Estimating the Region of Attraction of Ordinary Differential Equations by Quantified Constraint Solving

Henning Burchardt*
University of Oldenburg
Department of Computing Science
Germany

Stefan Ratschan†
Academy of Sciences of the Czech Republic
Institute of Computer Science
Czech Republic

Abstract: We formulate the problem of estimating the region of attraction using quantified constraints and show how the resulting constraints can be solved using existing software packages. We discuss the advantages of the resulting method in detail.

Key–Words: Region of attraction, constraint solving

1 Introduction

Given an ordinary differential equation $\dot{x} = f(x)$, an equilibrium of $\dot{x} = f(x)$, and a Lyapunov function V of $\dot{x} = f(x)$, we consider the problem of estimating the region of attraction around the equilibrium, that is, the problem of finding a set R such that the limit of every trajectory of $\dot{x} = f(x)$ starting in R is the equilibrium point.

Usually this problem is attacked by solving the optimization problem

$$\min \left\{ V(x) \mid \dot{V}(x) = 0, x \neq 0 \right\}. \tag{1}$$

which yields the corresponding sublevel set of V as an estimate for the region of attraction. This method has several drawbacks, which we remove by re-formulating the problem as a quantified constraint solving problem and applying corresponding solvers [12, 13, 3]. Among other advantages, the resulting method can not only estimate regions of attraction to equilibrium points but also to more general sets. Moreover, the method is easy to employ, since it relies on existing solvers, and does not need the implementation of complicated algorithms.

The classical overview of the general problem of estimating the region of attraction is an article by Genesio and co-authors [6]. Newer work that solves the problem based on a given function V uses a conservative convex optimization approximation [16], Gröbner basis computation [5], the solution of the above optimization problem using the theorem of Ehlich and Zeller [17], or an LMI method based on the theory of moments [8].

In Section 2 we formally describe the problem solved in this paper; in Section 3 we review the classical method used to solve this problem; in Section 4 we introduce a new method that removes some of the drawbacks of the classical method; in Section 5 we illustrate this improvement on some examples; and in Section 6 we conclude the paper.

2 Problem Description

Consider an ordinary differential equation

$$\dot{x} = f(x), \tag{2}$$

where f is continuous and $f(0) = 0$, that is, there is an equilibrium point of the system at the origin. Denote by $\dot{V}(x) = \nabla V^T f(x)$ the time-derivative of V in direction $f(x)$. We would like to estimate the region of attraction of f around the equilibrium, that is, we want to find a set R, such that for every trajectory $\phi(t)$ of $\dot{x} = f(x)$ that starts in R, the limit $\lim_{t\to\infty} \phi(t)$ is the equilibrium point. We call such a set R an *attraction region*. The existence of such an attraction region is ensured by the following corollary of Lyapunov's stability theorem:

Theorem 1 *Let (2) have an equilibrium at the origin and $D \subseteq \mathbb{R}^n$ be a connected set containing this equilibrium. Let $V : D \to \mathbb{R}$ be a continuously differen-*

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tiable function such that

$$V(0) = 0,$$
$$\textit{for all } x \in D - \{0\}, V(x) > 0$$
$$\textit{for all } x \in D - \{0\}, \dot{V}(x) < 0$$

Then there is an attraction region.

In such a case, the differential equation is called *asymptotically stable* and V is called a *Lyapunov function*. Note that the theorem is non-constructive, that is, it only ensures the existence of an attraction region, but does not provide it. In this paper, we consider the constructive version of the above theorem, that is, the problem of finding such an attraction region. We restrict ourselves to the case where both f and V are elementary functions.

Note that a Lyapunov function can often be found by solving the so-called Lyapunov equation for the linearization $\dot{x} = Ax$, where A is the Jacobian matrix of f evaluated at the origin. This captures the local behavior of the differential equation around the equilibrium. Hence, this will only allow us to find attraction regions that only include the part of the state space where the linearization behaves in a similar way as the original, non-linear system.

The problem of finding a Lyapunov function of a given non-linear ODE that captures the non-linear behavior of f around the equilibrium is highly non-trivial and usually relies on engineering intuition. Nonetheless, recently, tools for supporting this arise [10, 11, 14, 9, 7].

