394+684.334s & -662.314s \\ -0.0000251658 & 1.302818+0.105696196s & 0.541693s+s^2 \end{pmatrix}$$

Expanding this determinant of the following quadratic equation is obtained;

$$684.334\,(0.00465358+0.00453985s+s^2)\,(1.54231+1.16507s+s^2) =0$$

A common way to write these kinds of quadratic equations is to indicate to natural frequency and the damping ratio as;

$$(s^2+2\xi_p\omega_{np}s+\omega_{np}^2)(s^2+2\xi_s\omega_{ns}s+\omega_{ns}^2)=0$$

According to the equation () Short period oscillations' natural frequency and damping ratio are found as;

$$\left.\begin{aligned} \omega_{ns} &= 1{,}2419\,rad/\sec \\ \xi_s &= 0{,}469071 \end{aligned}\right\} \textit{short period oscillation}$$

$$\left.\begin{aligned} \omega_{np} &= 0{,}068217\,rad/\sec \\ \xi_p &= 0{,}033275 \end{aligned}\right\} \textit{phugoid oscillation}$$

Furthermore for short period and phugoid mode one half amplitudes are:

$$\textit{For}\ \text{Short Period Mode}: \tau_{1/2} = 1{,}71663\sec.$$

$$\textit{For}\ \text{Phugoid Mode} \quad : \tau_{1/2} = 440{,}53\sec. = 7{,}3421\min.$$

5 Transfer Functions for the Elevator Displacement

Taking the Laplace Transform of the longitudinal linearized equations with nonzero initial conditions yields:

♦ The calculation of transfer function of $\frac{`u(s)}{\delta_e(s)}$;

$$\frac{`u(s)}{\delta_e(s)} = \frac{-0.595983\,(-47.7053+s)\,(0.93987356+s)}{(0.00465356+0.00453985\,s+s^2)\,(1.5423+1.16507\,s+s^2)}$$

♦ The calculation of transfer function of $\frac{`\alpha(s)}{\delta_e(s)}$;

$$\frac{`\alpha(s)}{\delta_e(s)} = \frac{-0.0373287\,(44.4533+s)\,(0.0053117+0.00640441`\,s+s^2)}{(0.00465356`+0.00453985\,s+s^2)\,(1.5423+1.16507\,s+s^2)}$$

♦ The calculation of transfer function of $\frac{\theta(s)}{\delta_e(s)}$;

$$\frac{\theta(s)}{\delta_e(s)} = \frac{-1.68971\,(0.0119211+s)\,(0.486136+s)}{(0.00465356+0.00453985\,s+s^2)\,(1.5423+1.16507\,s+s^2)}$$

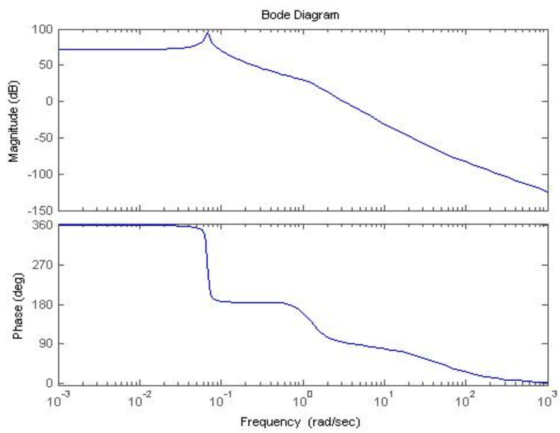

Figure 5.1 Magnitude plot for $\frac{`u(s)}{\delta_e(s)}$ transfer function versus ω for s=j ω

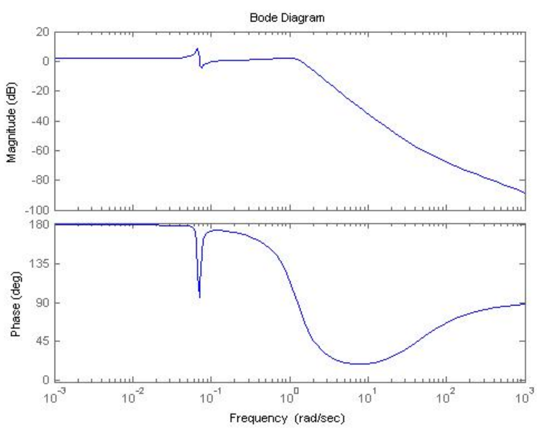

Figure 5.2 Magnitude plot for $\frac{`\alpha(s)}{\delta_e(s)}$ transfer function versus ω for s=j ω

