Numerical Accuracy and Hardware Trade-Offs for Fixed-Point CORDIC Processor for Digital Signal Processing System

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Abstract: - In this paper, the complete error analysis of the conventional CORDIC (COordinate Rotation DIgital Computer) algorithm and the CORDIC with expanded convergence range is presented. All the computational error consisting of the approximation error and truncation error in both the rotation mode and vectoring mode has been derived systematically. It has been shown that the computation errors of CORDIC processor are dependent on the word length and number of iterations. The main contribution of this paper is a complete set of formulas describing the computation errors of CORDIC have been summarized in tabular form. By referring to these tables, one can design a cost-effective digital signal processing system using CORDIC processor in terms of areas and performances.

Key-Words: - CORDIC, digital signal processing system, convergence range, approximation error, truncation error.

1 Introduction

CORDIC (COordinate Rotation DIgital Computer) is an efficient algorithm for the evaluation of various elementary functions such as trigonometric functions, hyperbolic functions, exponentials, logarithm, and square-roots [1]-[2]. As the hardware implementation of CORDIC processor may require only simple adders and shifters, it has received a lot of attention for many applications in which intensive calculation of the aforesaid functions are needed.

Though, CORDIC has been and will continue to be greatly applied to many applications, there is few literature on the analysis of computational error and numerical accuracy. It is highly probable that a designer of CORDIC processor may simply increase the number of iterations to reduce the amount of computational error without taking account of the numbers of fractional binary digits and mantissa bits. The error analysis of CORDIC arithmetic proposed by Hu [3] leads to a different viewpoint of the design of CORDIC processors. In the well-known work of Kota [4], only the error analysis of the circular CORDIC algorithm had been taken into consideration. Moreover, the mean-square error analyses of quantization and rounding have been studied for many digital signal processing applications such as FFT and DCT [5]-[8]. Sung [9]-[11] had analyzed the computation errors of CORDIC, and evaluated these errors in terms of hardware implementations. Recently, Sung and Hsin [12] proposed a double rotation algorithm to speed up the convergence of CORDIC processor for fast sine and cosine generation; the FPGA and VLSI implementations with numerical accuracy analysis can be found in [13].

In hardware implementation, the computation errors of CORDIC are dependent on the number of iterations and word length, which need to be taken into account to design an optimal CORDIC processor. From the design point of view, the prediction of computational error is therefore desirable. A complete set of formulas describing the computation errors of CORDIC has been derived and empirically evaluated in this paper, which can be used as a reference for the design of an optimal CORDIC processor.

The remainder of this paper proceeds as follows. In section 2, an overview of CORDIC is presented. In Section 3, proofs of the domain of convergence are given. In section 4, the error analysis of the CORDIC algorithm is derived. In section 5, the numerical accuracy and hardware trade-offs for implementation is discussed. Conclusion can be found in section 6.

2 Overview of the CORDIC Algorithm

The basic formula of CORDIC algorithm [2] is as follows,

$$x(i+1) = x(i) + m\delta(i)y(i)2^{-s(m,i)} \quad (1)$$

$$y(i+1) = y(i) - \delta(i)x(i)2^{-s(m,i)} \quad (2)$$

$$z(i+1) = z(i) - \delta(i)\alpha_m(i) \quad (3)$$

where $s(m,i)$ is a non-decreasing "shift-sequence" of integers satisfying $s(m,i) \le s(m,i+1) \le s(m,i)+1$,

$$\alpha_m(i) = m^{-1/2} \tan^{-1}[m^{1/2} 2^{-s(m,i)}] \quad (4)$$

is the angle rotated in the i^{th} iteration, $m = +1, -1\, or\, 0$ corresponds to the circular, hyperbolic, or linear coordinate system, respectively, and the direction of rotation: $\delta(i) = +1$ or -1 representing the different directions. Specifically, $s(0,i) = 1,2,....,n$; $s(1,i) = 0,1,....,n-1$; $s(-1,i) = 0,1,2,....,n-1$ with repeated $4,13,40,....,i,3i+1,......$

3 Domain of Convergence

As CORDIC is an iterative algorithm, the domain of convergence needs to be determined. Walther [2] proposed a converging criterion without proof. In this section, we prove this criterion with more theorems for convergence.