3 Classical Method

In this section we sketch a widely-used classical approach for finding an attraction region based on a given Lyapunov function V. In this approach, the attraction region is determined as a sublevel set of V, that is, a set $R_{V<c} \doteq \{x \mid V(x) < c\}$ for a certain value c. As long as the conditions that Theorem 1 enforces on D also hold on this set, we know that no trajectory leaves $R_{V<c}$. Moreover, these conditions ensure that every trajectory eventually reaches the equilibrium.

In order to deal with the case that $R_{V<c}$ is not connected, in the above process, one only considers the connected component of $R_{V<c}$ that contains the equilibrium.

One usually chooses c as the solution of Problem (1). If $V : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function such that $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$, and if the condition

$$\exists x_1 \in R_{V<c} : \dot{V}(x_1) < 0 \quad (3)$$

holds, then $\dot{V}(x) < 0$ for all $x \in R_{V<c}$. This can be seen as follows: From the choice of c it follows that for every $x \in R_{V<c}$, $\dot{V}(x) \neq 0$. This implies, because of the continuity of $V(x)$, and because of (3), that $\dot{V}(x) < 0$ for all $x \in R_{V<c}$. Hence, all preconditions of Theorem 1 hold for $D = R_{V<c}$, and V is a Lyapunov function.

This classical method has several drawbacks:

1. The found minimizer might not lie on the connected component of $\{x \mid \dot{V}(x) = 0\}$ that surrounds the equilibrium. This might result in a very conservative underapproximation of the region of attraction.

2. Sublevel sets of the Lyapunov function sometimes do not approximate the region of attraction well. Again, as a result, we might arrive at a very conservative underaproximation of the region of attraction.

3. If the found minimizer is only local but not global, then the resulting sublevel set is *not* an attraction region, that is, the method computes an incorrect result.

4. When excluding the equilibrium point from the minimization problem, one has to take care not to exclude other points with $\dot{V}(x) = 0$ also.

5. The procedure needs an equilibrium and cannot estimate regions of attraction to other sets, for example limit cycles.

4 Removing the Drawbacks

We observe that a large part of the reasoning above is done on sets. In the following we will describe these sets using constraints in a certain formal language (the first-order predicate language [4]) and then apply a corresponding solver.

First of all, we will use a constraint to describe the set of points that the trajectories eventually should reach. For example, this can be (in the 2-dimensional case) a ball described by $x_1^2 + x_2^2 < 1$, or a rectangle $-1 < x_1 < 1 \wedge -1 < x_2 < 1$. In the following, we will use $Target(x)$ as a short-cut for the used constraint.

Moreover, we will describe the attraction regions using such constraints. For example, in the case when one wants to use sublevel sets of Lyapunov functions this is the constraint $V(x) \leq c$ (or $V(x) < c$). In the following we will use $Region(c, x)$ as a short-cut for the used constraint, where x is the vector of state-space variables, and c the vector of parameters that can be changed to define different regions.

Now we can define the constraint that describes the c such that all elements of the region of attraction not yet in a target region are attracted to the region:

$$\forall x : \left[[Region(c, x) \wedge \neg Target(x)] \rightarrow \dot{V}(x) < 0 \right] \quad (4)$$

which can also be written as

$$\forall x : \left[\neg Region(c, x) \vee Target(x) \vee \dot{V}(x) < 0 \right] \quad (5)$$

Theorem 2 *If for a given c, Constraint (4) holds, the set $T \doteq \{x \mid Target(x)\}$ is open, and the set $R \doteq \{x \mid Region(c, x)\}$ is*

- *closed and bounded,*
- *contains the target set, and*
- *is invariant, that is, if for a given trajectory $\phi : \mathbb{R} \rightarrow \mathbb{R}^n$, and a given $t \geq 0$, $Region(c, \phi(t))$, then for all $t' > t$ also $Region(c, \phi(t'))$,*

then for every trajectory $\phi : \mathbb{R} \rightarrow \mathbb{R}^n$ with $Region(c, \phi(0))$, there is a $t > 0$ such that $Target(\phi(t))$.

Proof. Let $T \doteq \{x \mid Target(x)\}$, and let $R \doteq \{x \mid Region(c, x)\}$. Since R is bounded and $R \setminus T$ is closed, due to the continuity of $\dot{V}$, we know that $\dot{V}$ has a maximum ϵ in $R \setminus T$. Obviously, $\epsilon < 0$.

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}^n$ be a trajectory starting in R. We assume that for all $t \in \mathbb{R}_{\geq 0}$, not $\phi(t) \in T$, and derive a contradiction. Since R is invariant, and due to Constraint (4), for all $t \in \mathbb{R}_{\geq 0}$, $\dot{V}(\phi(t)) \leq \epsilon$. This implies that as t goes to infinity, $V(\phi(t))$ goes to minus infinity, contradicting the fact that V is bounded in $R \setminus T$.