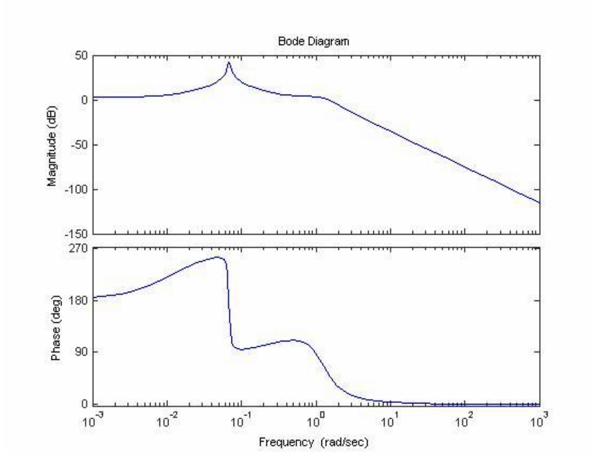

Figure 5.3 Magnitude plot for $\frac{\theta(s)}{\delta_e(s)}$ transfer function versus ω for s=j ω

6 Short Period Approximation

The short period oscillation occurs at almost constant forward speed; therefore let $`u = 0$ in the equations of motion. By neglecting $C_{z_{\dot\alpha}}$ and C_{z_q} and inserting $C_{z_{\delta e}}$ and $C_{m_{\delta e}}$.

♦ For the short period approximation, the transfer function of $\frac{`\alpha(s)}{\delta_e(s)}$ is;

$$\frac{`\alpha(s)}{\delta_e(s)} = \frac{-1.71422\ s - 0{,}0379574\ s^2}{1.58874\ + 1.175235\ s + s^2}$$

♦ For the short period approximation, the transfer function of $\frac{\theta(s)}{\delta_e(s)}$ is;

$$\frac{\theta(s)}{\delta_e(s)} = \frac{-0{,}844535\ -1{,}68964\ s}{s(1.58874\ + 1.175235\ s + s^2)}$$

♦ The natural frequency for the short period approximation;

$$\omega_{nSP} = \sqrt{\frac{Z_\alpha M_q}{U} - M_\alpha}$$

$$\omega_{nSP} = 1{,}26045\,rad/\sec$$

♦ The damping ratio for the short period approximation;

$$\zeta_{SP} = \frac{-\left(M_q + \frac{Z_\alpha}{U} + M_{\dot\alpha}\right)}{2\,\omega_{nSP}}$$

$$\zeta_{SP} = 0{,}466195$$

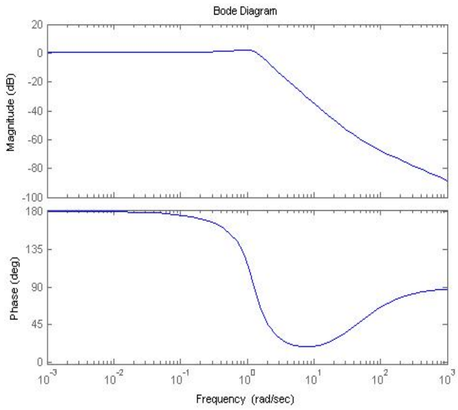

Figure 6.1 Magnitude plot for $\frac{`\alpha(s)}{\delta_e(s)}$ transfer function versus ω for s=j ω

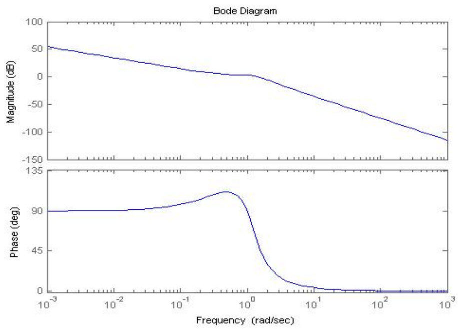

Figure 6.2 Magnitude plot for $\frac{\theta(s)}{\delta_e(s)}$ transfer function versus ω for s=j ω

7 Phugoid Approximation

The phugoid oscillation takes place at almost constant angle of attack, thus $`\alpha$ can be set to zero. Furthermore as phugoid oscillation is of long period, θ is varying quite slowly; additionally the inertia forces can be neglected.