Theorem 1:

$$\alpha_m(i) - \sum_{j=i+1}^{n-1} \alpha_m(j) \le \alpha_m(n-1);\ 0 \le i < n-1 \quad (5)$$

***Proof*:**

For the linear CORDIC algorithm, the rotation angle $\alpha_0(i)$ is equal to 2^{-i} . Thus, we have the following inequality:

$$\alpha_0(i) - \alpha_0(i+1) = 2^{-i} - 2^{-(i+1)} = 2^{-(i+1)} \le 2^{-(i+1)} = \alpha_0(i+1)$$

For the circular CORDIC algorithm, the rotation angle $\alpha_1(i)$ is equal to $\tan^{-1}(2^{-i})$.

Let $\alpha_1(i) = \theta_1$ *and* $\alpha_1(i+1) = \phi_1$, since

$$\tan(\theta_1 - \phi_1) = \frac{\tan\theta_1 - \tan\phi_1}{1 + \tan\theta_1 \tan\phi_1} = \frac{2^{-(i+1)}}{1 + 2^{-2i-1}} \le 2^{-(i+1)} = \tan\phi_1 ,$$

we have $\theta_1 - \phi_1 \le \phi_1$,i.e. $\alpha_1(i) - \alpha_1(i+1) \le \alpha_1(i+1)$.

For the hyperbolic CORDIC algorithm, the rotation angle $\alpha_{-1}(i)$ is equal to $\tanh^{-1}(2^{-i})$. It does not converge with this sequence. However, a modified sequence with repeated rotations at some specific iteration stages may be used for convergence [2]. Thus, the convergence criterion becomes $\alpha_{-1}(i) - \sum_{j=i+1}^{n-1} \alpha_{-1}(j) - \alpha_{-1}(3j+1) \le \alpha_{-1}(n-1)$, which leads to the sequence with repeated rotation angles at iteration number $4, 13, 40, ..., j, 3j+1, ...$

***Theorem 2**:*

$$|z(i)| \le \alpha_m(n-1) + \sum_{j=i}^{n-1} \alpha_m(j) \quad (6)$$

***Proof*:**

It is noted that $|z(i+1)| \le \alpha_m(i)$, together with *Theorem 1*, we have $|z(i+1)| \le \alpha_m(n-1) + \sum_{j=i+1}^{n-1} \alpha_m(j)$, and consequently the proof is complete.

***Lemma 1*:** The maximum input angle $|Z_{\max}(0)|$ called the range of convergence is given by

$$|Z_{\max}(0)| = \alpha_m(n-1) + \sum_{j=0}^{n-1} \alpha_m(j) \quad (7)$$

***Proof*:**

By *Theorem 2*, we have $|Z(0)| \le \alpha_m(n-1) + \sum_{j=0}^{n-1} \alpha_m(j)$.

As $Z(0)$ can be considered the initial input angle, it follows that the range of convergence is $|Z_{\max}(0)| = \alpha_m(n-1) + \sum_{j=0}^{n-1} \alpha_m(j)$.

The convergence criterion of the CORDIC algorithm with expanded convergence range is discussed as follows [14]:

In the linear coordinate system, $\frac{\alpha_0(i)}{\alpha_0(i+1)} \le 2$ satisfying the convergence criterion, thus

$$|Z_0(0)| \le \theta_{0(\max)} = 2^{M+1} \quad (8)$$

where $M > 0$.

In the circular coordinate system, let the iteration sequence $j = 0,0,0,1,2,....,n$, to avoid left shift in the design of CORDIC for hardware simplicity. Together with $\frac{\alpha_1(i)}{\alpha_1(i+1)} \le 2$, which satisfies the converging criterion, we have

$$\theta_{1(\max)} = \tan^{-1}(2^{-n}) + 2\tan^{-1}(2^0) + \sum_{j=0}^{n} \tan^{-1} 2^{-j} \cong 189.9^\circ > 180^\circ \quad (9)$$

By equation (9), the range of input can be unlimited.