Thus, there exists a $t \in \mathbb{R}_{\geq 0}$ such that $\phi(t) \in T$. □

One can either use a constraint $Region(c, x)$ for which the above conditions holds by construction (e.g., ensuring invariance by defining $Region(c, x)$ as sublevel sets of the Lyapunov function), or by formulating appropriate conditions that can be added to the above constraint.

In which cases, and how can these constraints be solved? In the case where all terms occuring in the constraints are polynomial, this is ensured by the fact that the predicate-logical theory of the real-numbers admits quantifier elimination [15]. This means that there is an algorithm that, for every expression that contains the logical quantifiers $\forall$ and $\exists$, the Boolean connectives $\wedge$, $\vee$, $\neg$, the predicate symbols $\leq$, $<$, and the function symbols $\times$ for multiplication and $+$ for addition, computes an equivalent, but quantifier free expression. There are also software packages implementing such algorithms (e.g., QEPCAD [1], the `Resolver` function implemented in *Mathematica*). In practice, however, the complexity for doing this is huge, and corresponding software packages can only solve examples of moderate size.

To be able to solve larger examples, and to also allow functions given by expressions with symbols like $\sin$, $\cos$, or $\exp$, one can resort to a method that can compute approximate solutions [13]. This method is implemented in the solver RSOLVER [12]. Given a quantified constraint and closed intervals restricting every variable in the input constraint, it computes a set of *boxes* (i.e., hyper-rectangles) containing only solutions to the input constraint and a set of boxes containing only non-solutions. To retain correctness of the method, it is necessary that the box B restricting the universally quantified variables of constraint (4) contains the found attraction region. Since the computation times of RSOLVER increase, as the size of B increases, one can for example first try to solve an example with a small box, and if the resulting region intersects B one can accordingly increase its size. The precision of the computed approximation can be adjusted by allowing that not the whole box restricting the free variables of the input constraint has to be covered by the set of boxes computed by RSOLVER . The ratio of the size of the region that is left uncovered relative to the size of the box restricting the free variables is called the *remnant value*.

The method described above solves the problems listed in the previous sections as follows:

- Problems 1 and 2 are solved or at least alleviated by the fact that one can freely define $Region(c, x)$, and hence one can approximate the region of attraction more closely than with level sets.
- Problems 3 and 4 are solved by the fact that the given constraints are solved exactly, by taking into account global, and not only local information, and by not suffering from correctness problems due to rounding errors.
- Problem 5 is solved by the fact that one can now freely define the target region $Region$.

5 Examples

This section contains three examples that illustrate the method described in the previous section and how it solves some of the problems of the classical method. All computations have been performed with both RSOLVER and QEPCAD on a Linux PC with an Intel Pentium 4 2.7 GHz CPU and 1 GB memory. For the RSOLVER calculations, the execution time depends highly on the boxes B restricting the domain for the bound variable x and the free variable c, so they are also listed.

5.1 Example 1

This is an example where the optimization problem (1) has more than one local minimum. The system dynamics is given by the differential equation

$$\dot{x} = \left[-x_2 - x_1 x_2 + x_1^2 x_2,\ 2x_1 - x_2\right]^T \qquad (6)$$

If the system is linearized at the origin, the right-hand side of (6) becomes $[-x_2,\ 2x_1 - x_2]^T$. Stability of the linearized system can be shown by using the Lyapunov function

$$V(x) = 7x_1^2 - 2x_1x_2 + 3x_2^2$$

For this example, we define

$$Region(c, x) \doteq V(x) \le c$$

As Figure 1 indicates, the optimization problem (1) has two local minima, one on the component of $\dot{V} = 0$ in the lower left part of the figure ($c \approx 1.5$), and one on the component in the upper right part of the figure ($c \approx 15$). Using a remnant value of 0.001 and the restrictions $x \in [-5, 5] \times [-5, 5]$, $c \in [0, 20]$, RSOLVER requires a time of about 28 seconds to prove constraint (4) for $c \in [0, 1.495]$ and to disprove it for $c \in [1.512, 20]$. As the volume of $R = \{x \mid Region(c, x)\}$ grows with increasing value c, the optimal solution has to lie in the interval $[1.495, 1.512]$. Figure 1 shows the set R for the value $c = 1.5$.