- ♦ For the phugoid approximation, the transfer function of $\frac{`u(s)}{\delta_e(s)}$ is;

$$\frac{'u(s)}{\delta_e(s)} = \frac{821.894}{-3.54923 - 4.32088\, s - 662.314\, s^2}$$

- ♦ For the phugoid approximation, the transfer function of $\frac{\theta(s)}{\delta_e(s)}$ is;

$$\frac{\theta(s)}{\delta_e(s)} = \frac{-0.166655 - 25.5453\, s}{-3.54923 - 4.32088\, s - 662.314\, s^2}$$

- ♦ The natural frequency for the phugoid approximation;

$$\omega_{nSP} = 0{,}0726205 rad / \sec$$

- ♦ The damping ratio for the phugoid approximation;

$$\zeta_{SP} = 0{,}036751$$

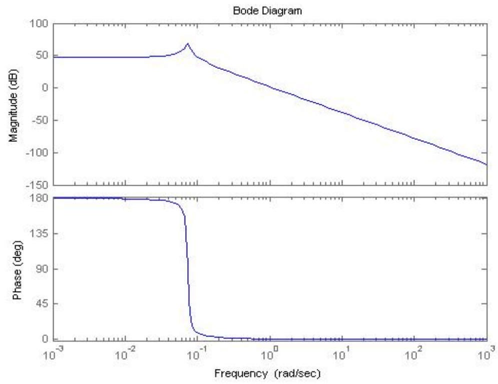

Figure 7.1 Magnitude plot for $\frac{`u(s)}{\delta_e(s)}$ transfer function versus ω for s=j ω

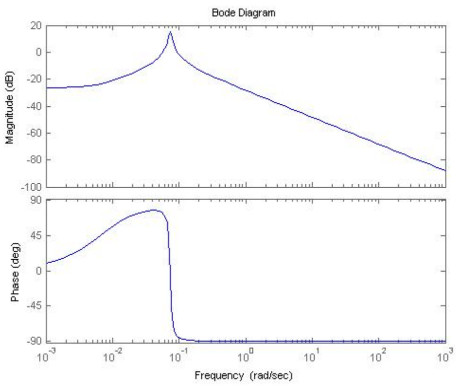

Figure 7.2 Magnitude plot for $\frac{\theta(s)}{\delta_e(s)}$ transfer function versus ω for s=j ω

8 Transient Response of the Aircraft

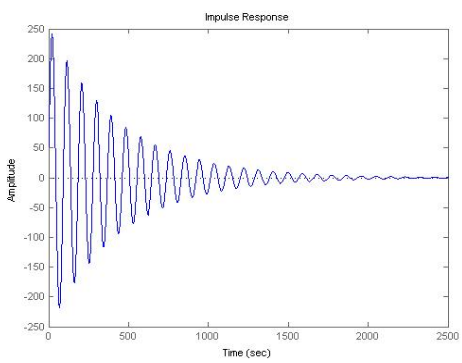

Figure 8.1 Transient response of the aircraft for $`u$

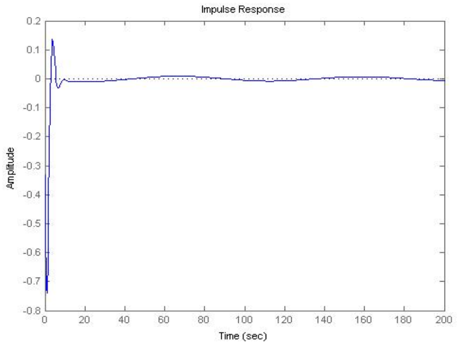

Figure 8.2 Transient response of the aircraft for `α

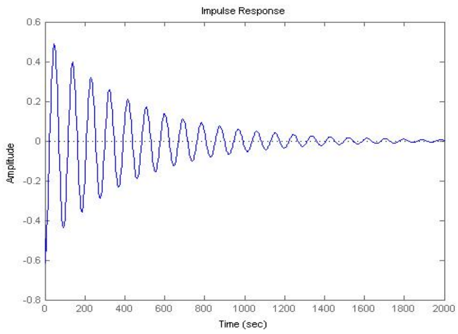

Figure 8.3 Transient response of the aircraft for θ

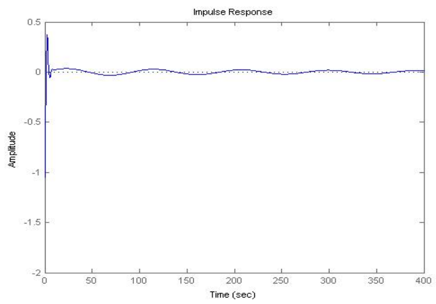

Figure 8.4 Transient response of the aircraft for θ

9 Conclusion

The equations of motion are derived and longitudinal stability equations are found and linearized. It is dedicated to stability derivatives of longitudinal dynamic model of Boeing 747-400. Then, transfer functions of elevator displacement are calculated and Bode diagrams are drawn.

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