In the hyperbolic coordinate system, $\tanh^{-1}(1 - 2^{-2^{-i+1}})$ is a complex value for $i \le 0$ [14]. As $\alpha_{-1}(i) = \tanh^{-1}(1 - 2^{-2^{-i+1}})$ for $i \le 0$,

$\frac{\alpha_{-1}(i)}{\alpha_{-1}(i+1)} = \frac{\tanh^{-1}(1 - 2^{-2^{-i+1}})}{\tanh^{-1}(1 - 2^{-2^{-(i+1)+1}})} \le 2$ is satisfied for the converging criterion, and we have

$$\theta_{-1(\max)} = \sum_{j=-M}^{0} \tanh^{-1}(1 - 2^{-2^{-j+1}}) + \tanh^{-1} 2^{-n} + \sum_{j=1}^{n} \tanh^{-1}(2^{-j}) \quad (10)$$

4 Numerical Analysis of the CORDIC Algorithm

Three types of representation are taken into consideration for the CORDIC algorithm given by equations (1)-(4), namely mathematical representation, hypothetical representation and real-word representation [2]. In mathematical representation, $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\boldsymbol{v}}_m, \theta_m)$ denotes the variables involved with infinite precision, where $\tilde{\boldsymbol{v}}_m = [\tilde{x}\ \tilde{y}]^T$. In hypothetical representation, the variables denoted by $(x, y, z, \boldsymbol{v}_m, \alpha_m)$ are represented by an infinite number of bits, and updated in the iterative manner with a sequence of appropriate rotation directions: $\delta(i)$'s. Both the iterations and word lengths of the variables denoted by $(\hat{x}, \hat{y}, \hat{z}, \hat{\boldsymbol{v}}_m, \phi_m)$ are finite in real-world representation. The truncation error in the rotation mode can be defined by the use of L_2-norm, i.e. $\|\boldsymbol{v}_m(n) - \hat{\boldsymbol{v}}_m(n)\|$, and the overall approximation error [4] is given by

$$\|\tilde{\boldsymbol{v}}_m(n) - \hat{\boldsymbol{v}}_m(n)\| \le \|\tilde{\boldsymbol{v}}_m(n) - \boldsymbol{v}_m(n)\| + \|\boldsymbol{v}_m(n) - \hat{\boldsymbol{v}}_m(n)\| \quad (11)$$

In the vectoring mode, $|\alpha_m - \phi_m|$ denotes the truncation error, and the overall approximation error is as follows.

$$|\theta_m - \phi_m| \le |\theta_m - \alpha_m| + |\alpha_m - \phi_m| \quad (12)$$

In the rotation mode, $\|\tilde{\boldsymbol{v}}_m(n) - \boldsymbol{v}_m(n)\|$ and $\|\tilde{\boldsymbol{v}}_m(n) - \boldsymbol{v}_m(n)\| / \|\tilde{\boldsymbol{v}}_m(n)\|$ denote the absolute and relative approximation errors, respectively. The error analysis is performed on the basis of the given numbers of iterations, fractional binary digits and binary digits of mantissa, which are denoted by n, b, and b_v, respectively. As

$$\left|\theta_m - \sum_{i=0}^{n-1} \delta(i)\alpha_m(i)\right| \le \alpha_m(n-1) \text{ [2], we have}$$

$$\left|\theta_m - \sum_{i=0}^{n-1} \delta(i)Q[\alpha_m(i)]\right| \le Q[\alpha_m(n-1)] \quad (13)$$

where Q denotes the quantization operator defined by the use of truncation with a given number of fractional binary digits for implementation simplicity. In addition,

$$\alpha_m - \sum_{i=0}^{n-1} \delta(i)Q[\alpha_m(i)] = \sum_{i=0}^{n-1} \delta(i)e(i) \quad (14)$$

where $\alpha_m = \sum_{i=0}^{n-1} \delta(i)\alpha_m(i)$,

and $e(i) = \alpha_m(i) - Q[\alpha_m(i)]$ denotes the quantization error of $\alpha_m(i)$. It is noted that $e(i) < 2^{-b} \equiv \mu$, which is essentially the truncation error with b fractional binary digits. Hence,

$$\left|\alpha_m - \sum_{i=0}^{n-1} \delta(i)Q[\alpha_m(i)]\right| < n\mu \quad (15)$$