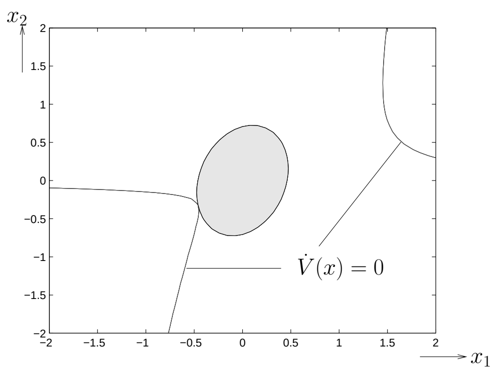

Figure 1: Attraction Region for Example 1

QEPCAD computes in approximately one second the equivalent, but quantifier free constraint

$$[[C_1 \le 0 \wedge C_2 \ge 0] \vee [C_3 > 0 \wedge C_2 \le 0]],$$

where

$$\begin{aligned}
C_1 &= 3c - 20,\\
C_2 &= 194481c^4 - 3016440c^3 + 6051600c^2 - 33024000c\\
&\quad + 40960000,\\
C_3 &= 1058841c^10 + 3756465342c^9 + 3924133090465c^8\\
&\quad - 89272758849652c^7 + 652631620171440c^6\\
&\quad - 4245121175691200c^5 + 19742893494400000c^4\\
&\quad - 54774380646400000c^3 + 49295495987200000c^2\\
&\quad + 150798336000000c + 152043520000000
\end{aligned}$$

The solution set of the above constraint can be closely approximated by RSOLVER in negligible time. Moreover, QEPCAD can compute a similar quantifier-free constraint in the case where the target set consists of a single point.

5.2 Example 2

Our method can also be applied when no stable equilibrium exists, like in the following example. Let the system dynamics be given by

$$\dot{x} = \left[x_2 + 0.2x_1 - 0.1x_1^3 + 0.01x_1^2x_2^2, -x_1\right]^T$$

and let further be

$$V(x) = 7x_1^2 - 2x_1x_2 + 3x_2^2 \text{ and}$$
$$Region(c, x) \doteq V(x) \le c$$

We define not only the attraction region but also the target region as a sublevel set of this Lyapunov function. Hence the target region is defined by a constraint of the form $V(x) < d$, where d is a new parameter, and the full constraint we try to solve is

$$\forall x \in B : d \le V(x) \le c \Rightarrow \dot{V} < 0. \qquad (7)$$

We will use the constraint solvers to find a c that is as large as possible and a d that is as small as possible. In order to reduce the problem dimension we split the problem into three subproblems:

1. Determine a value c_0 such that the constraint $\forall x \in B : V(x) = c_0 \Rightarrow \dot{V} < 0$ holds,
2. determine the possible values for d by solving
$$\forall x \in B : d \le V(x) \le c_0 \Rightarrow \dot{V} < 0 \qquad (8)$$
3. and for c by solving
$$\forall x \in B : c_0 \le V(x) \le c \Rightarrow \dot{V} < 0. \qquad (9)$$

Note that the number of free variables of both (8) and (9) is only half the number of free variables of the original problem (7). For each subproblem the remnant value was set to a value of 0.01 and x was restricted to the interval $[-10, 10] \times [-10, 10]$. The variables c_0, d and c were bounded by the intervals $[0, 500]$, $[0, 100]$ and $[100, 500]$, respectively. After a negligable computation time RSOLVER came up with some boxes solving the first problem, from which we chose the value 100 for c_0. RSOLVER needed 26 minutes in total to prove constraint (8) for $d \in [34.32, 100]$ and constraint (9) for $c \in [100, 220.1]$. RSOLVER also disproved (8) for $d \in [0, 33.09]$ and (9) for $c \in [223.8, 500]$, so – with a similar reasoning as for example 1 – the optimal solution for d lies inside interval $[33.09, 34.32]$ and the optimal solution for c lies inside $[220.1, 223.8]$. Figure 2 shows the target region and the attraction region for the values $c = 222$ and $d = 34$: The light shaded area depicts the target region, while the union of the light shaded area and the dark shaded area depicts the attraction region. Unlike RSOLVER, QEPCAD could not solve this problem.