By equations (13) and (15), we have

$$|\theta_m - \alpha_m| \le Q[\alpha_m(n-1)] + n\mu \quad (16)$$

4.1 Rotation Mode

4.1.1 Relative Approximation Errors

Theorem 3: In the rotation mode, the relative approximation error $\|\tilde{\boldsymbol{v}}_m(n) - \boldsymbol{v}_m(n)\| / \|\tilde{\boldsymbol{v}}_m(n)\|$ of the circular, hyperbolic and linear CORDIC are bounded by $\tan^{-1} 2^{-(n-1)} + n \cdot 2^{-b}$, $\left|e^{\tanh^{-1} 2^{-(n-r)}} + n \cdot 2^{-b} - 1\right|$ and $2^{-(n-1)} + n \cdot 2^{-b}$, respectively.

Proof:

In the circular CORDIC algorithm, define $\boldsymbol{M}_1 = \begin{bmatrix} \cos\theta_1 - \cos\alpha_1 & -(\sin\theta_1 - \sin\alpha_1) \\ \sin\theta_1 - \sin\alpha_1 & \cos\theta_1 - \cos\alpha_1 \end{bmatrix}$ such that $\tilde{\boldsymbol{v}}_1(n) - \boldsymbol{v}_1(n) = K_1 \cdot \boldsymbol{M}_1 \cdot \boldsymbol{v}_1(0)$, and

$$\|\tilde{\boldsymbol{v}}_1(n) - \boldsymbol{v}_1(n)\| = K_1 \|\boldsymbol{M}_1\| \cdot \|\boldsymbol{v}_1(0)\| \le K_1 \|\boldsymbol{v}_1(0)\| \cdot |\theta_1 - \alpha_1| \quad (17)$$

where $\|\boldsymbol{M}_1\| = \sqrt{2 - 2\cos(\theta_1 - \alpha_1)} \le |\theta_1 - \alpha_1|$. Together with equation (17), we have

$$\begin{aligned} &\|\tilde{\boldsymbol{v}}_1(n) - \boldsymbol{v}_1(n)\| / \|\boldsymbol{v}_1(n)\| \\ &\le Q[\alpha_1(n-1)] + n\mu \le \tan^{-1} 2^{-(n-1)} + n \cdot 2^{-b} \end{aligned} \quad (18)$$

Similarly, define the corresponding matrix $\boldsymbol{M}_{-1}$ in the hyperbolic CORDIC algorithm such that

$$\|\tilde{\boldsymbol{v}}_{-1}(n) - \boldsymbol{v}_{-1}(n)\| = K_{-1} \|\boldsymbol{M}_{-1}\| \|\boldsymbol{v}_{-1}(0)\| \quad (19)$$

It follows that

$$\|\tilde{\boldsymbol{v}}_{-1}(n) - \boldsymbol{v}_{-1}(n)\| / \|\tilde{\boldsymbol{v}}_{-1}(n)\| \le \left|e^{\tanh^{-1} 2^{-(n-r)} + n2^{-b}} - 1\right| \quad (20)$$

where r is the number of repeated iterations.

In the linear CORDIC algorithm, we have

$$\|\tilde{\boldsymbol{v}}_0(n) - \boldsymbol{v}_0(n)\| = \left\| \begin{bmatrix} 0 & 0 \\ \theta_0 - \alpha_0 & 0 \end{bmatrix} \right\| \|\boldsymbol{v}(0)\| = |\theta_0 - \alpha_0| \|\boldsymbol{v}(0)\| \quad (21)$$

and $\|\tilde{\boldsymbol{v}}_0(n)\|$ can be written by

$$\|\tilde{\boldsymbol{v}}_0(n)\| = \left[\frac{\theta_0^2 + \sqrt{\theta_0^4 + 4\theta_0^2} + 2}{2} \right]^{1/2} \|\boldsymbol{v}_0(0)\| \ge \|\boldsymbol{v}_0(0)\| \quad (22)$$

From equations (11) and (22), we obtain

$$\|\tilde{\boldsymbol{v}}_0(n) - \boldsymbol{v}_0(n)\| / \|\tilde{\boldsymbol{v}}_0(n)\| \le |\theta_0 - \alpha_0| \le 2^{-(n-1)} + n \cdot 2^{-b} \quad (23)$$

where $-\pi \le \theta_0 \le \pi$

4.1.2 Truncation Errors

Theorem 4: In the rotation mode, the fixed-point truncation error $\|Q[\hat{\boldsymbol{v}}_m(n)] - \boldsymbol{v}_m(n)\|$ of the circular, hyperbolic and linear CORDIC algorithms is bounded by

$\sqrt{2}\varepsilon\left[1+\sum_{j=1}^{n-1}(\prod_{i=j}^{n-1}\sqrt{1+2^{-2i}})\right]$,

$\sqrt{2}\varepsilon\left\{1+\sum_{j=1}^{n-1}[\prod_{i=j}^{n-1}\sqrt{1-2^{-2i}}]\right\}\cdot e^{\sum_{i=j}^{n-1}\tanh^{-1}2^{-i}}$, and $n\cdot\varepsilon$,

respectively.

Proof:

In the circular CORDIC algorithm, the relation between vectors $\boldsymbol{v}_1(i+1)$ and $\boldsymbol{v}_1(i)$ is given by $\boldsymbol{v}_1(i+1)=\boldsymbol{p_1}(i)\boldsymbol{v}_1(i)$,

where $\boldsymbol{p_1}(i)=\begin{bmatrix}1 & -\delta(i)2^{-i}\\ \delta(i)2^{-i} & 1\end{bmatrix}$ for the i^{th} rotation.

Thus,

$$\left\|Q[\tilde{\boldsymbol{v}}_1(n)]-\boldsymbol{v}_1(n)\right\|\le\|e(n)\|+\sum_{j=1}^{n-1}\left\|\prod_{i=j}^{n-1}\boldsymbol{p_1}(i)\right\|\cdot\|e(j)\|$$
$$\le\sqrt{2}\varepsilon\left[1+\sum_{j=1}^{n-1}(\prod_{i=j}^{n-1}\sqrt{1+2^{-2i}})\right] \quad (24)$$

where $\|e(i)\|\le\sqrt{2}\varepsilon$ for some ε.

Similarly, in the hyperbolic CORDIC algorithm with the corresponding transition matrix $\boldsymbol{p}_{-1}(i)$ for the i^{th} rotation, it can be shown that

$$\left\|Q[\hat{\boldsymbol{v}}_{-1}(n)]-\boldsymbol{v}_{-1}(n)\right\|$$
$$\le\sqrt{2}\varepsilon\left\{1+\sum_{j=1}^{n-1}\left[\prod_{i=j}^{n-1}\sqrt{1-2^{-2i}}\right]\cdot e^{\sum_{i=j}^{n-1}\tanh^{-1}2^{-i}}\right\} \quad (25)$$

In the linear CORDIC algorithm, it is noted that $Q[\hat{y}(i+1)]=Q[\hat{y}(i)]+\delta(i)\alpha_0(i)Q[\hat{x}(i)]-e(i+1)$ and

$$Q[\hat{y}(n)]=y(0)+\alpha_0\cdot x(0)-\sum_{i=1}^{n}e(i) \quad (26)$$

Thus, we have

$$\left|Q[\hat{y}(n)]-y(n)\right|\le\sum_{i=1}^{n}|e(i)|\le n\cdot\varepsilon \quad (27)$$

4.2 Vectoring Mode

4.2.1 Approximation Errors

Theorem 5**:** In the vectoring mode, the approximation error $|\theta_m-\alpha_m|$ of the circular, hyperbolic and linear CORDIC algorithms is bounded by

$\sin^{-1}(Q[\alpha_1(n-1)]+n\mu)$, $\sinh^{-1}\left|e^{|\alpha_{-1}-\theta|}-1\right|$ and

$Q[\alpha_0(n)]+n\mu$, respectively.