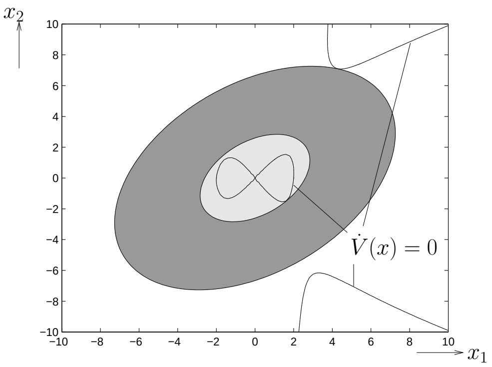

Figure 2: Attraction Region for Example 2

5.3 Example 3

The following example illustrates how to approximate the region of attraction in a better way than with sublevel sets.

Consider the differential equation

$$\dot{x} = \left[-2x_1,\ -x_2 + x_2^2\right]^T$$

Furthermore, let

$$Region(r, x) \doteq V(x) \leq r \wedge g(x) \leq 0,$$

where $g(x) = x_1^2 + x_2$, $V(x) = x_1^2 + x_2^2$.

For a given r, the set $R \doteq \{x \mid Region(r, x)\}$ is bounded because it is a subset of the bounded set $\{x \mid V(x) \leq r\}$ and it is closed because it is an intersection of the two closed sets $\{x \mid V(x) \leq r\}$ and $\{x \mid g(x) \leq 0\}$. We try to find the maximum r that fulfills the following conditions:

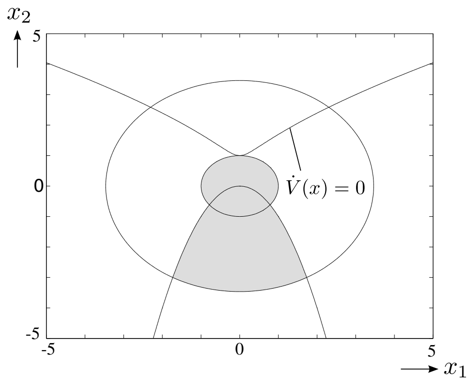

Figure 3: Attraction Region for Example 3

$$\forall x \in B: \ Region(r, x) \wedge \neg Target(x) \Rightarrow \dot{V}(x) < 0 \quad (10)$$

$$\forall x \in B: \ V(x) \leq r \wedge g(x) = 0 \wedge \neg Target(x) \Rightarrow \dot{g}(x) = \frac{\partial g}{\partial x} f(x) \leq 0 \quad (11)$$

Constraint (11) models the invariance requirement of Theorem 2. The union of R and $\{x \mid V \leq c\}$ – where c is the maximum value satisfying (4) – can be used as a approximation of the region of attraction. Letting x be bounded by the interval $[-5, 5] \times [-5, 5]$ and c and r bounded by $[0, 20]$, RSOLVER needed less than one second to solve (4) and about 86 seconds to solve (10-11). In both cases, a remnant value of 0.01 was used. Before solving (10-11), the equality $g(x) = 0$ in (11) was replaced by $g(x) < 10^{-4} \wedge g(x) > -10^4$. Following the procedure described in sections 5.1 and 5.2, the maximum value for c and r could be shown to lie in the intervals $[0.999, 1.145]$ and $[11.999, 12.005]$, respectively. Refer to Figure 3 for a visualization of set $\{x \mid V \leq c\} \cup R$ for $c = 1$ and $r = 12$: The small ellipse depicts the set $\{x \mid V(x) = c\}$, the large ellipse the set $\{x \mid V(x) = r\}$ and the parabola-shaped curve depicts the set $\{x \mid g(x) = 0\}$. The set $\{x \mid V \leq c\} \cup R$ is indicated by the shaded area.

QEPCAD computes in 4min 20s the equivalent, but quantifier free formula

$$400d^2 - 4760d + 17 < 0 \vee d^3 - 10d^2 + 21d - 36 \leq 0,$$

whose solution set can be closely approximated by RSOLVER in neglible time.

6 Conclusion

We have provided a method for estimating the region of attraction of ordinary differential equations, that removes some of the drawbacks of the classical approach by formulating the problem as a quantified constraint solving problem. As the examples in the previous section show, we always computed the whole set of feasible values for the free variables instead of calculating only the optimal value. In future work we will exclude solutions that are known not to be optimal from the beginning, so problems with more variables and more complex dynamics can be handled. A possible means to achieve this goal is the method of Lagrange multipliers. Another possible extension of our approach is to estimate regions of attraction for hybrid systems, i.e., systems with modes each of which having different dynamics.

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