Proof**:**

In the circular CORDIC algorithm, by equation (17) and $\tilde{y}(n)=0$ we have

$$|\theta_1-\alpha_1|<\sin^{-1}(Q[\alpha_1(n-1)]+n\mu) \quad (28)$$

In the hyperbolic CORDIC algorithm, by equation (19) and $\tilde{y}(n)=0$ we have

$$|\theta_{-1}-\alpha_{-1}|\le\sinh^{-1}\left|e^{|\alpha_{-1}|}-e^{|\theta|}\right|\le\sinh^{-1}\left|e^{|\alpha_{-1}-\theta|}-1\right| \quad (29)$$

In the linear CORDIC algorithm, by equations (16) and (21), and $\tilde{y}(n)=0$ we have

$$|y(n)|\le|x(0)|(Q[\alpha_0(n-1)]+n\mu) \quad (30)$$

Together with $|\theta_0-\alpha_0|=\left|\frac{y(n)}{x(0)}\right|$, we have

$$|\theta_0-\alpha_0|\le Q[\alpha_0(n)]+n\mu \quad (31)$$

4.2.2 Truncation Errors

Theorem 6**:** In the vectoring mode, the fixed-point truncation error of the circular, hyperbolic and linear CORDIC algorithms is bounded by $n\cdot\mu$, where $\mu=2^{-b}$.

Proof**:**

In the vectoring mode, from equation (15) we have

$$|\alpha_m-\phi_m|=\left|\alpha_m-\sum_{i=0}^{n-1}\delta(i)Q[\alpha_m(i)]\right|\le n\cdot\mu \quad (32)$$

where $\mu=2^{-b}$.

Finally, the overall error of the total CORDIC output is the sum of truncation error and approximation error. The formulas of the overall computation error in different coordinate systems and modes are summarized in Table 1. Similarly, the formulas of the overall computation error of the CORDIC with expanded convergence range are presented in Table 2 [14].

5 Numerical Accuracy and Hardware Trade-offs for Implementation of CORDIC Processor

Table 3 shows another contribution of this paper compared to the works of Hu [3], Kota [4] and Park [8]. In this section, hardware trade-offs between numerical accuracy and system parameters (i.e. the numbers of iterations n and word length b) are taken into account for the development of optimal CORDIC processors from the design point of view.

After extensive simulations, the numerical accuracy of CORDIC may not be improved any more by simply increasing the number of iterations. In order to develop an optimal CORDIC processor in terms of the trade-offs between the cost-benefit and desired performances, especially for the fixed-point applications, it is desirable to obtain the optimal numbers of iterations *n* and word length *b* from Tables 1and 2. For example, Figure 1 shows the two-dimensional error analysis of the circular CORDIC with expanded convergence range. Where, the relative error is set to 10^{-6}, the vertical and horizontal axes are the numbers of iterations *n* and word length *b*, respectively, and the vertex of the plot represents the optimal *n* and *b*. In addition, the prediction of computational error based on the numbers of iterations and word length is desirable.

6 Conclusion

The error analysis consisting of the absolute approximation error, relative approximation error and (absolute) truncation error of the conventional CORDIC algorithm and the CORDIC algorithm with expanded convergence range has been completed in this paper.

In the rotation mode of CORDIC, the maximum computation error is defined by the relative approximation error. On the other hand, as the truncation error is independent of the initial value, the fixed-point truncation error is defined in the absolute manner. It is worth noting that the number of iterations should not exceed the number of mantissa bits. Otherwise, the truncation error is to be increased as the iteration process proceeds. All the derived formulas describing the computational error of CORDIC are dependent on the word length and iteration number. Thus, a complete set of twelve formulas describing the fixed-point computation errors of CORDIC has been presented. By referring to these tables, one can design a cost-effective digital signal processing system using CORDIC processor in terms of areas and performances.

References:

[1] J. E. Volder, CORDIC Trigonometric Computing Technique, *IRE Transactions on Electronic Computers*, Vol. EC-8, No. 3, 1959, pp. 330-334.

[2] J. S. Walther, A Unified Algorithm for Elementary Functions, *Proc. Spring Joint Computer Conf.* 5, 1979, pp. 379-38.

[3] Y. H. Hu, The Quantization Effects of the CORDIC Algorithm, *IEEE Transactions on Signal Processing*, Vol. 40, No. 4, 1992, pp. 834-844.

[4] K. Kota, J. R. Callaro, Numerical Accuracy and Hardware Trade-offs for CORDIC Arithmetic for Special-Purpose Processors, *IEEE Transactions on Computers*, Vol. 42, No. 7, 1993, pp.769-779.

[5] Y. Ma, An Accurate Error Analysis Model for Fast Fourier Transform, *IEEE Trans. Signal Processing*, Vol. 45, June 1997, pp. 1641-1645.

[6] I. D. Yun, S. U. Lee, On the Fixed-Point Error Analysis of Several Fast DCT Algorithm, *IEEE Trans. Circuits Syst. Video Technol.,* Vol. 3, Feb. 1993, pp. 27-41.

[7] M. Bekooij, J. Huisken, K. Nowak, Numerical Accuracy of Fast Fourier Transform with CORDIC Arithmetic, *J. VLSI Signal Processing*, Vol. 25, No. 2, June 2000, pp. 187-193.

[8] S. Y. Park, N. I. Cho, Fixed-Point Error Analysis of CORDIC Processor Based on the Variance Propagation Formula, *IEEE Trans. on Circuits and Systems-I*, Vol. 51, No. 3, March 2004, pp.573-584.

[9] T.-Y. Sung, Y.-H. Sung, The Quantization Effects of CORDIC Arithmetic for Digital Signal Processing Applications, *The 21st Workshop on Combinatorial Mathematics and Computation Theory*, Taichung, Taiwan, May 21~22, 2004, pp. 16-25.

[10] T.-Y. Sung, K.-J. Lin, "The Quantization Effects of CORDIC Arithmetic with Expanding the Convergence Range," *Chung Hua Journal of Science and Engineering*, College of Engineering, Chung Hua University, Hsinchu, Taiwan, Vol.3, No.36, 2005, pp. 39-46.

[11] T.-Y. Sung, C.-S. Chen, M.-C. Shih, The Double Rotation CORDIC Algorithm: New Results for VLSI Implementation of Fast Sine/Cosine Generation, *2004 IEEE International Computer Symposium (ICS-2004)*, Taipei, Taiwan, Dec. 2004, pp.1285-1290.

[12] T.-Y. Sung, C.-S. Chen, M.-C. Shih, FPGA Implementation of a High-Speed CORDIC-Based Sine and Cosine Generator for OFDM Systems, *2005 IEEE International Conference on Systems and Signals (ICSS-2005)*, Kaohsiung, Taiwan, April 28-29, 2005, pp.1105-1109.

[13] T.-Y. Sung, H.-C. Hsin, Design and Simulation of Reusable IP CORDIC Core for Special-Purpose Processors, to appear in *IEE Proceedings – Comput. Digit. Tech.*.

[14] X. Hu, R. G. Harber, S. C. Bass, Expanding the range of the Convergence of the CORDIC Algorithm, *IEEE Trans. on Computers*, Vol. 40, No. 1, 1991, pp. 13-21.

Table 1 Overall computation errors of the conventional CORDIC algorithm

Mode	Linear Coordinate System
a. Rotation Mode:	$[2^{-n} + n \cdot 2^{-b}]\Vert\tilde{v}_0(n)\Vert + n \cdot 2^{-b}$
b. Vectoring Mode:	$[2^{-n} + n \cdot 2^{-b}] + n \cdot 2^{-b}$
Mode	**Circular Coordinate System**
a. Rotation Mode:	$[\tan^{-1} 2^{-(n-1)} + n \cdot 2^{-b}]\Vert\tilde{v}_1(n)\Vert + \sqrt{2} \cdot 2^{-b}\left[1 + \sum_{j=1}^{n-1}(\prod_{i=j}^{n-1}\sqrt{1+2^{-2i}})\right]$
b. Vectoring Mode:	$[\sin^{-1}(\tan^{-1} 2^{-(n-1)} + n \cdot 2^{-b})] + n \cdot 2^{-b}$
Mode	**Hyperbolic Coordinate System**
a. Rotation Mode:	$[\left\vert e^{\tanh^{-1} 2^{-(n-r)} + n2^{-b}} - 1\right\vert]\Vert\tilde{v}_{-1}(n)\Vert + \sqrt{2} \cdot 2^{-b}\left\{1 + \sum_{j=2}^{n}\left[\prod_{i=j}^{n}\sqrt{1-2^{-2i}}\right] \cdot e^{\sum_{i=j}^{n}\tanh^{-1} 2^{-i}}\right\}$
b. Vectoring Mode:	$[\sinh^{-1}(\left\vert e^{\tanh^{-1} 2^{-(n-r)} + n2^{-b}} - 1\right\vert)] + n \cdot 2^{-b}$

Table 2 Overall computation errors of the CORDIC with expanded convergence range

Mode	Linear Coordinate System
a. Rotation Mode:	$[2^{-n} + (n+M+1) \cdot 2^{-b+M+1}]\Vert\tilde{v}_0(n)\Vert + (n+M+1) \cdot 2^{-b}$
b. Vectoring Mode:	$[2^{-n} + (n+M+1) \cdot 2^{-b+M+1}] + (n+M+1) \cdot 2^{-b+M+1}$
Mode	**Circular Coordinate System**
a. Rotation Mode:	$[\tan^{-1} 2^{-(n-1)} + (n+2) \cdot 2^{-b+1}]\Vert\tilde{v}_1(n)\Vert + \sqrt{2} \cdot 2^{-b_v+1}\left[1 + \sum_{j=0,0,1}^{n-1}(\prod_{i=j}^{n-1}\sqrt{1+2^{-2i}})\right]$
b. Vectoring Mode:	$[\sin^{-1}(\tan^{-1} 2^{-(n-1)} + (n+2) \cdot 2^{-b+1})] + (n+2) \cdot 2^{-b+1}$
Mode	**Hyperbolic Coordinate System**
a. Rotation Mode:	$[\left\vert e^{\tanh^{-1} 2^{-(n-r)} + (n+M+1)2^{-b+M}} - 1\right\vert]\Vert\tilde{v}_{-1}(n)\Vert + \sqrt{2} \cdot 2^{-b}\left\{1 + \sum_{j=M+1}^{n}\left[\prod_{i=j}^{n}\sqrt{1-2^{-2i}}\right] \cdot e^{\sum_{i=j}^{n}\tanh^{-1} 2^{-i}}\right\}$
b. Vectoring Mode:	$[\sinh^{-1}(\left\vert e^{\tanh^{-1} 2^{-(n-r)} + (n+M+1)2^{-b+M}} - 1\right\vert)] + (n+M+1) \cdot 2^{-b+M}$

Table 3 Comparisons with the previous works
(*: Neglect the effect of sequence; **: Analyze only the circular class)

Analysis of computation errors	This work	Hu[3]	Kota[4]	Park [8]
The conventional CORDIC algorithm	Yes	*	**	Yes
CORDIC with expanded convergence range	Yes	No.	No.	No.

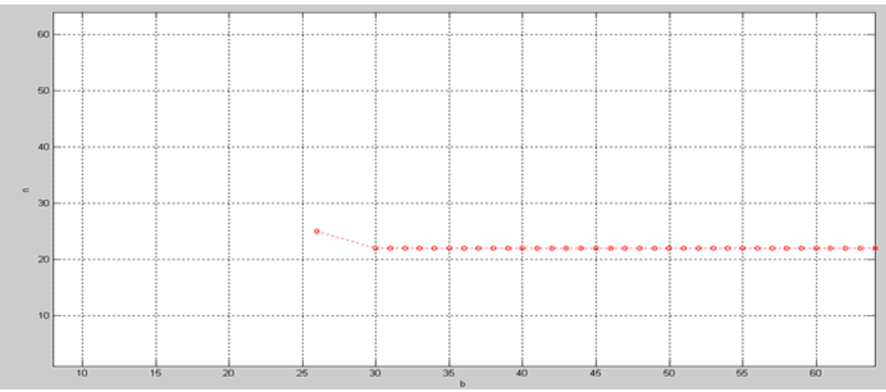

Fig.1. Two-dimensional error analysis of the circular CORDIC with expanded convergence range in the rotation mode for a given relative error of 10^{-6} (with respect to n and b)